Czechoslovak Mathematical Journal, 56 (131) (2006), 1011–1018

ON WEAK-OPEN π-IMAGES OF METRIC SPACES

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(Received July 15, 2004)

Abstract. In this paper, we give some characterizations of metric spaces under weak-open π-mappings, which prove that a space is g-developable (or Cauchy) if and only if it is a weak-open π-image of a metric space.

Keywords: weak-open mappings, π-mappings, g-developable spaces, Cauchy spaces, cs-covers, sn-covers, weak-developments, point-star networks

MSC 2000: 54E40, 54E99, 54C10, 54D55

1. Introduction and definitions

To find internal characterizations of certain images of metric spaces is one of the central problems in General Topology. Some characterization for certain quotient π-images (open π-images, pseudo-open π-images) of metric spaces are obtained in [5], [11], [12], [13], [14], [15], [18]. Recently, S. Xia [4] introduced the concept of weak-open mappings. By using it, certain g-first countable spaces are characterized as images of metric spaces under various weak-open mappings. Furthermore, we prove that a space is g-metrizable if and only if it is a weak-open σ-image of a metric space in [18].

The purpose of this paper is to give some characterizations of weak-open π-images of metric spaces. We prove that a space is g-developable (or Cauchy) if and only if it is a weak-open π-image of a metric space, and generalize the result of R. W. Heath in [12].

In this paper, all spaces are Hausdroff, all mappings are continuous and surjective. $\mathbb{N}$ denotes the set of all natural numbers. $\tau(X)$ denotes a topology on X. For the

This work was supported by the NSF of China (No. 10471020, No. 10471035) and the NSF of Hunan Province in China.

usual product space $\prod_{i\in\mathbb{N}} X_i$, π_i denotes the projection from $\prod_{i\in\mathbb{N}} X_i$ onto X_i. For a sequence $\{x_n\}$ in X, denote $\langle x_n\rangle = \{x_n\colon\ n\in\mathbb{N}\}$.

Definition 1.1. Let $\mathscr{P}=\bigcup\{\mathscr{P}_x\colon\ x\in X\}$ be a collection of subsets of a space X. $\mathscr{P}$ is called a weak-base for X if

(1) for each $x\in X$, $\mathscr{P}_x$ is a network of x in X,

(2) if $U,V\in\mathscr{P}_x$, then $W\subset U\cap V$ for some $W\in\mathscr{P}_x$.

(3) $G\subset X$ is open in X if and only if for each $x\in G$, there exists $P\in\mathscr{P}_x$ such that $P\subset G$.

$\mathscr{P}_x$ is called a weak neighborhood base of x in X, and every element of $\mathscr{P}$ is called a weak neighborhood of x in X.

Definition 1.2. Let $f\colon\ X\to Y$ be a mapping.

(1) f is called a weak-open mapping [4], if there exists a weak-base $\mathscr{B}=\bigcup\{\mathscr{B}_y\colon\ y\in Y\}$ for Y, and for each $y\in Y$ there exists $x_y\in f^{-1}(y)$ satisfying the following condition: for each open neighborhood U of $x_y, B_y\subset f(U)$ for some $B_y\in\mathscr{B}_y$.

(2) f is called a π-mapping [2], if (X,d) is a metric space, and for each $y\in Y$ and its open neighborhood V in Y, $d(f^{-1}(y), M\setminus f^{-1}(V))>0$.

It is easy to check that a weak-open mapping is quotient and a compact mapping on metric spaces is a π-mapping.

Definition 1.3 [8]**.** Let X be a space, and $P\subset X$. Then,

(1) A sequence $\{x_n\}$ in X is called eventually in P, if $\{x_n\}$ converges to x, and there exists $m\in\mathbb{N}$ such that $\{x\}\cup\{x_n\colon\ n\geqslant m\}\subset P$.

(2) P is called a sequential neighborhood of x in X, if whenever a sequence $\{x_n\}$ in X converges to x, then $\{x_n\}$ is eventually in P.

(3) P is called sequential open in X, if P is a sequential neighborhood of each of its points.

(4) X is called a sequential space, if any sequential open subset of X is open in X.

