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A SIMPLE METHOD FOR CONSTRUCTING NON-LIOUVILLIAN FIRST INTEGRALS OF AUTONOMOUS PLANAR SYSTEMS

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Abstract. We show that a transformation method relating planar first-order differential systems to second order equations is an effective tool for finding non-liouvillian first integrals. We obtain explicit first integrals for a subclass of Kukles systems, including fourth and fifth order systems, and for generalized Liénard-type systems.

Keywords: planar polynomial systems, Kukles systems, generalized Liénard systems, non-liouvillian first integrals

MSC 2000: 81U15, 34C, 33

1. Introduction

Planar autonomous systems

$$x'(t) = P(x(t), y(t)),\\ y'(t) = Q(x(t), y(t)), \tag{1}$$

where P and Q are polynomials, are of considerable interest mainly for two reasons: first, they occur in many areas of applied mathematics and physics; secondly, their investigation is motivated by Hilbert's 16th problem [[12], in which one seeks a connection between the number of limit cycles and the polynomial degree of P and Q and, furthermore, criteria for an isolated singularity of (1) to be a center. These questions are closely related to the integrability of the system, because existence of a first integral determines its phase portrait completely. Unfortunately, first integrals are in general hard to find, and thus numerous methods have been developed in order to track them down. These methods include the Darboux theory of integrability [5], the use of Lie symmetries [3], the use of Lax pairs [13], the compatibility analysis [18] and many more, see [9] for an overview. Especially the Darboux theory of integrability received recent attention in the context of results obtained by Prelle and Singer: they showed that if (1) has an elementary first integral, then one can construct it with the Darboux theory of integrability [16]. Furthermore, Singer proved that if (1) has a liouvillian first integral, then its integrating factor must be given by Darbouxian functions [17]. Using these results, in many cases one can compute an integrating factor or first integral in a straightforward manner (see for example [14] and references therein).

However, there is a restriction to all the methods mentioned above: they are not applicable to non-liouvillian first integrals. In other words, if a first integral contains special, non-liouvillian functions (e.g. the Bessel or the hypergeometric ones), it cannot be found with the above methods. In the present note we want to do a step towards finding non-liouvillian first integrals. We consider a transformation method introduced very recently [6] that relates planar first-order differential systems to second order equations. The latter equations are sometimes called "special function equations" as they are very often solved by special (non-liouvillian) functions. Using the transformation method we obtain interesting results on general Kukles systems, including explicit first integrals for the following fourth and fifth order systems:

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + b_{40}x^4 + b_{22}x^2y^2 + b_{04}y^4, \end{aligned}$$

and

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + b_{50}x^5 + b_{32}x^3y^2 + b_{14}xy^4, \end{aligned}$$

which Giné [7] recently showed to be integrable, but without giving the corresponding first integrals. Furthermore, we obtain results on generalized Liénard systems, including an explicit first integral on a special case of a system recently studied by Hayashi [11]:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - g_q(x) - f_n(x)y^4. \end{aligned}$$

In Section 2 we summarize the transformation method and give some remarks. Section 3 is devoted to the application on Kukles systems and in Section 4 we apply our method to generalized Liénard systems.

2. Statement of the method

The transformation method we shall apply is derived in [6]. For the sake of completeness, we summarize it here and give some remarks.

Theorem 1. *Consider the system of differential equations*

$$x'(t) = P(x(t), y(t)),$$
$$y'(t) = Q(x(t), y(t)). \tag{2}$$

If there are functions c, φ and V such that the ratio of Q and P satisfies

$$\frac{Q(x,y)}{P(x,y)} = -\frac{c'(x)\big(V(c(x)) + \varphi^2(c(x),y) + \varphi_{c(x)}(c(x),y)\big)}{\varphi_y(c(x),y)}, \tag{3}$$

where the indices denote partial differentiation, then a first integral I of the system (2) *reads*

$$I(x,y) = \frac{-\varphi(c(x),y)u_2(c(x)) + u_2'(c(x))c'(x)}{\varphi(c(x),y)u_1(c(x)) - u_1'(c(x))c'(x)}, \tag{4}$$

where u_1 and u_2 are linearly independent solutions of the "special function equation"

$$u''(x) + V(x)u(x) = 0. \tag{5}$$

Let us add a few remarks.

