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Douglas Bridges; Hajime Ishihara; Ray Mines; Fred Richman; Peter Schuster; Luminiţa Vîţă

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ALMOST LOCATEDNESS IN UNIFORM SPACES

Douglas Bridges, Christchurch, Hajime Ishihara, Ishikawa,
Ray Mines, Las Cruces, Fred Richman, Boca Raton,
Peter Schuster, München, and Luminiţa Vîţă, Christchurch

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Abstract. A weak form of the constructively important notion of locatedness is lifted from the context of a metric space to that of a uniform space. Certain fundamental results about almost located and totally bounded sets are then proved.

Keywords: uniform structure, located, constructive

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1. Uniform spaces

A *uniform structure* on a set X is a filter $\mathscr{U}$ of subsets, called *entourages*, of $X \times X$ such that for each $U \in \mathscr{U}$,

U1 $U^{-1} \in \mathscr{U}$,
U2 U contains the diagonal Δ of $X \times X$, and
U3 there exists $V \in \mathscr{U}$ such that $V \circ V \subset U$.

For more details see, for example, [7], Chapter 2.

The foregoing axiomatic properties of a uniform structure correspond to properties of a pseudometric ϱ. For example, closure under finite intersection corresponds to the statement

$$B_r(x) \cap B_s(x) = B_{\inf(r,s)}(x),$$

where $B_r(x)$ is the closed ball with center x and radius r. Likewise, property U1 corresponds to the symmetry of the pseudometric: $\varrho(x,y) = \varrho(y,x)$; U2 to the property $\varrho(x,x) = 0$; and U3 to the triangle inequality, $\varrho(x,z) \leqslant \varrho(x,y) + \varrho(y,z)$.

Each uniform space X has a natural equality defined by

$$x = y \Leftrightarrow \forall U \in \mathscr{U}\ ((x,y) \in U)\,.$$

Note that for the constructive theory of uniform spaces we require that $\mathscr{U}$ satisfy the following, classically trivial, axiom:

S For each $U \in \mathscr{U}$ there exists $V \in \mathscr{U}$ such that $X \times X = U \cup (X \times X) \setminus V$.

In that case, $V \subset U$. Metric spaces, locally convex linear spaces, and spaces whose topology is defined by a family of pseudometrics (see [3]) are uniform spaces with the property **S**.[1]

Each uniform space X also has a natural inequality defined by

$$x \neq y \Leftrightarrow \exists U \in \mathscr{U} \left((x, y) \notin U\right).$$

For each subset R of X there is a natural *apartness complement*

$$-R = \{x \in X : \exists U \in \mathscr{U} \forall y \in R \left((x, y) \notin U\right)\}.$$

For each $S \subset X$ we usually write $S - R$ instead of $S \cap (-R)$.

In [9], it is assumed from the outset that every uniform space is equipped with an imposed inequality; the purely constructive axioms for a uniform space are then phrased in terms of that inequality, with the help of the *complement* of U,

$$\sim U = \{t \in X \times X : \forall u \in U \, (t \neq u)\}.$$

Those axioms are

- For each $U \in \mathscr{U}$ there exists $V \in \mathscr{U}$ such that $V \circ V \subset U$ and $X \times X = U \cup \sim V$.
- If $x \neq y$ in X, then there exists U in $\mathscr{U}$ such that $(x, y) \in \sim U$.

It turns out that the imposed inequality is an apartness that is necessarily the same as the natural inequality; moreover, the single axiom **S** above results in exactly the same notion of uniform space.

[1] Grayson [8] calls a uniform space *weakly separated* if it is equipped with the natural equality; such a space is the uniform counterpart of a metric, as opposed to a pseudometric, space. He calls a uniform space *strongly separated* if it is weakly separated and satisfies property **S**.

2. Almost located subsets

A subset S of a metric space (X, ϱ) is said to be *located* (in X) if the *distance*

$$\varrho(x, S) = \inf \{\varrho(x, y) \colon\ y \in S\}$$

exists for each $x \in X$. Locatedness plays a vital role in the constructive theory of metric and normed spaces: for example, for a nonzero bounded linear functional u on a normed space X, the *norm*

$$\|u\| = \sup \{|u(x)| \colon\ x \in X, \|x\| \leqslant 1\}$$

exists if and only if the kernel

$$\ker(u) = \{x \in X \colon\ u(x) = 0\}$$

is located; and the Hahn-Banach extension theorem requires that the kernel of the functional be located ([2], Chapter 7, Theorem 4.6).