Definition 1.4. Let $\mathscr{P}$ be a cover of a space X.

(1) $\mathscr{P}$ is called a cs-cover for X, if every convergent sequence in X is eventually in some element of $\mathscr{P}$.

(2) $\mathscr{P}$ is called a sn-cover for X, if every element of $\mathscr{P}$ is a sequential neighborhood of some point in X, and for any $x\in X$ there exists a sequential neighborhood P of x in X such that $P\in\mathscr{P}$.

Definition 1.5. Let $\{\mathscr{P}_n\}$ be a sequence of covers of a space X.

(1) $\{\mathscr{P}_n\}$ is called a point-star network for X, if for each $x\in X$, $\langle\mathrm{st}(x,\mathscr{P}_n)\rangle$ is a network of x in X.

(2) $\{\mathscr{P}_n\}$ is called a weak-development for X, if for each $x \in X$, $\langle \mathrm{st}(x, \mathscr{P}_n) \rangle$ is a weak neighborhood base of x in X.

If in a weak-development $\{\mathscr{P}_n\}$ for X each $\mathscr{P}_n$ satisfies property C, $\{\mathscr{P}_n\}$ is called a C weak-development for X.

Definition 1.6 ([6]). Let (X, d) is a symmetrizable space. Then,

(1) A sequence $\{x_n\}$ in X is called d-Cauchy if, for each $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $n, m > k$.

(2) X is called Cauchy (respectively weak Cauchy), if each convergent sequence is d-Cauchy (respectively each convergent sequence has a d-Cauchy subsequence).

For a space X, let g be a mapping defined on $\mathbb{N} \times X$ to the power-set of X such that $x \in g(n, x)$ and $g(n+1, x) \subset g(n, x)$ for each $n \in \mathbb{N}$ and $x \in X$, and a subset U of X is open if for each $x \in U$, there exists $n \in \mathbb{N}$ such that $g(n, x) \subset U$. We call such a mapping a CWC-mapping (i.e., countable weakly-open covering mapping).

Definition 1.7 ([7]). A space X is g-developable if X has a CWC-mapping g with the following property: If $x, x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$, then sequence $\{x_n\}$ converges to x.

2. Results

Theorem 2.1. *The following are equivalent for a space X:*

(1) *X is a weak-open π-image of a metric space.*

(2) *X has a cs-cover weak-development.*

(3) *X has a sn-cover weak-development.*

(4) *X is a Cauchy space.*

(5) *X is a g-developable space.*

Proof. (1) $\Rightarrow$ (2): Suppose X is an image of a metric space (M, d) under a weak-open π-mapping f. For each $n \in \mathbb{N}$, put $\mathscr{P}_n = \{f(B(z, 1/n))\colon z \in M\}$, where $B(z, 1/n) = \{y \in M\colon d(z, y) < 1/n\}$. Then $\{\mathscr{P}_n\}$ is a point-star network for X. In fact, for each $x \in X$, and its open neighborhood U, since f is a π-mapping, there exists $n \in \mathbb{N}$ such that $d(f^{-1}(x), M \setminus f^{-1}(U)) > 1/n$. We can pick $m \in \mathbb{N}$ such that $m \geqslant 2n$. If $z \in M$ with $x \in f(B(z, 1/m))$, then

$$f^{-1}(x) \cap B(z, 1/m) \neq \emptyset.$$

If $B(z, 1/m) \not\subset f^{-1}(U)$, then

$$d(f^{-1}(x), M \setminus f^{-1}(U)) \leqslant 2/m \leqslant 1/n,$$

a contradiction. Thus $B(z,1/m) \subset f^{-1}(U)$, so $f(B(z,1/m)) \subset U$. Hence $\mathrm{st}(x,\mathscr{P}_m) \subset U$. Therefore $\{\mathscr{P}_n\}$ is a point-star network for X.

