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ON THE DIVISIBILITY OF POWER LCM MATRICES BY POWER GCD MATRICES

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Abstract. Let $S=\{x_1,\ldots,x_n\}$ be a set of n distinct positive integers and $e\geqslant 1$ an integer. Denote the $n\times n$ power GCD (resp. power LCM) matrix on S having the e-th power of the greatest common divisor (x_i,x_j) (resp. the e-th power of the least common multiple $[x_i,x_j]$) as the (i,j)-entry of the matrix by $((x_i,x_j)^e)$ (resp. $([x_i,x_j]^e)$). We call the set S an odd gcd closed (resp. odd lcm closed) set if every element in S is an odd number and $(x_i,x_j)\in S$ (resp. $[x_i,x_j]\in S$) for all $1\leqslant i,j\leqslant n$. In studying the divisibility of the power LCM and power GCD matrices, Hong conjectured in 2004 that for any integer $e\geqslant 1$, the $n\times n$ power GCD matrix $((x_i,x_j)^e)$ defined on an odd-gcd-closed (resp. odd-lcm-closed) set S divides the $n\times n$ power LCM matrix $([x_i,x_j]^e)$ defined on S in the ring $M_n(\mathbb{Z})$ of $n\times n$ matrices over integers. In this paper, we use Hong's method developed in his previous papers [J. Algebra 218 (1999) 216–228; 281 (2004) 1–14, Acta Arith. 111 (2004), 165–177 and J. Number Theory 113 (2005), 1–9] to investigate Hong's conjectures. We show that the conjectures of Hong are true for $n\leqslant 3$ but they are both not true for $n\geqslant 4$.

Keywords: GCD-closed set, LCM-closed set, greatest-type divisor, divisibility

MSC 2000: 11C20, 11A25, 15A36

1. Introduction

Let f be an arithmetical function. It was first stated by H. Smith in 1876 in his famous paper [19] that if $[f(i,j)]$ is an $n\times n$ matrix having f evaluated at the greatest common divisor (i,j) of i and j as the (i,j)-entry of the matrix, then

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$\det[f(i,j)] = \prod_{k=1}^{n} (f*\mu)(k)$, where μ is the Möbius function and $f*\mu$ is the Dirichlet convolution of f and μ. This result was generalized by Apostol [1] in 1972 and in 1988, McCarthy [18] extended the results of both Smith and Apostol to the class of even functions of m (mod r), where m and r are positive integers. Here we call a complex-valued function $\beta(m,r)$ an *even function of* m (mod r) if $\beta(m,r) = \beta((m,r),r)$ for all values of m, and we notice that the functions considered by Smith and Apostol are in fact even functions of m (mod r). The results of Smith, Apostol, and McCarthy were subsequently extended further by Bourque and Ligh [5] in 1993. The results of Smith, Apostol, McCarthy, Bourque and Ligh have been generalized by Hong [10] in 2002 to certain classes of arithmetical functions.

For the set $S = \{x_1, \ldots, x_n\}$ of n distinct positive integers, we denote the $n \times n$ matrix on S having f evaluated at the greatest common divisor (x_i, x_j) of the entries x_i and x_j by $(f(x_i, x_j))$ and we use $(f[x_i, x_j])$ to denote the $n \times n$ matrix on the set S having f evaluated at the least common multiple $[x_i, x_j]$ of the entries x_i and x_j, respectively. Then some factorization theorems on the divisibility of the matrix $(f[x_i, x_j])$ by the matrix $(f(x_i, x_j))$ were obtained by Bourque and Ligh [6] and also by Hong in [9] and [11]. Furthermore, Hong has also given some theorems on the nonsingularity of the matrices $(f(x_i, x_j))$ and $(f[x_i, x_j])$ in [13].