It is reasonable to ask if we can lift locatedness to the context of a uniform space and then prove significant analogues of metric-space theorems. The absence of a distance function makes this question nontrivial within constructive mathematics—mathematics with intuitionistic logic [1], [2], [4], [11]. In this paper we introduce and examine a weak analogue of locatedness for subsets of a uniform space.

For $x \in X$ and $U \in \mathscr{U}$, let $U[x] = \{y \in X \colon\ (x, y) \in U\}$. We say that a subset S of a uniform space $(X, \mathscr{U})$ is *almost located* if for each $U \in \mathscr{U}$ there exists $V \in \mathscr{U}$ such that

$$\forall x \in X\, (S \cap V[x] = \emptyset \vee S \cap U[x] \neq \emptyset)\,. \tag{1}$$

We may take V to be a subset of U here because $U \cap V$ is also an entourage.

Proposition 1. *A located set in a metric space is almost located.*

Proof. Let S be located in the metric space (X, ϱ), and let U be an entourage in the standard metric uniform structure on X. Choose a positive number β such that

$$\{(x, y) \in X \times X \colon\ \varrho(x, y) < \beta\} \subset U.$$

For any positive number $\alpha < \beta$, let

$$V = \{(s, t) \in X \times X \colon\ \varrho(s, t) < \alpha\} \in \mathscr{U}.$$

For each $x \in X$, either $\varrho(x, S) > \alpha$ or $\varrho(x, S) < \beta$. In the first case, $S \cap V[x] = \emptyset$. In the second case, $S \cap U[x] \neq \emptyset$. $\square$

Returning now to a general uniform space $(X, \mathscr{U})$, given $U \in \mathscr{U}$, we define a subset Y of X to be *U-small* if $Y \times Y \subset U$. We say that a subset S of X is *totally bounded* if for each $U \in \mathscr{U}$ there is a finite covering of S by U-small sets, each of which has nonempty intersection with S.

Lemma 2. *Let S be a subset of a uniform space $(X, \mathscr{U})$. In order that S be totally bounded, it is necessary and sufficient that for each $U \in \mathscr{U}$ there exist a finitely enumerable subset $\{s_1, \ldots, s_n\}$ of S such that $S \subset \bigcup_{i=1}^{n} U[s_i]$.*

P r o o f. The proof is left as an exercise. □

An *n-chain of entourages* of a uniform space X is an n-tuple $(U_1, \ldots, U_n)$ of entourages such that

$$U_k \circ U_k \subset U_{k-1} \quad \text{and} \quad X \times X = U_{k-1} \cup (X \times X) \setminus U_k$$

for $k = 2, \ldots, n$. The axioms for a uniform space ensure that for each $U \in \mathscr{U}$ and each positive integer n there exists an n-chain $(U_1, \ldots, U_n)$ of entourages with $U_1 = U$.

Lemma 3. *Let V be an entourage of a uniform space X, and S an almost located subset of X. Then there exists an entourage W of X such that (V, W) is a 2-chain and*

$$\forall x \in X \, (S \cap W[x] = \emptyset \vee S \cap V[x] \neq \emptyset) \, .$$

P r o o f. Choose an entourage E such that $E \circ E \subset V$ and $X \times X = V \cup (X \times X) \setminus E$. Since S is almost located, there exists an entourage $W \subset E$ such that

$$\forall x \in X \, (S \cap W[x] = \emptyset \vee S \cap E[x] \neq \emptyset) \, .$$

Since $E \subset V$, the desired conclusion follows. □

Proposition 4. *An almost located subset of a totally bounded uniform space is totally bounded.*

P r o o f. Let S be an almost located subset of a totally bounded uniform space $(X, \mathscr{U})$, and let $U \in \mathscr{U}$. Choose $V \in \mathscr{U}$ so that (U, V) is a 2-chain. By Lemma 3, there exists $W \in \mathscr{U}$ such that (U, V, W) is a 3-chain and

$$\forall x \in X \, (S \cap W[x] = \emptyset \vee S \cap V[x] \neq \emptyset) \, .$$