We shall prove that every $\mathscr{P}_k$ is a cs-cover for X. Since f is weak-open, there exists a weak-base $\mathscr{B} = \bigcup\{\mathscr{B}_x \colon x \in X\}$ for X, and for each $x \in X$ there exists $m_x \in f^{-1}(x)$ satisfying the following condition: for each open neighborhood U of m_x in $M, B \subset f(U)$ for some $B \in \mathscr{B}_x$. For each $k \in \mathbb{N}$, if $\{x_n\}$ converges to $x \in X$ in X, there exists $B \in \mathscr{B}_x$ such that $B \subset f(B(m_x,1/k))$ since f is weak-open. Since B is a weak-neighborhood of x in X, B is a sequential neighborhood of x in X by Corollary 1.6.18 in [9], so $f(B(m_x,1/k))$ is too. Thus $\{x_n\}$ is eventually in $f(B(m_x,1/k))$. This implies each $\mathscr{P}_k$ is a cs-cover for X.

For each $x \in X$ and $k \in \mathbb{N}$, since $f(B(m_x,1/k))$ is a sequential neighborhood of x in X, $\mathrm{st}(x,\mathscr{P}_k)$ is too. Obviously, X is a sequential space. So $\langle \mathrm{st}(x,\mathscr{P}_k)\rangle$ is a weak neighborhood base of x in X.

In other words, $\{\mathscr{P}_n\}$ is a cs-cover weak-development for X.

(2) $\Rightarrow$ (3): Suppose $\{\mathscr{P}_n\}$ is a cs-cover weak-development for X. We can assume that $\mathscr{P}_{n+1}$ refines $\mathscr{P}_n$ for each $n \in \mathbb{N}$. For each $x, y \in X$, denoting

$$t(x,y) = \min\{n \colon x \notin \mathrm{st}(y,\mathscr{P}_n)\} \quad (x \neq y),$$

we define

$$d(x,y) = \begin{cases} 0, & x = y, \\ 2^{-t(x,y)}, & x \neq y, \end{cases}$$

then $d \colon X \times X \to [0,+\infty)$ is a symmetric function on X.

Claim. *For each $x, y \in X$, $x \in \mathrm{st}(y,\mathscr{P}_n)$ if and only if $t(x,y) > n$.*

In fact, the if part is obvious. For the only if part, suppose $x \in \mathrm{st}(y,\mathscr{P}_n)$ but $t(x,y) \leqslant n$. Since $\mathscr{P}_n$ refines $\mathscr{P}_{t(x,y)}$, $\mathrm{st}(y,\mathscr{P}_n) \subset \mathrm{st}(y,\mathscr{P}_{t(x,y)})$. Note that $x \notin \mathrm{st}(y,\mathscr{P}_{t(x,y)})$, so $x \notin \mathrm{st}(y,\mathscr{P}_n)$, a contradiction.

For each $x \in X$ and $n \in \mathbb{N}$, $\mathrm{st}(x,\mathscr{P}_n) = B(x,1/2^n)$ by the Claim. Because $\{\mathscr{P}_n\}$ is a point-star network for X, (X,d) is symmetrizable. And d has the following property: for each $x \in X$ and $\varepsilon > 0$, there exists $\delta = \delta(x,\varepsilon) > 0$ such that $d(x,y) < \delta$ and $d(x,z) < \delta$ imply $d(y,z) < \varepsilon$. Otherwise, there exist $\varepsilon_0 > 0$ and two sequences $\{y_n\}$ and $\{z_n\}$ in X such that $d(y_n,z_n) \geqslant \varepsilon_0$ whenever $d(x,y_n) < 1/2^n$ and $d(x,z_n) < 1/2^n$. Since $\mathscr{P}_n$ is a point-star network for X, $\{y_n\}$ and $\{z_n\}$ all converge to x. We choose $k \in \mathbb{N}$ such that $1/2^k < \varepsilon_0$. Since $\mathscr{P}_k$ is a cs-cover for X, $\{y_m, z_m\} \subset P$ for some $m \in \mathbb{N}$ and $P \in \mathscr{P}_k$. Thus $y_m \in \mathrm{st}(z_m,\mathscr{P}_k)$. By the Claim, $t(y_m,z_m) > k$. Thus, $d(y_m,z_m) = 1/2^{t(y_m,z_m)} < 1/2^k < \varepsilon_0$, a contradiction.