Now, for any given integer $e \geqslant 1$, we let ξ_e be the arithmetical function defined for any positive integer x by $\xi_e(x) = x^e$. We then call $(\xi_e(x_i, x_j))$ (abbreviated by $((x_i, x_j)^e)$) and $(\xi_e[x_i, x_j])$ (abbreviated by $([x_i, x_j]^e)$) the $n \times n$ *power greatest common divisor (GCD) matrix* on S and the $n \times n$ *power least common multiple (LCM) matrix* on S respectively. If $e = 1$, then we simply call them the *greatest common divisor (GCD) matrix* and *the least common multiple (LCM) matrix,* respectively. Naturally, we call the set S *factor closed (FC)* if it contains all divisors of x for any $x \in S$. The set S is called *gcd closed* if $(x_i, x_j) \in S$ for all $1 \leqslant i, j \leqslant n$. Obviously, any FC set is gcd closed but the converse is not necessarily true. In this aspect, Bourque and Ligh first generalized Smith's result in [19] and also Beslin and Ligh showed in [2] that the determinant of the power GCD matrix $((x_i, x_j)^e)$ on a gcd-closed set $S = \{x_1, \ldots, x_n\}$ is the product $\prod_{k=1}^{n} \alpha_{e,k}$, where

$$\alpha_{e,k} = \sum_{\substack{d \mid x_k \\ d \nmid x_t, x_t < x_k}} J_e(d).$$

In the above equality, we call $J_e := \xi_e * \mu$ the Jordan totient function. Hong [10] proved that the determinant of the LCM matrix $([x_i, x_j]^e)$ on a gcd-closed set

$S = \{x_1, \ldots, x_n\}$ is equal to $\prod_{k=1}^{n} x_k^{2e} \cdot \beta_{e,k}$, where

$$\beta_{e,k} = \sum_{\substack{d \mid x_k \\ d \nmid x_t, x_t < x_k}} \Big(\frac{1}{\xi_e} * \mu\Big)(d).$$

On the other hand, Hong has also obtained two important results in [12] on the nonsingularity of the power LCM matrix $(\xi_e[x_i, x_j])$. It was first noticed by Bourque and Ligh in [4] that the power GCD matrix $(\xi_e(x_i, x_j))$ on any set S is positive definite, and then Hong and Loewy [15] made some progress on the asymptotic behavior of the eigenvalues of the power GCD matrix $(\xi_e(x_i, x_j))$ on the set S. The eigenvalues of another kind of power GCD matrix were investigated by Wintner [20] as well as Lindqvist and Seip [17].

In studying the GCD and LCM matrices, Bourque and Ligh [3] showed that if the set $S = \{x_1, \ldots, x_n\}$ is FC then the GCD matrix $((x_i, x_j))$ on S always divides the LCM matrix $([x_i, x_j])$ on S in the ring $M_n(\mathbb{Z})$ of $n \times n$ matrices over the integers. It was noticed by Hong in [9] that the factorization theorem on LCM and GCD matrices is in general not true. We now call the set S an *odd gcd closed* set if S is gcd closed and every element in S is an odd number. Naturally, we call the set S an *even gcd closed* set if S is not an odd gcd closed set. By [9] we know that there exists an even-gcd-closed set $S = \{x_1, \ldots, x_n\}$ such that the GCD matrix $((x_i, x_j))$ on S does not divide the LCM matrix $([x_i, x_j])$ on S in the ring $M_n(\mathbb{Z})$. However, it is not clear whether there exists an odd-gcd-closed set $S = \{x_1, \ldots, x_n\}$ such that its GCD matrix $((x_i, x_j))$ on S does not divide the LCM matrix $([x_i, x_j])$ on S in the ring $M_n(\mathbb{Z})$? Consequently, Hong [12] proposed the following conjecture.

Conjecture 1.1. *Let $e \geqslant 1$ be a positive integer and $S = \{x_1, \ldots, x_n\}$ an odd-gcd-closed set. Then the power GCD matrix $((x_i, x_j)^e)$ on S divides the power LCM matrix $([x_i, x_j]^e)$ on S in the ring $M_n(\mathbb{Z})$.*

For the above conjecture, He and Zhao [7] have recently given a counterexample so that the above Conjecture 1.1 is not true for $e = 1$ and $n = 4$. In this paper, by using the reduced formulas given in [12] and [13] and by using Hong's method developed in [8] for finding a solution of the Bourque-Ligh conjecture in [3], we are able to show that for any given integer $e \geqslant 1$, Conjecture 1.1 is true for $n \leqslant 3$, but it is not true for $n \geqslant 4$. Thus Hong's Conjecture 1.1 is completely solved.