Let $\{x_1, \ldots, x_n\}$ be a finitely enumerable set such that $X = \bigcup_{i=1}^{n} W[x_i]$. Write $\{1, \ldots, n\}$ as a union of sets P, Q such that $S \cap V[x_i] \neq \emptyset$ whenever $i \in P$, and $S \cap W[x_i] = \emptyset$ whenever $i \in Q$. For each $i \in P$ construct y_i in $S \cap V[x_i]$. Consider any $y \in S$. There exists i such that $y \in W[x_i]$; so $i \notin Q$ and therefore $i \in P$. We have $(y, x_i) \in W$ and $(x_i, y_i) \in V$; whence $(y, y_i) \in W \circ V \subset V \circ V \subset U$. Thus $S \subset \bigcup_{i \in P} U[y_i]$. It follows from Lemma 2 that S is totally bounded. □

Corollary 5. *In a totally bounded metric space, locatedness and almost locatedness coincide.*

P r o o f. This follows from Propositions 1 and 4, with reference to [2] (Chapter 2, Proposition (4.4)). □

Here is a converse of Proposition 4.

Proposition 6. *A totally bounded subset of a uniform space is almost located.*

P r o o f. Let S be a totally bounded subset of the uniform space $(X, \mathscr{U})$. Let $U \in \mathscr{U}$ and choose $W \subset V \subset U$ in $\mathscr{U}$ such that $W \circ W \subset V$ and $X \times X = U \cup (X \times X) \setminus V$. As S is totally bounded, there are $s_1, \ldots, s_n \in S$ such that $S \subset \bigcup_{i=1}^{n} W[s_i]$. Given x in X, either $x \in U[s_i]$ for some i, or $x \notin V[s_i]$ for all i. In the former case, $U[x] \cap S \neq \emptyset$; in the latter case, $W[x] \cap S = \emptyset$. □

Sometimes we can get along with the following weaker version of almost located: a subset S of a uniform space X is said to be *pointwise almost located* if for each $x \in X$ and $U \in \mathscr{U}$, either $x \in -S$ or $U[x] \cap S \neq \emptyset$. Every almost located subset, and every singleton subset, is pointwise almost located. In [9], [5] a subset S of a uniform space X is defined to be *weakly located* if

$$\forall x \in X \ \forall R \subset X \, (x \in -R \Rightarrow (x \in -S \vee S - R \neq \emptyset)) \, .$$

Weak locatedness was introduced by Troelstra [10] in the context of a general topological space. On pages 359–360 of [11] it is shown that the proposition 'every weakly located subset of a metric space is located' is essentially nonconstructive.

Proposition 7. *A subset S of a uniform space $(X, \mathscr{U})$ is pointwise almost located if and only if it is weakly located.*

P r o o f. Let S be a pointwise almost located subset of a uniform space $(X, \mathscr{U})$. Let $x \in X$, and let R be a subset of X such that $x \in -R$. There exists a 3-chain (U, V, W) such that $(\{x\} \times R) \cap U = \emptyset$. Since S is pointwise almost located, there exists $E \in \mathscr{U}$ such that either $S \cap E[x] = \emptyset$ or $S \cap V[x] \neq \emptyset$. In the first case we get $(\{x\} \times S) \cap E = \emptyset$; that is, $x \in -S$. In the second case let $y \in S \cap V[x]$ and $r \in R$. Then either $(y, r) \notin W$ or $(y, r) \in V$. In the latter event, since $(x, y) \in V$, it follows that $(x, r) \in V \circ V \subset U$, a contradiction. Hence $(\{y\} \times R) \cap W = \emptyset$, and so $y \in S - R$.

