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SIGNPOST SYSTEMS AND SPANNING TREES OF GRAPHS

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Abstract. By a ternary system we mean an ordered pair (W, R), where W is a finite nonempty set and $R \subseteq W \times W \times W$. By a signpost system we mean a ternary system (W, R) satisfying the following conditions for all $x, y, z \in W$: if $(x, y, z) \in R$, then $(y, x, x) \in R$ and $(y, x, z) \notin R$; if $x \neq y$, then there exists $t \in W$ such that $(x, t, y) \in R$. In this paper, a signpost system is used as a common description of a connected graph and a spanning tree of the graph. By a ct-pair we mean an ordered pair (G, T), where G is a connected graph and T is a spanning tree of G. If (G, T) is a ct-pair, then by the guide to (G, T) we mean the ternary system (W, R), where $W = V(G)$ and the following condition holds for all $u, v, w \in W$: $(u, v, w) \in R$ if and only if $uv \in E(G)$ and v belongs to the $u - w$ path in T. By Proposition 1, the guide to a ct-pair is a signpost system. We say that a signpost system is tree-controlled if it satisfies a certain set of four axioms (these axioms could be formulated in a language of the first-order logic). Consider the mapping φ from the class of all ct-pairs into the class of all signpost systems such that $\varphi((G, T))$ is the guide to (G, T) for every ct-pair (G, T). It is proved in this paper that φ is a bijective mapping from the class of all ct-pairs onto the class of all tree-controlled signpost systems.

Keywords: signpost system, path, connected graph, tree, spanning tree

MSC 2000: 05C38, 05C05, 05C12, 05C99

1. SIGNPOST SYSTEMS AND CT-PAIRS

Following [7], we say that S is a *ternary system* if $S = (W, R)$, where W is a finite nonempty set and $R \subseteq W \times W \times W$.

Let $S = (W, R)$ be a ternary system. We denote $V(S) = W$. Moreover, if $u, v, w \in V(S)$, then instead of $(u, v, w) \in R$ we will write $uvSw$ and instead of $(u, v, w) \notin R$ we will write $\neg(uvSw)$.

Let S be a ternary system. We denote by $\mathcal{A}_S$ the binary relation on $V(S)$ defined as follows: $(u, v) \in \mathcal{A}_S$ if and only if

$$u \neq v \text{ and if } utSv, \text{ then } t = v \text{ for every } t \in V(S)$$

for all $u, v \in V(S)$. Moreover, we denote by $S^{\mathcal{A}}$ the ternary system defined as follows: $V(S^{\mathcal{A}}) = V(S)$ and

$$xyS^{\mathcal{A}}z \text{ if and only if } xySz \text{ and } (x, y) \in \mathcal{A}_S$$

for all $x, y, z \in V(S)$.

By a *partial signpost system* we mean a ternary system S satisfying the following axioms (sp1) and (sp2):

(sp1) if $xySz$, then $yxSx$ for all $x, y, z \in V(S)$;
(sp2) if $xySz$, then $\neg(yxSz)$ for all $x, y, z \in V(S)$.

Lemma 1. *Let S be a partial signpost system, let $u, v, w \in V(S)$, and let $uvSw$. Then*
(a) *$uvSv$,*
(b) *$u \neq v$, and*
(c) *$u \neq w$.*

Proof. Axiom (sp1) implies (a) and axiom (sp2) implies (b). Combining axioms (sp1) and (sp2), we get (c). □

By a graph we mean a finite undirected graph without loops or multiple edges (notions and symbols not defined here can be found in [1]). Let S be a partial signpost system. According to axiom (sp1), $xySy$ if and only if $yxSx$ for all $x, y \in V(S)$. By the *underlying* graph of S we mean the graph G defined as follows: $V(G) = V(S)$ and

$$uv \in E(G) \text{ if and only if } uvSv \text{ for all } u, v \in V(S).$$

By a *signpost system* we mean a partial signpost system S satisfying the following axiom (sp3):

(sp3) if $x \neq y$, then there exists $t \in V(S)$ such that $xtSy$ for all $x, y \in V(S)$.

The term "signpost system" appeared for the first time in [2]. Nonetheless, signpost systems were implicitly studied already in [3] and [4].

Let T be a tree and let u and v be adjacent vertices of T. Then by $T(u, v)$ we denote the component of $T - u$ which contains v. Recall that if S is a partial signpost system, $u, v, w \in V(S)$, and $uvSw$, then u and v are adjacent in the underlying graph of S. The next lemma will be used also in Section 2.

Lemma 2. *Let S be a partial signpost system, let T be a tree, and let T be a component of the underlying graph of S. Assume that there exist $u, v, w \in V(S)$ such that $uvSw$, $u \in V(T)$, and w does not belong to $T(u, v)$. Then S is not a signpost system.*

P r o o f. Put $n = |V(F(u, v))|$. Without loss of generality we assume that

$$\text{if } |V(T(u^*, v^*)| < n, \text{ then } w^* \in V(T(u^*, v^*))$$

for every $u^*, v^*, w^* \in V(S)$ such that $u^*v^*Sw^*$ and $u^* \in V(T)$.

