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AN ANSWER TO A QUESTION OF CAO, REILLY AND XIONG

Z. Ercan, Ankara, S. Onal

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Abstract. We present a simple proof of a Banach-Stone type Theorem. The method used in the proof also provides an answer to a conjecture of Cao, Reilly and Xiong.

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MSC 2000: 46B42, 54C35

In this paper we use the standard terminology and notation of the Riesz spaces theory (see [1]). In particular, the Banach lattice (under pointwise operations, order and supremum norm) of continuous functions from a compact Hausdorff space K into a Banach lattice E is denoted by $C(K,E)$. If $E=\mathbb{R}$ then we write $C(K)$ instead of $C(K,E)$. $\mathbf{1}_K$ stands for the unit function in $C(K)$.

One version of the Banach-Stone theorem states:

Theorem 1. *Let X and Y be compact Hausdorff spaces. Then $C(X)$ and $C(Y)$ are Riesz isomorphic if and only if X and Y are homeomorphic.*

More precisely, if $\pi\colon C(X)\longrightarrow C(Y)$ is a Riesz isomorphism then there exists a homeomorphism $\sigma\colon Y\longrightarrow X$ and $h\in C(Y)$ such that $\pi(f)(y)=h(y)f(\sigma(y))$ and $0<h(y)$ for each $y\in Y$. An elementary proof of this theorem can be found in [3]. This theorem is generalized in [2] as follows.

Theorem 2. *Let X and Y be compact Hausdorff spaces and E a Banach lattice. If $\pi\colon C(X,E)\longrightarrow C(Y)$ is a Riesz isomorphism such that $\pi(f)$ has no zeros whenever f has no zero, then X and Y are homeomorphic and E is Riesz isomorphic to $\mathbb{R}$.*

The proof of Theorem 2 is given without using Theorem 1 in [2] and it is conjectured that Theorem 2 follows from Theorem 1. We present an elementary proof of

Theorem 2 which also yields an affirmative answer to this conjecture. First we need the following lemma.

Lemma 3. *Let X, Y and M be compact Hausdorff spaces such that $X \times M$ and Y are homeomorphic. Suppose that for a given $f \in C(X \times M)$, $f(x,m) \neq 0$ for all $(x,m) \in X \times M$ if and only if for each $x \in X$ there exists $m \in M$ such that $f(x,m) \neq 0$. Then X and Y are homeomorphic and $M = \{m\}$.*

Proof. Suppose that there exist $m_1, m_2 \in M$ and $m_1 \neq m_2$. Choose $g \in C(M)$ with $g(m_1) = 0$ and $g(m_2) = 1$. Let $f \in C(X \times M)$ with $f(x,m) = g(m)$. This is impossible, so $M = \{m\}$ and X is homeomorphic to Y. □

Now we are ready to give an elementary proof of Theorem 2. The technique of the proof provides an answer to the conjecture mentioned above.

Proof of Theorem 2. Clearly $\pi^{-1}(\mathbf{1}_Y)$ is a strong order unit of $C(X,E)$. Then $0 < \pi^{-1}(\mathbf{1}_Y)(x)$ is a strong order unit of E for each $x \in X$. By the Kakutani Representation Theorem (see [3] for a direct and simple proof) there exits a compact Hausdorff space M such that E and $C(M)$ are Riesz isomorphic spaces. Let $a \in X$ be fixed and let $\pi_0\colon C(M) \longrightarrow E$ be a Riesz isomorphism such that $\pi_0(\mathbf{1}_M) = \pi^{-1}(\mathbf{1}_Y)(a)$. Then $C(X,E)$, $C(X,C(M))$ and $C(X \times M)$ are Riesz isomorphic spaces under Riesz isomorphisms

$$\pi_1\colon C(X \times M) \longrightarrow C(X, C(M)) \quad \text{and} \quad \pi_2\colon C(X, C(M)) \longrightarrow C(X,E)$$

defined as $\pi_1(f)(x)(m) = f(x,m)$ and $\pi_2(f)(x) = \pi_0(f(x))$. By Theorem 1, there exist a homeomorphism $\sigma\colon Y \longrightarrow X \times M$ and $h \in C(Y)$ such that $0 < h(y)$ for each $y \in Y$ and $\pi\pi_2\pi_1(f)(y) = h(y)f(\sigma(y))$. Since $\pi(f)$ has no zeros whenever f has no zeros, we have that for a given $f \in C(X \times M)$, $f(x,m) \neq 0$ for all $(x,m) \in X \times M$ whenever for each $x \in X$ there exists $m \in M$ such that $0 \neq f(x,m)$. To see this claim, let $f \in C(X \times M)$ be such that for each $x \in X$ there exits $m_x \in M$ such that $f(x, m_x) \neq 0$. Let $x_0 \in X$. Define $f_{x_0}\colon M \longrightarrow \mathbb{R}$ by $f_{x_0}(m) = f(x_0, m)$. Then $\mathbf{1}_X \otimes \pi_0(f_{x_0}) \in C(X,E)$ is a non-zero constant function, where $\mathbf{1}_X \otimes \pi_0(f_{x_0})(x) = \pi_0(f_{x_0})$ for each $x \in X$. Choose $p \in C(X \times M)$ such that $\pi_2\pi_1(p) = \mathbf{1}_X \otimes \pi_0(f_{x_0})$. From the hypothesis we obtain

$$0 \neq \pi(\mathbf{1}_X \otimes \pi_0(f_{x_0}))(y) = \pi\pi_2\pi_1(p)(y) = h(y)p(\sigma(y)).$$

This shows that there exists $\varepsilon > 0$ such that $\varepsilon\mathbf{1}_Y \leqslant \pi(\mathbf{1}_X \otimes |\pi_0(f_{x_0})|)$, that is, $\pi^{-1}(\varepsilon\mathbf{1}_Y) \leqslant \mathbf{1}_X \otimes |\pi_0(f_{x_0})|$, hence

$$\varepsilon\pi_0(\mathbf{1}_M) = \pi^{-1}(\varepsilon\mathbf{1}_Y)(a) \leqslant \mathbf{1}_X \otimes |\pi_0(f_{x_0})|(a) = |\pi_0(f_{x_0})| = \pi_0(|f_{x_0}|).$$

This implies that $\varepsilon \mathbf{1}_M \leqslant |f_{x_0}|$. Hence $0 \neq f(x_0, m)$ for each m. From the previous lemma, we have $M = \{m\}$, hence X and Y are homeomorphic. Since $C(M)$ is a Riesz space isometrically isomorphic to $\mathbb{R}$, the proof is completed. $\square$

References

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Author's address: Zafer Ercan, Middle East Technical University, Department of Mathematics, 06531 Ankara, Turkey, e-mail: zercan@metu.edu.tr.