Conversely, suppose that S is weakly located and that $x \in X$ and $U \in \mathscr{U}$. Choose $V \in \mathscr{U}$ so that

$$X \times X = U \cup (X \times X) \setminus V,$$

and let $R = X \setminus V[x]$. Note that

$$X = U[x] \cup (X \setminus V[x]) = U[x] \cup R.$$

Clearly $x \in -R$, so either $x \in -S$, and we are done, or else $S - R \neq \emptyset$. Thus we may assume that $S - R \neq \emptyset$. But $-R \subset U[x]$, because $X \setminus R \subset U[x]$, and therefore $S \cap U[x] \neq \emptyset$. □

We now show, by means of a mixed recursive and Brouwerian example, that not every pointwise almost located subset is almost located. Assuming Church's Thesis, we will construct a subset of $[0, 1]$ that is pointwise almost located but not located; whence, by Corollary 5, it is not almost located. Let (s_n) be a Specker sequence—that is, an increasing sequence of rational numbers in $[0, 1]$ such that s_n is eventually bounded away from each (recursive) real number; Church's Thesis ensures that such sequences exist (see [4], Chapter 3). Let (a_n) be a binary sequence with at most one term equal to 1, and let $S = \{s_n \colon a_n = 1\}$. To see that S is a pointwise almost located subset of $[0, 1]$, consider any $x \in [0, 1]$ and choose N and $\delta > 0$ such that $|x - s_n| \geqslant \delta$ for all $n \geqslant N$. If $a_n = 0$ for all $n < N$, then $d(x, S) \geqslant \delta$; if there exists $n < N$ with $a_n = 1$, then S is a singleton and hence pointwise almost located. Now assume that S is located, and compute $d = d(1, S)$. Either $d > 1$ and therefore $a_n = 0$ for all n, or else $d < 2$, in which case S is nonempty and therefore there exists n with $a_n = 1$.

A subset S of a uniform space X is *locally totally bounded* if there exists an entourage V_0 such that for each x in X, the set $V_0[x] \cap S$ is contained in a totally bounded subset of S.

A uniform space $(X, \mathscr{U})$ is *first countable* if it has a countable basis of entourages $U_1, U_2, \ldots$. We may assume that $U_{n+1} \circ U_{n+1} \subset U_n$ for each n. In that case, X is a first countable topological space in the usual sense.

The rest of this section is devoted to the proof of the uniform analogue of a theorem about metric spaces in [4] (Chapter 2, Theorem 4.11).

Theorem 8. *The following hold for a nonempty subset Y of a uniform space X.*

(i) *If Y is locally totally bounded, it is almost located.*

(ii) *If X is first countable and locally totally bounded, and Y is almost located, then Y is locally totally bounded.*

This depends on a proposition, a corollary, and a lemma, each of which holds some intrinsic interest.

Proposition 9. *Let X be a first countable, totally bounded uniform space with a countable basis of entourages $U_1, U_2, \ldots$, and let ξ be a point of X. Then for each positive integer n there exists a closed, totally bounded subset K of X such that $U_{n+4}[\xi] \subset K \subset U_n[\xi]$.*

Proof. We may assume that $U_{k+1} \circ U_{k+1} \subset U_k$ for each k. Fixing the positive integer n, and taking $F_1 = \{\xi\}$, we construct an increasing sequence $(F_k)_{k=1}^{\infty}$ of finitely enumerable subsets of X such that for each k,

$$\forall x \in F_{k+1}\ \exists y \in F_k((x, y) \in U_{n+k+1}) \tag{2}$$

and

$$\forall x \in U_{n+4}[\xi]\ \exists y \in F_k((x, y) \in U_{n+k+3}). \tag{3}$$

To this end, assume that $F_1, \ldots, F_k$ have been constructed with properties (2) and (3). Let $\{x_1, \ldots, x_N\}$ be a U_{n+k+4}-approximation to X, and write $\{1, \ldots, N\}$ as a union of subsets A, B such that

$$i \in A \Rightarrow \exists y \in F_k((x_i, y) \in U_{n+k+1}),$$
$$i \in B \Rightarrow \forall y \in F_k((x_i, y) \notin U_{n+k+2}).$$

Setting

$$F_{k+1} = \{x_i :\ i \in A\} \cup F_k,$$

we see immediately that F_{k+1} satisfies (2). Let $x \in U_{n+4}[\xi]$. By our induction hypothesis, there exists $y \in F_k$ with $(x, y) \in U_{n+k+3}$. Choosing i such that $(x, x_i) \in U_{n+k+4}$, we have

$$(x_i, y) \in U_{n+k+4} \circ U_{n+k+3} \subset U_{n+k+2}.$$

Thus i cannot belong to B, and so $x_i \in F_{k+1}$. As $(x, x_i) \in U_{n+k+4}$, the set F_{k+1} satisfies (3). This completes the inductive construction of the sequence $(F_k)_{k=1}^{\infty}$.