Suppose, to the contrary, that S satisfies axiom (sp3). There exists $t \in V(S)$ such that $vtSw$. Obviously, v and t are adjacent in T. Axiom (sp2) implies that $t \neq u$. Hence $n \geqslant 2$ and $|V(T(v, t))| < n$. This means that $w \in V(T(v, t))$ and therefore $w \in V(T(u, v))$, which is a contradiction. Thus S does not satisfy axiom (sp3), which completes the proof. □

Corollary 1. *Let S be a signpost system, and let G be the underlying graph of S. If G is disconnected, then no component of G is a tree.*

Lemma 3. *Let S be a signpost system, let $u, v \in V(S)$, and let $(u, v) \in \mathcal{A}_S$. Then $uvSv$.*

P r o o f. Since $(u, v) \in \mathcal{A}_S$, we have $u \neq v$. Since S satisfies axiom (sp3), there exists $t \in V(S)$ such that $utSv$. This implies that $t = v$ and therefore $uvSv$. □

Corollary 2. *Let S be a signpost system, and let $u, v \in V(S)$. Then $(u, v) \in \mathcal{A}_S$ if and only if $uvS^{\mathcal{A}}v$.*

Let G be a connected graph, and let d denote the distance function of G. By the *step system* of G we mean a ternary system S such that $V(S) = V(G)$ and

$$uvSw \text{ if and only if } d(u, v) = 1 \text{ and } d(v, w) = d(u, w) - 1 \text{ for all } u, v, w \in V(S).$$

It is easily shown that if S is the step system of a connected graph, then S is a signpost system and $S^{\mathcal{A}} = S$.

Remark 1. Let $W = \{w_1, w_2, w_3\}$, where w_1, w_2, and w_3 are pairwise distinct. Put $w_4 = w_1$ and $w_5 = w_2$. Let $S = (W, R)$ denote the ternary system such that R is the set of the following nine elements: $w_iw_{i+1}Sw_{i+1}$, $w_{i+1}w_iSw_i$, $w_iw_{i+1}Sw_{i+2}$ for $i = 1, 2, 3$. It is easy to see that S is a signpost system but $S^{\mathcal{A}}$ does not satisfy axiom (sp1).

As was proved in [3], if S is a signpost system such that the underlying graph of S is connected, then S is the step system of a connected graph if and only if S satisfies a finite set A of certain axioms (all the axioms in A could be formulated in a language of the first order logic). A shorter proof of this result can be found in [6]. A stronger result was found for modular graphs and median graphs in [2]. (Without connections to step systems, signpost systems were studied in [7]).

The present paper brings a new view on signpost systems. We will show that a signpost system of a certain kind can be used as a common description of a connected graph and a spanning tree of the graph.

By a ct-*pair* we mean an ordered pair (G, T), where G is a connected graph and T is a spanning tree of G. Let $P = (G, T)$ be a ct-pair. By the *guide* to P we mean the ternary system S defined as follows: $V(S) = V(G)$ and

$$uvSw \text{ if and only if } uv \in E(G) \text{ and } v \text{ belongs to the } u - w \text{ path in } T$$

for all $u, v, w \in V(G)$.

Proposition 1. *Let $P = (G, T)$ be a* ct-*pair and let S denote the guide to P. Then S is a signpost system.*

P r o o f. Consider arbitrary $u, v, w \in V(S)$. If $u \neq v$, then there exists $t \in V(S)$ such that t belongs to the $u - v$ path in T and $ut \in E(G)$, which means that $utSv$. Hence S satisfies axiom (sp3). Let now $uvSw$. Then $uv \in E(G)$ and v belongs to the $u - w$ path in T. Obviously, $u \neq v$. It is clear that v belongs to the $v - u$ path in T and therefore $vuSu$. Since $u \neq v$, u does not belong to the $v - w$ path in T and therefore $\neg(vuSw)$. We see that S satisfies axioms (sp1) and (sp2). Hence S is a signpost system. □

It will be proved in Section 3 that if S is the guide to a ct-pair, then $S^{\mathcal{A}}$ is also a signpost system.

Let G and H be graphs, and let S be a signpost system. We will prove that (G, H) is a ct-pair and S is the guide to (G, H) if and only G is the underlying graph of S, H is the underlying graph of $S^{\mathcal{A}}$, and S satisfies a certain set of four axioms.

2. Tree-like signpost systems and trees

We say that a signpost system S is *tree-like* if it satisfies the following axioms (tl1) and (tl2):

(tl1) if $x \neq y$, then there exists at most one $t \in V(S)$ such that $xtSy$ for all $x, y \in V(S)$;

(tl2) if $xySy$, then $xySz$ or $yxSz$ for all $x, y, z \in V(S)$.

It was shown in [5] that, simply saying, every tree can be considered as a finite nonempty set with a certain binary operation. Some ideas of [5] will be used in the proof of the following theorem.