Remark 1. The underlying idea of Theorem 1 is first to transform an equation that is solved by special functions into a first order Riccati equation; next, a combined transformation of the dependent and the independent variable takes the Ricatti equation into a more general equation of the first order. The solution of the latter equation then determines a first integral of an associated planar system of the first order (2) and contains special functions, provided the original second order equation has special function solutions.

Remark 2. The fact that there is no first derivative in $u'' + Vu = 0$ is not a restriction: it is well known that if functions u_1 and u_2 are solutions of the more general equation $fu'' + gu' + hu = 0$, then the transformation

$$\hat{u}(x) = (C_1u_1(x) + C_2u_2(x)) \exp\Big(\int \frac{g(x)}{f(x)}\, \mathrm{d}x\Big)$$

yields the equation $\hat{u}'' + V\hat{u} = 0$ with

$$V(x) = \frac{h(x)}{f(x)} - \Big(\frac{g(x)}{2f(x)}\Big)^2 - \Big(\frac{g(x)}{2f(x)}\Big)'.$$

Remark 3. Given two functions P and Q, for which φ, V and c do they fulfill relation (3)? To answer this question, one regards (3) as a partial differential equation for φ and tries to solve it. Unfortunately, an explicit solution is in general not available. In fact, if we try the method of characteristics, we see that it involves a solution of $y' = Q/P$, which is equivalent to the original underlying equation (2) whose solution we seek. This fact, of course, restricts the practical applicability of our method.

Remark 4. The search for exact, closed-form solutions of equation (5) is supported from two directions: First, there has been recent interest in solving equations of the type (5) in terms of special functions [1], [4], [20]. Secondly, equation (5) appears in the context of nonrelativistic quantum mechanics as the famous Schrödinger equation, whose exact solvability has been very well studied, especially for rational functions V [19]. Newer results on solutions in terms of special functions can be found, for example, in [10] and [15].

3. Application to Kukles systems

In this section we want to demonstrate how Theorem 1 can be practically applied for solving a class of Kukles systems, including the important fourth and fifth-order cases that were investigated recently [7]. The general strategy is to first try to match the terms containing y, since the terms containing x can always be controlled by an arbitrary function V. Obviously, systems of arbitrary order can be treated with our method, but, in order to keep it simple here, we aim at the following Kukles system:

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + b_{k4}x^k y^4 + b_{m2}x^m y^2 + F(x), \end{aligned} \tag{6}$$

where k and m are integers and F is (for now) an arbitrary function. In a Kukles system, we have $P(x, y) = -y$. Hence, in order to fulfil (3), the idea is to choose φ such that the denominator of (3) equals y. This yields

$$\varphi_y(c(x), y) = y \Rightarrow \varphi(c(x), y) = \frac{y^2}{2} + R(c(x)). \tag{7}$$

Inserting the latter into the right-hand side of (3), we get

$$\frac{Q(x,y)}{P(x,y)} = \frac{c'(x)\big(\frac{1}{4}y^4 + V(c(x)) + R(c(x))y^2 + R^2(c(x)) + R'(c(x))\big)}{-y}. \tag{8}$$

Comparing with (6) we see that the coefficient of y^4 must be $b_{k4}x^k$, which determines c as follows:

$$\frac{c'(x)}{4} = b_{k4}x^k \Rightarrow c(x) = \frac{4b_{k4}}{k+1}x^{k+1}. \tag{9}$$

By inserting the latter into (8), we obtain

$$\frac{Q(x,y)}{P(x,y)} = \frac{b_{k4}x^ky^4 + 4b_{k4}x^k\big(R(c(x))y^2 + R^2(c(x)) + R'(c(x)) + V(c(x))\big)}{-y}, \tag{10}$$

where c is given by (9). Next we fix the coefficient of y^2, which has to be $b_{m2}x^m$. We obtain

$$4b_{k4}x^kR(c(x)) = b_{m2}x^m \Rightarrow R(c(x)) = \frac{b_{m2}}{4b_{k4}}x^{m-k}. \tag{11}$$