Now let K be the closure of $\bigcup_{k=1}^{\infty} F_k$ in X. We see from (3) that $U_{n+4}[\xi] \subset K$. On the other hand, if $m \geqslant k$ and $y \in F_m$, then by (3), we can find points $y_m = y, y_{m-1} \in F_{m-1}, \ldots, y_k \in F_k$ such that $(y_{i+1}, y_i) \in U_{n+i+1}$ for $k \leqslant i \leqslant m-1$. Thus

$$(y, y_k) \in U_{n+m} \circ \ldots \circ U_{n+k+1} \subset U_{n+k}. \tag{4}$$

It follows that F_k is a U_k-approximation to K. Finally, taking $k = 1$ in (4), we see that $(y, \xi) \in U_n$ for each $y \in K$. Thus $K \subset U_n[\xi]$. □

Corollary 10. *If X is a first countable, totally bounded uniform space, then for each entourage U of X there exist totally bounded U-small sets $K_1, \ldots, K_n$ such that $X = \bigcup_{i=1}^{n} K_i$.*

P r o o f. Let $U_1, U_2, \ldots$ be a countable basis of entourages. Without loss of generality, assume that $U_{k+1} \circ U_{k+1} \subset U_k$ for each k. Pick ν such that $U_\nu \circ U_\nu \subset U$, and then points $x_1, \ldots, x_n$ of X such that

$$X = \bigcup_{i=1}^{n} U_{\nu+4}[x_i].$$

For each i $(1 \leqslant i \leqslant n)$, choose a totally bounded subset K_i of X such that $U_{\nu+4}[x_i] \subset K_i \subset U_\nu[x_i]$. Then $K_i \times K_i \subset U_\nu \circ U_\nu \subset U$, so K_i is U-small. Also, clearly, $X = \bigcup_{i=1}^{n} K_i$. □

Lemma 11. *Let L be an almost located subset of a first countable uniform space X, and let T be a totally bounded subset of X. Then there exists a totally bounded set S such that $T \cap L \subset S \subset L$.*

P r o o f. Let $U_1, U_2, \ldots$ be a countable basis of entourages of X. We may assume that for each n,

$$U_{n+1} \circ U_{n+1} \subset U_n$$

and

$$\forall x \in X (L \cap U_{n+1}[x] = \emptyset \vee L \cap U_n[x] \neq \emptyset).$$

For each positive integer n let T_n be a finite U_{n+3}-approximation to T. Write T_n as a union of finite sets A_n and B_n such that

$$t \in A_n \Rightarrow U_{n+1}[t] \cap L \neq \emptyset,$$
$$t \in B_n \Rightarrow U_{n+2}[t] \cap L = \emptyset.$$

For each t in A_n choose s_t^n in L such that $(t, s_t^n) \in U_{n+1}$. Let

$$S_n = \{s_t^n \colon\ t \in A_n\}\,,$$

and let S be the closure of $\bigcup_{n=1}^{\infty} S_n$ in L. To prove S totally bounded, fix m, and consider a positive integer $n \geqslant m+2$ and any element s of S_n. There exist $t' \in A_n$ and $t \in T_m$ such that $(s, t') \in U_{n+1}$ and $(t', t) \in U_{m+3}$; whence

$$(s, t) \in U_{n+1} \circ U_{m+3} \subset U_{m+2}.$$

Thus $t \in A_m$ and

$$(s, s_t^m) \in U_{m+2} \circ U_{m+1} \subset U_m.$$

It follows that $\bigcup_{k=1}^{m+2} S_k$ is a finitely enumerable U_m-approximation to $\bigcup_{n=1}^{\infty} S_n$. So S is totally bounded.