On the other hand, we call the set S *lcm closed* if $[x_i, x_j] \in S$ for all $1 \leqslant i, j \leqslant n$. The set S is called *odd lcm closed* if S is lcm closed and every element in S is an odd number. Thus the set S is an *even lcm closed* set if it is not an odd lcm closed set. For example, the set $S = \{1, 2, 3, 6, 8, 24\}$ is an even lcm closed set. In fact, we can easily construct an *even-lcm-closed set* S such that the GCD matrix $((x_i, x_j))$ on S does not divide the LCM matrix $([x_i, x_j])$ on S in the ring $M_n(\mathbb{Z})$ (see [9]). However, it is not clear whether there exists an odd-lcm-closed set $S = \{x_1, \ldots, x_n\}$ such that the GCD matrix $((x_i, x_j))$ on the set S does not divide the LCM matrix $([x_i, x_j])$ on the set S in the ring $M_n(\mathbb{Z})$? For the lcm-closed sets, Hong [12] has also proposed the following conjecture.

Conjecture 1.2. *Let $e \geqslant 1$ be a positive integer and $S = \{x_1, \ldots, x_n\}$ an odd-lcm-closed set. Then the power GCD matrix $((x_i, x_j)^e)$ on S divides the power LCM matrix $([x_i, x_j]^e)$ on S in the ring $M_n(\mathbb{Z})$.*

For this conjecture, He and Zhao also gave a counterexample in [7] for $e = 1$ and $n = 4$. In this paper, we will show that for any given integer $e \geqslant 1$, Conjecture 1.2 is true for $n \leqslant 3$, but the conjecture is false for $n \geqslant 4$. Thus Conjecture 1.2 is also completely solved.

2. Preliminaries

In this section, we recall the reduced formulas of Hong for $\alpha_{e,k}$ and $\beta_{e,k}$. First we recall the concept of greatest-type divisor given by Hong.

Definition ([8]). Let T be a set of distinct positive integers. For any $a, b \in T$ and $a < b$, we call a a *greatest-type divisor* of b in T if $a \mid b$ and the conditions $a \mid c \mid b$ and $c \in T$ imply that $c \in \{a, b\}$.

Remark. The concept of greatest-type divisor played central roles in solving the Bourque-Ligh conjecture [3] (see Hong [8]) and in solving Sun's conjecture in [14].

Lemma 2.1 ([13]). *Let $S = \{x_1, \ldots, x_n\}$ be a gcd-closed set and $R_k = \{y_{k,1}, \ldots, y_{k,l_k}\}$ the set of the greatest-type divisors of x_k $(1 \leqslant k \leqslant n)$ in S, where $y_{k,1} < \ldots < y_{k,l_k}$, $l_1 = 0$, $l_2 = l_3 = 1$, and $1 \leqslant l_k \leqslant k - 2$ for $k \geqslant 4$. Then*

$$\alpha_{e,k} = x_k^e + \sum_{t=1}^{l_k} (-1)^t \sum_{1 \leqslant i_1 < \ldots < i_t \leqslant l_k} (x_k, y_{k,i_1}, \ldots, y_{k,i_t})^e.$$

Lemma 2.2 ([12]). *Let $S = \{x_1, \ldots, x_n\}$ be a gcd-closed set. Let $R_k = \{y_{k,1}, \ldots, y_{k,l_k}\}$ be the set of the greatest-type divisors of x_k $(1 \leqslant k \leqslant n)$ in S, where $y_{k,1} < \ldots < y_{k,l_k}, l_1 = 0, l_2 = l_3 = 1$, and $1 \leqslant l_k \leqslant k-2$ for $k \geqslant 4$. Then*

$$\beta_{e,k} = \frac{1}{x_k^e} + \sum_{t=1}^{l_k} (-1)^t \sum_{1 \leqslant i_1 < \ldots < i_t \leqslant l_k} \frac{1}{(x_k, y_{k,i_1}, \ldots, y_{k,i_t})^e}.$$

Remark. Lemmas 1.1 and 1.2 can be extended to posets (see Hong and Sun in [16]).

3. Solving conjecture 1.1

We first prove the following crucial lemma.