Substituting this into (10), we come to

$$\begin{aligned}\frac{Q(x,y)}{P(x,y)} &= -y^{-1}\Big(b_{k4}x^ky^4 + b_{m2}x^my^2 \\ &\quad + \frac{b_{m2}}{4b_{k4}}x^{m-k-1}(-k+m+b_{m2}x^{m+1}) + 4b_{k4}x^kV(c(x))\Big),\end{aligned} \tag{12}$$

where c is again given by (9). In the final step we absorb all terms depending only on x into V. We require

$$\begin{aligned}x + F(x) &= \frac{b_{m2}}{4b_{k4}}x^{m-k-1}(-k+m+b_{m2}x^{m+1}) + 4b_{k4}x^kV(c(x)) \\ \Rightarrow V(c(x)) &= x^{-2k-1}(b_{m2}(k-m)x^m - b_{m2}^2x^{2m+1} + 4b_{k4}x^{k+1}(x+F(x))).\end{aligned} \tag{13}$$

We finally obtain

$$\frac{Q(x,y)}{P(x,y)} = \frac{x + b_{k4}x^ky^4 + b_{m2}x^my^2 + F(x)}{-y},$$

corresponding to the form of our system (6). According to Theorem 1, we can give a first integral of (16) provided we find a solution of (5). The latter reads in the present case $u''(x) + V(x)u(x) = 0$, where $V(x)$ can easily be inferred from (13) as

$$\begin{aligned}V(x) = \frac{1}{16b_{k4}^2}\big\{&(c^{-1}(x))^{-2k-1}\big[b_{m2}(k-m)(c^{-1}(x))^m - b_{m2}^2(c^{-1}(x))^{2m+1} \\ &+ 4b_{k4}(c^{-1}(x))^{k+1}(c^{-1}(x) + F(c^{-1}(x)))\big]\big\},\end{aligned} \tag{14}$$

where

$$c^{-1}(x) = 4^{-1/(k+1)}\Big(\frac{(k+1)x}{b_{k4}}\Big)^{1/(k+1)}. \tag{15}$$

Hence, in all cases when equation (5) with V as given in (14) is solvable, we can state an exact first integral of the Kukles system (6), which is given by (4). Let us now have a look at two important special cases.

Example 1 (fifth order Kukles system). This fifth order Kukles system is obtained from (6) by setting

$$\begin{aligned} k &= 1,\\ m &= 3,\\ F(x) &= b_{50}x^5, \end{aligned}$$

yielding the system

$$\begin{aligned} \dot{x} &= -y,\\ \dot{y} &= x + b_{50}x^5 + b_{32}x^3y^2 + b_{14}xy^4. \end{aligned} \tag{16}$$

Consequently, we find by evaluating (15) and (14) after some elementary manipulations

$$\begin{aligned} c^{-1}(x) &= \sqrt{\frac{x}{2b_{14}}},\\ V(x) &= \frac{2b_{14}-b_{32}}{8b_{14}^2} + \frac{-b_{32}^2+4b_{14}b_{50}}{64b_{14}^4}x^2. \end{aligned}$$

In order to construct a first integral (4), we abbreviate

$$a = \frac{2b_{14}-b_{32}}{8b_{14}^2}, \tag{17}$$

$$b = \frac{-b_{32}^2+4b_{14}b_{50}}{64b_{14}^4}. \tag{18}$$

The general solution of $u''(x) + (a+bx^2)u(x) = 0$ then reads

$$u(x) = C_1u_1(x) + C_2u_2(x),$$

where

$$\begin{aligned} u_1(x) &= \sqrt{\frac{1}{x}}M_{\frac{-\mathrm{i}a}{4\sqrt{b}},\frac{1}{4}}(\mathrm{i}\sqrt{b}x^2),\\ u_2(x) &= \sqrt{\frac{1}{x}}W_{\frac{-\mathrm{i}a}{4\sqrt{b}},\frac{1}{4}}(\mathrm{i}\sqrt{b}x^2). \end{aligned}$$

The symbols M and W denote Whittaker functions, defined by the relations

$$\begin{aligned} M_{k,m}(x) &= \exp\Big(-\frac{x}{2}\Big)x^{m+\frac{1}{2}}{}_1F_1\Big(\frac{1}{2}+m-k, 1+2m, x\Big), \\ W_{k,m}(x) &= \exp\Big(-\frac{x}{2}\Big)x^{m+\frac{1}{2}}U\Big(\frac{1}{2}+m-k, 1+2m, x\Big), \end{aligned}$$

where ${}_1F_1$ and U are the well known linearly independent solutions of the confluent hypergeometric equation. Now, by inserting the above u_1 and u_2 together with (7), (9), (11), (17) and (18) into (4), we obtain that system (16) has the first integral I