If $x \in T \cap L$ and $n \geqslant 1$, then there exists t in T_n such that $(x, t) \in U_{n+3}$. So $t \in A_n$ and therefore

$$(x, s_t^n) \in U_{n+3} \circ U_{n+1} \subset U_n,$$

where $s_t^n \in S$. As x and n are arbitrary and S is closed, $T \cap L \subset S$. □

We now give the *proof of Theorem* 8.

P r o o f. Assume first that Y is locally totally bounded, and let U be any entourage of X. Choose an entourage W such that (U, W) is a 2-chain. Let V_0 be an entourage such that for each x in X, $V_0[x] \cap Y$ is contained in a totally bounded subset of Y. Choose an entourage V such that $V \subset V_0$ and $V^2 \subset W$. For each x in X there exist $x_1, \ldots, x_n \in Y$ such that

$$V_0[x] \cap Y \subset \bigcup_{i=1}^{n} V[x_i].$$

Either $(x, x_i) \in U$ for some i or else $(x, x_i) \notin W$ for all i. In the first case we have $U[x] \cap Y \neq \emptyset$. In the second case, if $y \in V[x] \cap Y$, then

$$y \in V[x] \cap Y \subset V_0[x] \cap Y \subset \bigcup_{i=1}^{n} V[x_i];$$

choosing i such that $(y, x_i) \in V$, as $(x, y) \in V$ we see that $(x, x_i) \in V^2 \subset W$, a contradiction. Thus $V[x] \cap Y = \emptyset$. This proves (i) of Theorem 8; part (ii) is a simple consequence of Lemma 11. □

Even for Hilbert spaces, almost locatedness is not as strong as locatedness, as the following example shows. Let P be an arbitrary proposition, and consider the subspace $X = X_1 \cup X_2$ of the Hilbert space $\mathbb{R}^2$, where

$$X_1 = \mathbb{R}\times \{0\}\,,\ X_2 = \{(x,y) \in \mathbb{R}^2 :\ P\}.$$

Let

$$V = \{(x,x) \in \mathbb{R}^2 :\ x \neq 0 \Rightarrow P\},$$

and note that $(0,0) \in V$. Let $(x,y) \in X$ and $0 < \varepsilon < 1$. If $(x,y) \in X_2$, then P holds, so V is the diagonal of $\mathbb{R}^2$ and is therefore located; whence

$$\forall v \in V\left(\|(x,y) - v\| > \varepsilon^2/64\right) \vee \exists v \in V\left(\|(x,y) - v\| < \varepsilon\right). \tag{5}$$

If $(x,y) \in X_1$, then $y = 0$ and either $|x| < \varepsilon$ or $|x| > 3\varepsilon/4$. In the first case,

$$\|(x,y) - (0,0)\| = |x| < \varepsilon.$$

In the second case, for each $(z,z) \in V$, we have either $|x - z| > \varepsilon/8$, when

$$\|(x,y) - (z,z)\|^2 \geqslant |x - z|^2 > \varepsilon^2/64$$

and therefore $\|(x,y) - (z,z)\| > \varepsilon/8$; or else $|x - z| < \varepsilon/4$. In that case, $|z| > \varepsilon/2$, so P holds, V is located, and therefore

$$\|(x,y) - (z,z)\| \geqslant |z| > \varepsilon/2.$$

Thus in all cases, (5) holds. It follows that V is almost located. However, if the distance from $(1,0)$ to V is less than 1, then P holds; while if the distance from $(1,0)$ to V is greater than $1\big/\sqrt{2}$, then P does not hold. Thus if, in a Hilbert space, almost locatedness implies locatedness, then we can prove the law of excluded middle.

In fact, almost locatedness cannot be equivalent to locatedness, because the former is a uniform invariant but the latter is not, even for subspaces of normed spaces. To see this, consider $\mathbb{R}^2$ with the ℓ_1-norm

$$\|(x,y)\| = |x| + |y|$$

(the taxicab norm) and also with the norm

$$\|(x,y)\|' = |x| + \frac{1}{2}\,|y|\,.$$

Let ϱ and ϱ' be the respective metrics. Note that

$$\|(x,y)\|' \leqslant \|(x,y)\| \leqslant 2\,\|(x,y)\|',$$

so the two norms are uniformly equivalent. Given an arbitrary proposition P, let

$$V = \left\{(x,y) \in \mathbb{R}^2 :\ y = 0 \vee P\right\}$$

and consider the subspace

$$S = \{(0,0)\} \cup \{(r,r):\ r \in \mathbb{R} \wedge P\}.$$

Then $\varrho((x,y),S) = |x-y|$ for each $(x,y) \in V$, so S is located with respect to ϱ. On the other hand, suppose that $\varrho'((1,0),S)$ exists. If $\varrho'((1,0),S) < 1$, then P; while if $\varrho'((1,0),S) > 1/2$, then $\neg P$.