For each $x \in X$ and $n \in \mathbb{N}$, we can pick $\delta = \delta(x,n)$ such that $d(y,z) < 1/n$ whenever $d(x,y) < \delta$ and $d(x,z) < \delta$. Let $g(n,x) = B(x,\delta(x,n))$. Since $\mathscr{P}_n$ is a cs-cover for X, $\mathrm{st}(x, \mathscr{P}_n)$ is a sequential neighborhood of x in X, so $g(n, x)$ is too. Put

$$\mathscr{F}_n = \{g(n, x)\colon\ x \in X\},$$

then every $\mathscr{F}_n$ is an sn-cover for X .

If $\{\mathscr{F}_n\}$ is not a point-star network for X, then there exist $x \in G \in \tau(X)$ and two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x \in g(n, y_n)$ and $x_n \in g(n, y_n) \setminus G$. So $\{x_n\}$ does not converge to x, and $d(y_n, x) < \delta(y_n, n)$, $d(y_n, x_n) < \delta(y_n, n)$. By the property above, $d(x, x_n) < 1/n$. This implies that $\{x_n\}$ converges to x, a contradiction. Hence $\{\mathscr{F}_n\}$ is a point-start network for X.

Since X is symmetrizable, X is a sequential space. For each $x \in X$ and $n \in \mathbb{N}$, by the above, $g(n, x)$ is a sequential neighborhood of x in X. By $g(n, x) \subset \mathrm{st}(x, \mathscr{F}_n)$, $\mathrm{st}(x, \mathscr{F}_n)$ is too. So $\langle \mathrm{st}(x, \mathscr{F}_n)\rangle$ is a weak neighborhood base of x in X. Hence $\{\mathscr{F}_n\}$ is a sn-cover weak-development for X.

(3) $\Rightarrow$ (1): Suppose $\{\mathscr{P}_n\}$ is a sn-cover weak-development for X. For each $i \in \mathbb{N}$, let $\mathscr{P}_i = \{P_\alpha\colon\ \alpha \in \Lambda_i\}$. Endow Λ_i with the discrete topology, then Λ_i is a metric space. Put

$$M = \left\{\alpha = (\alpha_i) \in \prod_{i \in \mathbb{N}} \Lambda_i\colon\ \langle P_{\alpha_i}\rangle \text{ forms a network at some point } x_\alpha \text{ in } X\right\},$$

and endow M with the subspace topology induced from the usual product topology of the collection $\{\Lambda_i\colon\ i \in \mathbb{N}\}$ of metric spaces, then M is a metric space. Since X is Hausdroff, x_α is unique in X. For each $\alpha \in M$, we define $f\colon\ M \to X$ by $f(\alpha) = x_\alpha$. For each $x \in X$ and $i \in \mathbb{N}$, there exists $\alpha_i \in \Lambda_i$ such that $x \in P_{\alpha_i}$. Since $\{\mathscr{P}_i\}$ is a point-star network for X, $\{P_{\alpha_i}\colon\ i \in \mathbb{N}\}$ is a network of x in X. Put $\alpha = (\alpha_i)$, then $\alpha \in M$ and $f(\alpha) = x$. Thus f is surjective. Suppose $\alpha = (\alpha_i) \in M$ and $f(\alpha) = x \in U \in \tau(X)$, then there exists $n \in \mathbb{N}$ such that $P_{\alpha_n} \subset U$. Put

$$V = \{\beta \in M\colon\ \text{the } n\text{th coordinate of } \beta \text{ is } \alpha_n\},$$

then $\alpha \in V \in \tau(M)$, and $f(V) \subset P_{\alpha_n} \subset U$. Hence f is continuous.