$$I(x,y) = \frac{I_1(x,y)}{I_2(x,y)},$$

where

$$\begin{aligned} I_1(x,y) = &-\left(\frac{y^2}{\sqrt{8b_{14}x}} + \frac{b_{32}x}{\sqrt{32b_{14}^3}}\right)W_{\frac{-\mathrm{i}a}{4\sqrt{b}},\frac{1}{4}}(4\mathrm{i}\sqrt{b}b_{14}^2x^4) \\ &+ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sqrt{2b_{14}x}}W_{\frac{-\mathrm{i}a}{4\sqrt{b}},\frac{1}{4}}(4\mathrm{i}\sqrt{b}b_{14}^2x^4)\right), \\ I_2(x,y) = &\left(\frac{y^2}{\sqrt{8b_{14}x}} + \frac{b_{32}x}{\sqrt{32b_{14}^3}}\right)M_{\frac{-\mathrm{i}a}{4\sqrt{b}},\frac{1}{4}}(4\mathrm{i}\sqrt{b}b_{14}^2x^4) \\ &- \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sqrt{2b_{14}x}}M_{\frac{-\mathrm{i}a}{4\sqrt{b}},\frac{1}{4}}(4\mathrm{i}\sqrt{b}b_{14}^2x^4)\right) \end{aligned}$$

and

$$\begin{aligned} a &= \frac{2b_{14}-b_{32}}{8b_{14}^2}, \\ b &= \frac{-b_{32}^2+4b_{14}b_{50}}{64b_{14}^4}. \end{aligned}$$

Example 2 (fourth order Kukles system)**.** We reduce (6) by choosing

$$\begin{aligned} k &= 0, \\ m &= 2, \\ F(x) &= b_{40}x^4, \end{aligned}$$

yielding the system

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + b_{40}x^4 + b_{22}x^2y^2 + b_{04}y^4. \end{aligned} \tag{19}$$

As before, we find by evaluating (15) and (14) that

$$c^{-1}(x) = \frac{x}{4b_{04}},$$
$$V(x) = \frac{2b_{04} - b_{22}}{32b_{04}^3}x + \frac{-b_{22}^2 + 4b_{04}b_{40}}{4096b_{04}^6}x^4.$$

Abbreviating

$$a = \frac{2b_{04} - b_{22}}{32b_{04}^3}, \tag{20}$$
$$b = \frac{-b_{22}^2 + 4b_{04}b_{40}}{4096b_{04}^6}, \tag{21}$$

we obtain the general solution of $u''(x) + (ax + bx^4)u(x) = 0$, which reads

$$u(x) = C_1 u_1(x) + C_2 u_2(x),$$

where

$$u_1(x) = \frac{1}{x} M_{\frac{-\mathrm{i}a}{6\sqrt{b}}, \frac{1}{6}}\Big(\frac{2\mathrm{i}\sqrt{b}}{3}x^3\Big), \tag{22}$$
$$u_2(x) = \frac{1}{x} W_{\frac{-ia}{6\sqrt{b}}, \frac{1}{6}}\Big(\frac{2\mathrm{i}\sqrt{b}}{3}x^3\Big). \tag{23}$$

Now, inserting the latter together with (7), (9), (11), (20) and (21) into (4), we obtain that the system (19) has the first integral I

$$I(x,y) = \frac{I_1(x,y)}{I_2(x,y)},$$

where

$$\begin{aligned} I_1(x,y) = &-\Big(\frac{y^2}{8b_{04}x} + \frac{b_{22}x}{16b_{04}^2}\Big) W_{\frac{-\mathrm{i}a}{6\sqrt{b}}, \frac{1}{6}}\Big(\frac{128\mathrm{i}\sqrt{b}b_{04}^3}{3}x^3\Big) \\ &+ \frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{1}{4b_{04}x} W_{\frac{-\mathrm{i}a}{6\sqrt{b}}, \frac{1}{6}}\Big(\frac{128\mathrm{i}\sqrt{b}b_{04}^3}{3}x^3\Big)\Big), \\ I_2(x,y) = &\Big(\frac{y^2}{8b_{04}x} + \frac{b_{22}x}{16b_{04}^2}\Big) M_{\frac{-\mathrm{i}a}{6\sqrt{b}}, \frac{1}{6}}\Big(\frac{128\mathrm{i}\sqrt{b}b_{04}^3}{3}x^3\Big) \\ &- \frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{1}{4b_{04}x} M_{\frac{-\mathrm{i}a}{6\sqrt{b}}, \frac{1}{6}}\Big(\frac{128\mathrm{i}\sqrt{b}b_{04}^3}{3}x^3\Big)\Big) \end{aligned}$$

and

$$a = \frac{2b_{04} - b_{22}}{32b_{04}^3},$$
$$b = \frac{-b_{22}^2 + 4b_{04}b_{40}}{4096b_{04}^6}.$$