Lemma 3.1. *Let $e \geqslant 1$, $n \geqslant 4$ be integers and $S = \{x_1, \ldots, x_n\}$. Suppose that*

$$x_k = a^{k-1}, \quad 1 \leqslant k \leqslant n-3, \quad x_{n-2} = qb, \quad x_{n-1} = pb, \quad x_n = p^2 qb, \tag{3.1}$$

where $b = a^{n-4}$, q and p are distinct primes, and $a > 1$ is an integer satisfying $(a, p^e q^e + q^e - 1) = 1$. If the determinant of the $n \times n$ power LCM matrix $([x_i, x_j]^e)$ defined on S is divisible by the $n \times n$ power GCD matrix $((x_i, x_j)^e)$ defined on S, then $p \mid (q^e - 1)$.

P r o o f. We first note that $\alpha_{e,1} = \beta_{e,1} = 1$. For $2 \leqslant k \leqslant n-3$, we have, by Lemmas 2.1 and 2.2,

$$\alpha_{e,k} = a^{e(k-1)} - a^{e(k-2)} = a^{e(k-2)})(a^e - 1)$$

and

$$\beta_{e,k} = \frac{1}{a^{e(k-1)}} - \frac{1}{a^{e(k-2)}} = \frac{1 - a^e}{a^{e(k-1)}},$$

respectively. Consequently, for $2 \leqslant k \leqslant n-3$, we can compute that

$$\frac{x_k^{2e} \beta_{e,k}}{\alpha_{e,k}} = \frac{a^{e(k-1)})(1 - a^e)}{a^{e(k-2)}(a^e - 1)} = -a^e. \tag{3.2}$$

Clearly, the greatest-type divisors of both $x_{n-2} = qb$ in S and $x_{n-1} = pb$ in S are b, so by using Lemmas 2.1 and 2.2 again, we have

$$\frac{x_{n-2}^{2e} \beta_{e,n-2}}{\alpha_{e,n-2}} = \frac{(qb)^{2e}(1/(qb)^e - 1/b^e)}{(qb)^e - b^e} = -q^e \tag{3.3}$$

and

$$\frac{x_{n-1}^{2e}\beta_{e,n-1}}{\alpha_{e,n-1}} = (pb)^{2e}(1/(pb)^e - 1/b^e)/(pb)^e - b^e = -p^e. \tag{3.4}$$

Since the greatest-type divisors of $x_n = p^2qb$ in S are qb and pb, it follows from Lemmas 2.1 and 2.2 that

$$\begin{aligned} \frac{x_n^{2e}\beta_{e,n}}{\alpha_{e,n}} &= \frac{(p^2qb)^{2e}(1/(p^2qb)^e - 1/(qb)^e - 1/(pb)^e + 1/b^e)}{(p^2qb)^e - (qb)^e - (pb)^e + b^e} \\ &= p^{2e}q^e \cdot \frac{p^eq^e - p^e - 1}{p^eq^e + q^e - 1}. \end{aligned} \tag{3.5}$$

Therefore, by Equations (3.2)–(3.5), we infer that

$$\begin{aligned} \frac{\det([x_i, x_j]^e)}{\det((x_i, x_j)^e)} &= \prod_{k=1}^{n} \frac{x_k^{2e}\beta_{e,k}}{\alpha_{e,k}} \\ &= (-a)^{e(n-4)} \cdot (-q^e) \cdot (-p^e) \cdot p^{2e}q^e \cdot \frac{p^eq^e - p^e - 1}{p^eq^e + q^e - 1} \\ &= (-1)^{en} \cdot q^{2e} \cdot p^{3e} \cdot a^{e(n-4)} \cdot \frac{p^eq^e - p^e - 1}{p^eq^e + q^e - 1}. \end{aligned}$$

It is now easy to see that $(q^{2e}, p^eq^e + q^e - 1) = (a^{e(n-4)}, p^eq^e + q^e - 1) = 1$. However, by our assumption, we can easily see that

$$\frac{\det([x_i, x_j]^e)}{\det((x_i, x_j)^e)} \in \mathbb{Z}.$$

So we have

$$p^{3e} \cdot \frac{p^eq^e - p^e - 1}{p^eq^e + q^e - 1} \in \mathbb{Z}.$$

Since $p^eq^e - p^e - 1 < p^eq^e + q^e - 1$, from the above equation, we can deduce that $p \mid (p^eq^e + q^e - 1)$. Hence it follows that $p \mid (q^e - 1)$, as desired. $\square$

Now we give below an answer to Conjecture 1.1.