In spite of the last two examples, almost locatedness looks like a promising property of subsets of a uniform space. Even in metric spaces, a hypothesis of locatedness can often be relaxed to one of (pointwise) almost locatedness: see, for example, the proof of Bishop's lemma in [5] (Proposition 12). There remains the problem of generalising almost locatedness to the context of apartness spaces, which, constructively, form a bigger class of spaces than uniform ones [6].

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References

[1] *Errett Bishop*: Foundations of constructive analysis. McGraw-Hill, 1967. Zbl 0183.01503

[2] *Errett Bishop and Douglas Bridges*: Constructive Analysis. Grundlehren der math. Wissenschaften Bd. 279, Springer, Heidelberg, 1985. Zbl 0656.03042

[3] *Douglas Bridges and Luminiţa Dediu (Vîţă)*: Constructive notes on uniform and locally convex spaces. Proceedings of 12th International Symposium, FCT'99, Iaşi, Romania, Springer Lecture Notes in Computer Science 1684 (1999), 195–203.

[4] *Douglas Bridges and Fred Richman*: Varieties of Constructive Mathematics. London Math. Soc. Lecture Notes 97, Cambridge University Press, London, 1987. Zbl 0618.03032

[5] *Douglas Bridges and Luminiţa Vîţă*: Cauchy nets in the constructive theory of apartness spaces. Scientiae Math. Japonicae *56* (2001), 123–132. Zbl 1005.54029

[6] *Douglas Bridges, Peter Schuster and Luminiţa Vîţă*: Apartness, topology, and uniformity: a constructive view in *Computability and Complexity in Analysis* (Proc. Dagstuhl Seminar 01461, 11–16 November 2001, V. Brattka, P. Hertling, M. Yasugi eds.). Math. Log. Quart. *48* (2002), Suppl. 1, 16–28. Zbl pre01850711

[7] *Nicolas Bourbaki*: General Topology (Part 1). Addison-Wesley, Reading, MA, 1966. Zbl 0301.54001

[8] *Robin J. Grayson*: Concepts of general topology in constructive mathematics and in sheaves, II. Ann. Math. Logic *23* (1982), 55–98. Zbl 0495.03040

[9] *Peter Schuster, Douglas Bridges and Luminiţa Vîţă*: Apartness as a relation between subsets. Combinatorics, Computability and Logic (Proceedings of DMTCS'01, Constanţa, Romania, 2–6 July 2001) (C. S. Calude, M. J. Dinneen, S. Sburlan, eds.). DMTCS Series 17, Springer-Verlag, London, 2001, pp. 203–214.

[10] *Anne S. Troelstra*: Intuitionistic General Topology. PhD. Thesis, University of Amsterdam, 1966.

[11] *Anne S. Troelstra and Dirk van Dalen*: Constructivism in Mathematics. Studies in Mathematical Logic and the Foundations of Mathematics, 121 and 123, North-Holland, Amsterdam, 1988. Zbl 0661.03047(Vol. II)

Authors' addresses: D. Bridges and L. Vîţă, Department of Mathematics & Statistics, University of Canterbury, Private Bag 4800, Christchurch, New Zealand, e-mail: `d.bridges@math.canterbury.ac.nz`; H. Ishihara, School of Information Science, Japan Advanced Institute of Science and Technology, Nomi, Ishihikawa 923–12, Japan; R. Mines, Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM 88003, USA; F. Richman, Department of Mathematics, Florida Atlantic University, Boca Raton, 33431 FL, USA; P. Schuster, Mathematisches Institut, Ludwig-Maximilians-Universität, Theresienstraße 39, 80333 München, Germany.