4. Application to generalized Liénard systems

The purpose of this section is to show how the transformation method can be applied to different kinds of generalized Liénard systems. We construct first integrals for classes of systems that have the form

$$\begin{aligned} \dot{x} &= \psi(y), \\ \dot{y} &= -f(x)h(y) - g(x). \end{aligned} \tag{24}$$

Such systems were treated for example in [2], [11] and [21]. In order to apply our method, the relation (3) must be fulfilled. Clearly, we want the denominator of the ratio in (3) to take the value $\psi(y)$. Therefore, we have to set

$$\begin{aligned} \varphi_y(c(x), y) &= \psi(y), \\ \varphi(c(x), y) &= \int \psi(y)\,\mathrm{d}y + R(c(x)), \end{aligned} \tag{25}$$

where R can be seen as a constant of integration. Using this setting, expression (3) simplifies to

$$\begin{aligned} \frac{Q(x,y)}{P(x,y)} = -\psi(y)^{-1}\Big(c'(x)\Big(\Big(\int \psi(y)\,\mathrm{d}y\Big)^2 + 2R\big(c(x)\big)\int \psi(y)\,\mathrm{d}y \\ + R^2\big(c(x)\big) + V\big(c(x)\big) + R'\big(c(x)\big)\Big)\Big). \end{aligned} \tag{26}$$

Next, the numerator in this ratio must take the form $f(x)h(y) + g(x)$. After a little reflection it becomes clear that this happens for

$$R(c(x)) = R = \text{constant}. \tag{27}$$

In fact, (26) simplifies to

$$\frac{Q(x,y)}{P(x,y)} = -\frac{c'(x)\big(\big(\int \psi(y)\,\mathrm{d}y\big)^2 + 2R\big(\int \psi(y)\,\mathrm{d}y\big) + R^2 + V(c(x))\big)}{\psi(y)}. \tag{28}$$

Now this ratio fits to the form of the generalized Liénard system (24); by comparison with the second equation in (24) we get the equalities

$$f(x) = c'(x), \tag{29}$$

$$g(x) = c'(x)(R^2 + V(c(x))), \tag{30}$$

$$h(y) = \Big(\int \psi(y)\,\mathrm{d}y\Big)^2 + 2R\Big(\int \psi(y)\,\mathrm{d}y\Big). \tag{31}$$

Summarizing, our transformation method is applicable to generalized Liénard systems (24) provided the functions f, g and h are given by (29), (30) and (31), respectively, and provided V is chosen in such a way that equation (5) is solvable.

Example 3 (Hayashi's system)**.** In his recent article [11] Hayashi studies local integrability of the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - g_q(x) - f_n(x)y^p, \end{aligned} \tag{32}$$

where $n, p \geqslant 1$, $q \geqslant 2$ and the functions f_n and g_q satisfy

$$f_n(x) = \sum_{k \geqslant n} f_k x^k, \tag{33}$$

$$g_q(x) = \sum_{k \geqslant q} g_k x^k. \tag{34}$$

Clearly, Hayashi's system (32) is a special case of the generalized Liénard system (24), where

$$\psi(y) = y, \tag{35}$$

$$f(x) = f_n(x), \tag{36}$$

$$g(x) = x + g_q(x), \tag{37}$$

$$h(y) = y^p. \tag{38}$$

We now want to apply our transformation method to a subclass of Hayashi's system. Employing (35) in (25), we obtain

$$\varphi(c(x), y) = \frac{1}{2}y^2 + R(c(x)).$$

This changes (28) to

$$\frac{Q(x,y)}{P(x,y)} = -\frac{c'(x)\left(\frac{1}{4}y^4 + Ry^2\right) + R^2 c'(x) + V(c(x))c'(x)}{y}. \tag{39}$$