Theorem 3.2. *Let $e \geqslant 1$ be an arbitrary given integer and $n \geqslant 1$. Then the following statements hold:*

(i) *If $n \leqslant 3$, then for any gcd-closed set $S = \{x_1, \ldots, x_n\}$, the power GCD matrix $((x_i, x_j)^e)$ on S divides the power LCM matrix $([x_i, x_j]^e)$ on S in the ring $M_n(\mathbb{Z})$.*

(ii) *For $n \geqslant 4$, there exists an odd-gcd-closed set $S = \{x_1, \ldots, x_n\}$ such that the power GCD matrix $((x_i, x_j)^e)$ on the set S does not divide the power LCM matrix $([x_i, x_j]^e)$ on the set S in the ring $M_n(\mathbb{Z})$.*

P r o o f. (i) Let $S=\{x_1,\dots,x_n\}$ be a gcd-closed set. Without loss of generality, we may assume that $1\leqslant x_1<\dots<x_n$. If $n=1$, then it is clear that the statement is true. If $n=2$, then because the set $S=\{x_1,x_2\}$ is gcd closed, we know that $x_1\mid x_2$. Now, we form the matrix

$$A=\begin{pmatrix}0 & 1\\ (x_2/x_1)^e & 0\end{pmatrix}.$$

Since $e\in\mathbb{Z}^+$ and $x_2/x_1\in\mathbb{Z}$, we deduce that $(x_2/x_1)^e\in\mathbb{Z}$, and, consequently, $A\in M_2(\mathbb{Z})$. We can also check that $([x_i,x_j]^e)=A\cdot((x_i,x_j)^e)$. Therefore, our result holds for the case of $n=2$. Now, we consider the case of $n=3$. Since the set $S=\{x_1,x_2,x_3\}$ is gcd closed, we can easily check that $x_1\mid x_i$ $(i=2,3)$, and $(x_2,x_3)=x_1$ or x_2. If $(x_2,x_3)=x_2$, then $x_1\mid x_2\mid x_3$. Now, we form the matrix

$$B=\begin{pmatrix}0 & 0 & 1\\ (x_2/x_1)^e & -1 & 1\\ (x_3/x_1)^e & 0 & 0\end{pmatrix}.$$

Since $(x_2/x_1)^e\in\mathbb{Z}$ and $(x_3/x_1)^e\in\mathbb{Z}$, we can see that $B\in M_3(\mathbb{Z})$. Also, we can easily check that $([x_i,x_j]^e)=B\cdot((x_i,x_j)^e)$. This shows that the statement in this case is still true. Now, we consider the case: $(x_2,x_3)=x_1$. For such case, we have $[x_2,x_3]=x_2x_3/x_1$. Let

$$C=\begin{pmatrix}-1 & 1 & 1\\ 0 & 0 & (x_2/x_1)^e\\ 0 & (x_3/x_1)^e & 0\end{pmatrix}.$$

Then we have $C\in M_3(\mathbb{Z})$. Now, we can easily check that $([x_i,x_j]^e)=C\cdot((x_i,x_j)^e)$. Hence the statement (i) in this case holds.

(ii) Let $n\geqslant 4$ be an integer and consider the set $S=\{x_1,\dots,x_n\}$ as in (3.1). Since q and p are distinct odd primes such that $p>q^e-1$ (for any given integer $e\geqslant 1$, such a pair $(p,\,q)$ always exists since there are infinitely many primes), and $b=a^{n-4}$ and $a>1$ is an odd number satisfying the situation $(a,p^eq^e+q^e-1)=1$ (such element a always exists, for example, we can take $a=2$, or q), S is clearly an odd gcd closed set. We now claim that

$$\frac{\det([x_i,x_j]^e)}{\det((x_i,x_j)^e)}\notin\mathbb{Z}. \tag{3.6}$$

For if otherwise, we will have $\det([x_i,x_j]^e)\mid\det((x_i,x_j)^e)$. Then by Lemma 3.1, we know that $p\mid(q^e-1)$, and thereby, $p\leqslant q^e-1$. This is of course absurd since $p>q^e-1$. Thus, our claim is established. It now follows from (3.6) that in the ring $M_n(\mathbb{Z})$, we have $((x_i,x_j)^e)\nmid([x_i,x_j]^e)$, as required. The proof of Theorem 3.2 is hence complete. $\square$

Remark. In Theorem 3.2, we see immediately that Conjecture 1.1 holds for $n \leqslant 3$ and that Conjecture 1.1 does not hold for $\geqslant 4$.