For each $\alpha, \beta \in M$, we define

$$d(\alpha, \beta) = \begin{cases} 0, & \alpha = \beta \\ \max\{1/k\colon\ \pi_k(\alpha) \neq \pi_k(\beta)\}, & \alpha \neq \beta, \end{cases}$$

then d is a distance in M. Because the topology of M is the subspace topology induced from the usual product topology of the collection $\{\Lambda_i\colon\ i \in \mathbb{N}\}$ of discrete spaces, d is metric in M. For each $x \in U \in \tau(X)$, note that $\{\mathscr{P}_n\}$ is a point-star network for X, hence there exists $n \in \mathbb{N}$ such that $\mathrm{st}(x, \mathscr{P}_n) \subset U$. For $\alpha \in f^{-1}(x)$, $\beta \in M$, if $d(\alpha, \beta) < 1/n$, then $\pi_i(\alpha) = \pi_i(\beta)$ for all $i \leqslant n$. So $x \in P_{\pi_n(\alpha)} = P_{\pi_n(\beta)}$. Thus

$$f(\beta) \in \bigcap_{i \in \mathbb{N}} P_{\pi_i(\beta)} \subset P_{\pi_n(\beta)} \subset U.$$

Hence

$$d(f^{-1}(x), M \setminus f^{-1}(U)) \geqslant 1/n.$$

Therefore f is a π-mapping.

We shall prove that f is weak-open. For each $x \in X$, since every $\mathscr{P}_i$ is an sn-cover for X, there exists $\alpha_i \in \Lambda_i$ such that P_{α_i} is a sequential neighborhood of x in X. Since $\{\mathscr{P}_i\}$ is a point-star network for X, $\langle P_{\alpha_i} \rangle$ is a network of x in X. Put $\beta_x = (\alpha_i) \in \prod_{i \in \mathbb{N}} \Lambda_i$, then $\beta_x \in f^{-1}(x)$.

Let $\{U_{m\beta_x}\}$ be a decreasing neighborhood base of β_x in M, and put

$$\mathscr{B}_x = \{f(U_{m\beta_x})\colon\ m \in \mathbb{N}\},$$
$$\mathscr{B} = \bigcup \{\mathscr{B}_x\colon\ x \in X\},$$

then $\mathscr{B}$ satisfies (1), (2) in Definition 1.1. Suppose G is open in X. For each $x \in G$, from $\beta_x \in f^{-1}(x)$, we see that $f^{-1}(G)$ is an open neighborhood of β_x in M. Thus $U_{m\beta_x} \subset f^{-1}(G)$ for some $m \in \mathbb{N}$, so $f(U_{m\beta_x}) \subset G$ and $f(U_{m\beta_x}) \in \mathscr{B}_x$. On the other hand, suppose that $G \subset X$ and for $x \in G$, there exists $B \in \mathscr{B}_x$ such that $B \subset G$. Denote $B = f(U_{m\beta_x})$ for some $m \in \mathbb{N}$. Let $\{x_n\}$ be a sequence converging to x in X. Since P_{α_i} is a sequential neighborhood of x in X for each $i \in \mathbb{N}$, $\{x_n\}$ is eventually in P_{α_i}. For each $n \in \mathbb{N}$, if $x_n \in P_{\alpha_i}$, let $\alpha_{in} = \alpha_i$; if $x_n \notin P_{\alpha_i}$, pick $\alpha_{in} \in \Lambda_i$ such that $x_n \in P_{\alpha_i n}$. Thus there exists $n_i \in \mathbb{N}$ such that $\alpha_{in} = \alpha_i$ for all $n > n_i$. So $\{\alpha_{in}\}$ converges to α_i. For each $n \in \mathbb{N}$, put

$$\beta_n = (\alpha_{in}) \in \prod_{i \in \mathbb{N}} \Lambda_i,$$

then $f(\beta_n) = x_n$ and $\{\beta_n\}$ converges to β_x. Since $U_{m\beta_x}$ is an open neighborhood of β_x in M, $\{\beta_n\}$ is eventually in $U_{m\beta_x}$, so $\{x_n\}$ is eventually in G. Hence G is a sequential neighborhood of x. So G is sequential open in X. Since X is a sequential space, G is open in X. This implies that $\mathscr{B}$ is a weak-base for X.

By the definition of $\mathscr{B}$, f is weak-open.