Since we need only a single power of y in the numerator of this ratio (see (32)), either the second or the fourth power of y must be removed. We cannot remove the second power as $c'(x) = 0$ is impossible, because c is a coordinate transformation. Hence, we have to set

$$R = 0, \tag{40}$$

simplifying (39) to

$$\frac{Q(x,y)}{P(x,y)} = -\frac{\frac{1}{4}c'(x)y^4 + V(c(x))c'(x)}{y}. \tag{41}$$

Now c and V can be used such that this expression fits to the system (32). A straightforward computation yields

$$c(x) = 4 \int f_n(x)\,\mathrm{d}x, \tag{42}$$

$$V(c(x)) = \frac{x + g_q(x)}{f_n(x)}. \tag{43}$$

Employing the abover settings, (41) matches the form of Hayashi's system (32) for $p = 4$:

$$\frac{Q(x,y)}{P(x,y)} = -\frac{x + g_q(x) + f_n(x)y^4}{y}.$$

At this point we cannot continue deriving first integrals of (32), because we are unable to specify $V(c(x))$ due to the lack of information on c^{-1}. Clearly, c^{-1} is not available in general, because equation (42) is not solvable for x. Thus, let us restrict ourselves to a simple special case, namely to

$$f_n(x) = f_1 x, \tag{44}$$

$$g_q(x) = g_3 x^3. \tag{45}$$

These choices correspond to special cases of the general functions f_n and g_q, as given in (33) and (34). Now we can compute c (42) and its inverse c^{-1}:

$$c(x) = 2f_1 x^2, \tag{46}$$

$$c^{-1}(x) = \sqrt{\frac{x}{2f_1}}, \tag{47}$$

where we have taken the positive sign in the inverse. Next, we compute the function V from (43). We find with (47):

$$V(c(x)) = \frac{1}{4f_1} + \frac{g_2}{4f_1}x^2 \Rightarrow V(x) = \frac{1}{4f_1} + \frac{g_2}{8f_1^2}x.$$

Now we have to solve equation (5), which reads in the present case

$$u''(x) + \Big(\frac{1}{4f_1} + \frac{g_2}{8f_1^2}x\Big)u(x) = 0.$$

Its solution contains special functions, namely Airy functions Ai and Bi. We have

$$u(x) = C_1 u_1(x) + C_2 u_2(x),$$

where

$$\begin{aligned} u_1(x) &= \mathrm{Ai}\Big(\frac{(-1)^{1/3}(2f_1 + g_3 x)}{2(f_1 g_3)^{2/3}}\Big), \\ u_2(x) &= \mathrm{Bi}\Big(\frac{(-1)^{1/3}(2f_1 + g_3 x)}{2(f_1 g_3)^{2/3}}\Big). \end{aligned}$$

The definition and properties of Airy functions can be found in [8]. Now we are in position to state a non-liouvillian first integral of the system (32) for $p = 4$ and functions f_n, g_q as given in (44) and (45), respectively. We collect the results (25), (40), (46) and the above two functions u_1, u_2 and insert them into (4). The first integral reads

$$I(x, y) = \frac{I_1(x, y)}{I_2(x, y)},$$

where

$$\begin{aligned} I_1(x, y) &= -\frac{1}{2} y^2 \mathrm{Bi}\Big(\frac{(-1)^{1/3}(4f_1^2 + g_3 x^2)}{2(f_1 g_3)^{2/3}}\Big) + \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Bi}\Big(\frac{(-1)^{1/3}(2f_1 + g_3 x)}{2(f_1 g_3)^{2/3}}\Big), \\ I_2(x, y) &= \frac{1}{2} y^2 \mathrm{Ai}\Big(\frac{(-1)^{1/3}(4f_1^2 + g_3 x^2)}{2(f_1 g_3)^{2/3}}\Big) - \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Ai}\Big(\frac{(-1)^{1/3}(2f_1 + g_3 x)}{2(f_1 g_3)^{2/3}}\Big). \end{aligned}$$

5. Concluding remarks

In this note we have presented applications for a simple method for finding planar first order systems that admit a first integral in terms of special functions. Although we know that the practical applicability of our method is in general limited, our approach is noteworthy for three reasons: First, so far there is little literature on non-liouvillian solutions of differential equations, and no literature referring directly to non-liouvillian first integrals of planar systems. Secondly, using our transformation method we could obtain first integrals of interesting, recently investigated systems of Kukles type and of generalized Liénard type. Thirdly, this work could inspire work on other types of transformations that lead to non-liouvillian first integrals, namely Darboux- and fractional Darboux transformations, which we will report on in a forthcoming paper.

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