4. Solving conjecture 1.2

In this section, we denote the least common multiple of all elements in S by $m = \mathrm{lcm}(S)$. We first prove the following lemmas.

Lemma 4.1. *Let $e, n \geqslant 1$ be integers and $S = \{x_1, \ldots, x_n\}$ a set of n distinct positive integers. Then we have the following equalities:*

$$((x_i, x_j)^e) = \frac{1}{m^e} \cdot \mathrm{diag}(x_1^e, \ldots, x_n^e) \cdot \left(\left(\frac{m}{x_i}, \frac{m}{x_j}\right)^e\right) \cdot \mathrm{diag}(x_1^e, \ldots, x_n^e)$$

and

$$([x_i, x_j]^e) = \frac{1}{m^e} \cdot \mathrm{diag}(x_1^e, \ldots, x_n^e) \cdot \left(\left[\frac{m}{x_i}, \frac{m}{x_j}\right]^e\right) \cdot \mathrm{diag}(x_1^e, \ldots, x_n^e).$$

Proof. We first observe the following equalities:

$$(x_i, x_j) = \frac{m}{\left[\frac{m}{x_i}, \frac{m}{x_j}\right]} = \frac{m \cdot \left(\frac{m}{x_i}, \frac{m}{x_j}\right)}{\frac{m}{x_i} \cdot \frac{m}{x_j}} = \frac{x_i x_j}{m} \cdot \left(\frac{m}{x_i}, \frac{m}{x_j}\right).$$

Since $e \geqslant 1$ is an integer, we have

$$(x_i, x_j)^e = \frac{x_i^e x_j^e}{m^e} \cdot \left(\frac{m}{x_i}, \frac{m}{x_j}\right)^e.$$

Therefore the first equation follows immediately. The second equation has been proved in [12]. $\square$

Definition ([12]). Let $S = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers. Then the reciprocal set of S, denoted by mS^{-1}, is defined by $mS^{-1} = \{\frac{m}{x_1}, \ldots, \frac{m}{x_n}\}$.

Lemma 4.2. *Let $S = \{x_1, \ldots, x_n\}$ be a set of distinct positive integers. Then S is an lcm-closed set if and only if the reciprocal set mS^{-1} is a gcd-closed set.*

Proof. One side of the equivalence has been proved by Hong in [12]. The converse implication can be proved similarly and hence we omit the details. $\square$

We now give an answer to Conjecture 1.2.

Theorem 4.3. *Let $e \geqslant 1$ be an arbitrary given integer and $n \geqslant 1$ an integer.*

(i) *If $n \leqslant 3$, then for any lcm-closed set $S = \{x_1, \ldots, x_n\}$, the power GCD matrix $((x_i, x_j)^e)$ on S divides the power LCM matrix $([x_i, x_j]^e)$ on S in the ring $M_n(\mathbb{Z})$.*

(ii) *For $n \geqslant 4$, there exists an odd-lcm-closed set $S = \{x_1, \ldots, x_n\}$ such that the power GCD matrix $((x_i, x_j)^e)$ on S does not divide the power LCM matrix $([x_i, x_j]^e)$ on S in the ring $M_n(\mathbb{Z})$.*

Proof. (i) Let $S = \{x_1, \ldots, x_n\}$ be an lcm-closed set. Without loss of generality, we may assume that $1 \leqslant x_1 < \ldots < x_n$. Let $n = 1$. Then it is clear that the statement (i) is true. Let $n = 2$. Since the set $S = \{x_1, x_2\}$ is lcm closed, we know that $x_1 \mid x_2$. Because the set S is also gcd closed, the result in this case follows immediately from Theorem 3.2 (i). Now let $n = 3$. Since $S = \{x_1, x_2, x_3\}$ is lcm closed, we know that $x_i \mid x_3$ $(i = 1, 2)$, and $[x_1, x_2] = x_2$ or x_3. If $[x_1, x_2] = x_2$, then $x_1 \mid x_2 \mid x_3$ and so the set S is gcd closed. Consequently, the result in this case follows from Theorem 3.2 (i). Now consider the case: $[x_1, x_2] = x_3$. For this case, we see that $(x_1, x_2) = x_1x_2/x_3$. Thus we have