(2) $\Rightarrow$ (4): Suppose $\{\mathscr{P}_i\}$ is a cs-cover weak-development for X. We can assume that $\mathscr{P}_{n+1}$ refines $\mathscr{P}_n$ for each $n \in \mathbb{N}$. Similarly as in the proof of (2) $\Rightarrow$ (3), we can define a symmetric distance function d on X such that $\mathrm{st}(x, \mathscr{P}_n) = B(x, 1/2^n)$ for each $x \in X$ and $n \in \mathbb{N}$. So (X, d) is symmetrizable. For each sequence $\{x_n\}$ in X

converging to $x \in X$ and $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $1/2^k < \varepsilon$. Since $\mathscr{P}_k$ is a cs-cover for X, there exist $P \in \mathscr{P}_k$ and $l \in \mathbb{N}$ such that $\{x\} \cup \{x_n\colon\ n \geqslant l\} \subset P$. If $n, m \geqslant l$, then $x_n, x_m \in P$, so $x_n \in \operatorname{st}(x_m, \mathscr{P}_k)$. Thus $t(x_n, x_m) > k$ by the Claim in (2) $\Rightarrow$ (3). Hence $d(x_n, x_m) = 1/2^{t(x_n,x_m)} < 1/2^k < \varepsilon$ whenever $n, m \geqslant l$. Therefore $\{x_n\}$ is d-Cauchy. This implies that X is a Cauchy space.

(4) $\Rightarrow$ (2): Suppose X is a Cauchy space. For each $n \in \mathbb{N}$, put

$$\mathscr{P}_n = \{A \subset X\colon\ \sup\{d(x, y)\colon\ x, y \in A\} < 1/n\}$$

then $\operatorname{st}(x, \mathscr{P}_n) = B(x, 1/n)$ for each $x \in X$, so $\{\mathscr{P}_n\}$ is a point-star network for X. It is clear that X is a sequential space. We need only prove that each $\mathscr{P}_n$ is a cs-cover for X. For each sequence $\{x_n\}$ converging to x in X, since $\{x_n\}$ is d-Cauchy and X is symmetrizable, there exists $m \in \mathbb{N}$ such that $d(x, x_i) < 1/(n+1)$ and $d(x_i, x_j) < 1/(n+1)$ for all $i, j \geqslant m$ by Lemma 9.3 in [16]. Put

$$P = \{x\} \cup \{x_i\colon\ i \geqslant m\}$$

then $P \in \mathscr{P}_n$. Hence each $\mathscr{P}_n$ is a cs-cover for X.

(4) $\Leftrightarrow$ (5) follows from Theorem 2.3 in [7]. $\square$

By Theorem 2.1, Proposition 2.2 in [7], Proposition 2.1.16 (3) in [9] and Proposition 2.1.16 in [9], we have

Proposition 2.2. *A space is developable if and only if it is a weak-open, π, pseudo-open image of a metric space.*

Corollary 2.3 ([12])**.** *A space is developable if and only if it is an open π-image of a metric space.*

We give examples illustrating Theorem 2.1 of this paper.

Example 2.4. Let X be the Arens space S_2 (see [9, Example 1.8.6]). Since X is Cauchy, X is a weak-open π-image of a metric space by Theorem 2.1. But X is not *an* open π-image of a metric space because X is not first countable. Thus the following holds:

A weak-open π-image of a metric space needn't be *an* open π-image of a metric space.

Example 2.5. Let X be the weak Cauchy space in [5, Example 2.14 (3)]. By Theorem 12 in [15], X is a quotient π-image of a metric space. But X is not Cauchy, X is not a weak-open π-image of a metric space by Theorem 2.1. Thus the following holds:

A quotient π-image of a metric space needn't be a weak-open π-image of a metric space.

Example 2.6. Let X be the Mrowka space $\psi(\mathbb{N})$ (see [9, Example 1.8.4]). Since X is developable, X is *an* open π-image of a metric space. But X has no point-countable cs*-networks. Thus X is not a quotient s-image of a metric space by Corollary 2.7.6 in [9]. Thus the following holds:

(1) A weak-open π-image of a metric space needn't be a weak-open compact image of a metric space.

(2) A weak-open π-image of a metric space needn't be a weak-open s-image of a metric space.

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