$$\begin{pmatrix} x_1^e & (\frac{x_1x_2}{x_3})^e & x_1^e \\ (\frac{x_1x_2}{x_3})^e & x_2^e & x_2^e \\ x_1^e & x_2^e & x_3^e \end{pmatrix}^{-1} = \begin{pmatrix} \frac{x_3^e}{x_1^e(x_3^e - x_1^e)} & 0 & \frac{1}{x_1^e - x_3^e} \\ 0 & \frac{x_3^e}{x_2^e(x_3^e - x_2^e)} & \frac{1}{x_2^e - x_3^e} \\ \frac{1}{x_1^e - x_3^e} & \frac{1}{x_2^e - x_3^e} & \frac{(x_1x_2)^e - x_3^{2e}}{x_3^e(x_3^e - x_2^e)(x_1^e - x_3^e)} \end{pmatrix}.$$

Since $(x_3/x_1)^e \in \mathbb{Z}$ and $(x_3/x_2)^e \in \mathbb{Z}$, we deduce that

$$\begin{aligned} ([x_i, x_j]^e)((x_i, x_j)^e)^{-1} &= \begin{pmatrix} x_1^e & x_3^e & x_3^e \\ x_3^e & x_2^e & x_3^e \\ x_3^e & x_3^e & x_3^e \end{pmatrix} \cdot \begin{pmatrix} x_1^e & (\frac{x_1x_2}{x_3})^e & x_1^e \\ (\frac{x_1x_2}{x_3})^e & x_2^e & x_2^e \\ x_1^e & x_2^e & x_3^e \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 0 & (\frac{x_3}{x_2})^e & 0 \\ (\frac{x_3}{x_1})^e & 0 & 0 \\ (\frac{x_3}{x_1})^e & (\frac{x_3}{x_2})^e & -1 \end{pmatrix} \in M_3(\mathbb{Z}). \end{aligned}$$

This shows that the statement (i) in this case holds and our proof of part (i) is complete.

(ii) Let $n \geqslant 4$ be an integer. Suppose that

$$x_1 = 1,\ x_2 = pq,\ x_3 = p^2,\ x_4 = p^2qa^i, \quad \text{where } 0 \leqslant i \leqslant n - 4,$$

where q and p are distinct odd primes such that $p > q^e - 1$ and $a > 1$ is an odd number satisfying $(a, p^eq^e + q^e - 1) = 1$. Now, we can easily see that the set $S = \{x_1, \ldots, x_n\}$

is odd lcm closed. By Lemma 4.2, the reciprocal set mS^{-1} is an odd gcd closed set, where $m = p^2 q a^{n-4}$. It now follows from Lemma 4.1 that

$$\frac{\det([x_i, x_j]^e)}{\det((x_i, x_j)^e)} = \frac{\det([\frac{m}{x_i}, \frac{m}{x_j}]^e)}{\det((\frac{m}{x_i}, \frac{m}{x_j})^e)}.$$

If we let

$$y_k = a^{k-1},\ 1 \leqslant k \leqslant n-3, \quad y_{n-2} = qb, \quad y_{n-1} = pb, \quad y_n = p^2 qb,$$

where $b = a^{n-4}$, then $T = \{y_1, \ldots, y_n\}$ is just a permutation of the set mS^{-1} and so we deduce that $\det([\frac{m}{x_i}, \frac{m}{x_j}]^e) = \det([y_i, y_j]^e)$ and $\det((\frac{m}{x_i}, \frac{m}{x_j})^e) = \det((y_i, y_j)^e)$. Therefore we have

$$\frac{\det([x_i, x_j]^e)}{\det((x_i, x_j)^e)} = \frac{\det([y_i, y_j]^e)}{\det((y_i, y_j)^e)}.$$

But by the proof of Theorem 3.2 (ii), we know that

$$\frac{\det([y_i, y_j]^e)}{\det((y_i, y_j)^e)} \notin \mathbb{Z}$$

and thereby, we infer that

$$\frac{\det([x_i, x_j]^e)}{\det((x_i, x_j)^e)} \notin \mathbb{Z}.$$

This shows immediately that in the ring $M_n(\mathbb{Z})$, we have $((x_i, x_j)^e) \nmid ([x_i, x_j]^e)$, as expected. Thus the proof of Theorem 4.3 is complete. $\square$

By Theorem 4.3 we see immediately that Conjecture 1.2 holds for $n \leqslant 3$ but does not hold for $n \geqslant 4$.

In closing this paper, we remark that although Conjectures 1.1 and 1.2 are in general not true, Hong has proved in [11] that for any given integer $e \geqslant 1$, if $S = \{x_1, \ldots, x_n\}$ is a *divisor chain* (that is, $x_1 \mid \ldots \mid x_n$), then the power GCD matrix $((x_i, x_j)^e)$ on S always divides the power LCM matrix $([x_i, x_j]^e)$ on S in the ring $M_n(\mathbb{Z})$. Note that a divisor chain is both gcd-closed and lcm-closed. However, the problem how to determine all gcd-closed (resp. lcm-closed) sets S such that the power GCD matrix $((x_i, x_j)^e)$ on S divides the power LCM matrix $([x_i, x_j]^e)$ on S in the ring $M_n(\mathbb{Z})$ remains open, where $e \geqslant 1$ is any given integer.

References

[1] *T. M. Apostol*: Arithmetical properties of generalized Ramanujan sums. Pacific J. Math. *41* (1972), 281–293.

[2] *S. Beslin and S. Ligh*: Another generalization of Smith's determinant. Bull. Austral. Math. Soc. *40* (1989), 413–415.

[3] *K. Bourque and S. Ligh*: On GCD and LCM matrices. Linear Algebra Appl. *174* (1992), 65–74.

[4] *K. Bourque and S. Ligh*: Matrices associated with arithmetical functions. Linear and Multilinear Algebra *34* (1993), 261–267.

[5] *K. Bourque and S. Ligh*: Matrices associated with classes of arithmetical functions. J. Number Theory *45* (1993), 367–376.

[6] *K. Bourque and S. Ligh*: Matrices associated with classes of multiplicative functions. Linear Algebra Appl. *216* (1995), 267–275.

[7] *C. He and J. Zhao*: More on divisibility of determinants of LCM Matrices on GCD-closed sets. Southeast Asian Bull. Math. *29* (2005), 887–893.

[8] *S. Hong*: On the Bourque-Ligh conjecture of least common multiple matrices. J. Algebra *218* (1999), 216–228.

[9] *S. Hong*: On the factorization of LCM matrices on gcd-closed sets. Linear Algebra Appl. *345* (2002), 225–233.

[10] *S. Hong*: Gcd-closed sets and determinants of matrices associated with arithmetical functions. Acta Arith. *101* (2002), 321–332.

[11] *S. Hong*: Factorization of matrices associated with classes of arithmetical functions. Colloq. Math. *98* (2003), 113–123.

[12] *S. Hong*: Notes on power LCM matrices. Acta Arith. *111* (2004), 165–177.

[13] *S. Hong*: Nonsingularity of matrices associated with classes of arithmetical functions. J. Algebra *281* (2004), 1–14.

[14] *S. Hong*: Nonsingularity of least common multiple matrices on gcd-closed sets. J. Number Theory *113* (2005), 1–9.

[15] *S. Hong and R. Loewy*: Asymptotic behavior of eigenvalues of greatest common divisor matrices. Glasgow Math. J. *46* (2004), 551–569.

[16] *S. Hong and Q. Sun*: Determinants of matrices associated with incidence functions on posets. Czechoslovak Math. J. *54* (2004), 431–443.

[17] *P. Lindqvist and K. Seip*: Note on some greatest common divisor matrices. Acta Arith. *84* (1998), 149–154.

[18] *P. J. McCarthy*: A generalization of Smith's determinant. Canad. Math. Bull. *29* (1986), 109–113.

[19] *H. J. S. Smith*: On the value of a certain arithmetical determinant. Proc. London Math. Soc. *7* (1875–1876), 208–212.

[20] *A. Wintner*: Diophantine approximations and Hilbert's space. Amer. J. Math. *66* (1944), 564–578.

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