Theorem 1. *Let H be a graph, and let S be a signpost system. Then the following two statements are equivalent:*

(I) *H is a tree and S is the step system of H;*

(II) *S is tree-like and H is the underlying graph of S.*

Proof. It is easy to prove that (I) implies (II). We will prove that (II) implies (I). Assume that (II) holds. Consider an arbitrary component F of H. We denote by S_F the ternary system defined as follows: $V(S_F) = V(F)$ and

$$uvS_Fw \text{ if and only if } uvSw \text{ for all } u, v, w \in V(F).$$

It is not difficult to see that S_F is a tree-like signpost system and F is the underlying graph of $S(F)$. Let d and S^{step} denote the distance function of F and the step system of F, respectively.

Consider arbitrary $u, v \in V(F)$. We now prove that

$$uxS^{\text{step}}v \text{ if and only if } uxS_Fv \text{ for all } x \in V(F). \tag{1}$$

We proceed by induction on $d(u, v)$. The case when $d(u, v) \leqslant 1$ is obvious. Assume that $d(u, v) \geqslant 2$ and the following statement holds for all $u^*, v^* \in V(F)$ such that $d(u^*, v^*) = d(u, v) - 1$:

$$u^*x^*S^{\text{step}}v^* \text{ if and only if } u^*x^*S_Fv^* \text{ for all } x^* \in V(F). \tag{2}$$

Consider an arbitrary $y \in V(F)$ and assume that $uyS^{\text{step}}v$. Obviously, uyS_Fy and thus, by axiom (tl2), yuS_Fv or uyS_Fv. Assume that yuS_Fv. Since $d(y, v) = d(u, v) - 1$, (2) implies that $yuS^{\text{step}}v$, which is a contradiction. Thus $uyS^{\text{step}}v$. We have proved that

$$\text{if } uyS^{\text{step}}v, \text{ then } uyS_Fv \text{ for } y \in V(F). \tag{3}$$

Consider an arbitrary $z \in V(F)$ and assume that uzS_Fv. There exists $t \in V(F)$ such that $utS^{\text{step}}v$. By (3), utS_Fv. Since S satisfies axiom (tl1), we get $t = z$. Hence $uzS^{\text{step}}v$ and the proof of (1) is complete. This means that $S^{\text{step}} = S_F$.

Assume that F contains a cycle. Let m denote the minimum length of a cycle in H. Consider a cycle C in H such that the length of C is m. Let d_C denote the distance function of C. It is easy to see that $d_C(u, v) = d(u, v)$ for all $u, v \in V(C)$. There exists $i \geqslant 2$ such that $m = 2i$ or $m = 2i - 1$. If $m = 2i$, then S^{step} does not satisfy axiom (tl1), which is a contradiction. Assume that $m = 2i - 1$. Then there exist $x, y, z \in V(C)$ such that $xy \in E(C)$ and $d(x, z) = i - 1 = d(y, z)$. Obviously, $xyS^{\text{step}}y$, $\neg xyS^{\text{step}}z$, and $\neg yxS^{\text{step}}z$. Hence S^{step} does not satisfy axiom (tl2), which is a contradiction. This means that F is a tree.

By virtue of Corollary 1, $F = H$. Hence H is a tree, which completes the proof.

□

Proposition 2. *For every tree T there exists exactly one signpost system S such that T is the underlying graph of S.*

Proof. Let S^{step} denote the step system of T. By Theorem 1, T is the underlying graph of S^{step}. Hence there exists at least one signpost system S such that T is the underlying graph of S.

Assume that there exists a signppost system S_0 different from S^{step} such that T is the underlying graph of S_0. It is easy to see that there exist $u, v, w \in V(T)$ such that v and w belong to distinct components of $T - u$ and uvS_0w. Lemma 2 implies that S_0 is not a signpost system, which is a contradiction. Thus the proposition is proved. □

The following lemma will be used in Section 3.

Lemma 4. *Let T be a tree, and let $u, v, w \in V(T)$ be such that $u \neq v$. Then the following two statements are equivalent:*

(I) *v belongs to the $u - w$ path in T;*

(II) *there exists $t \in V(T)$ such that $vt \in E(T)$, t belongs to the $v - u$ path in T, and v belongs to the $t - w$ path in T.*

Proof. It is clear that (I) implies (II).

Conversely, let (II) hold. Since $vt \in E(T)$ and t belongs to the $v - u$ path in T, we see that v does not belong to the $t - u$ path in T. If there exists $x \in V(T)$ different from t such that x belongs both to the $u - t$ path in T and to the $t - w$ path in T, then T contains a cycle; a contradiction. This means that t is the only common vertex of the $u - t$ path in T and the $t - w$ path in T. Since v belongs to the $t - w$ path in T, we see that (II) holds, which completes the proof. □

We say that a signpost system S is *tree-controlled* if it satisfies the following axioms (tc1), (tc2), (tc3), and (tc4):

(tc1) $(x,y) \in \mathcal{A}_S$ if and only if $(y,x) \in \mathcal{A}_S$ for all $x,y \in V(S)$;

(tc2) if $x \neq y$, then there exists exactly one $t \in V(S)$ such that $xtSy$ and $(x,t) \in \mathcal{A}_S$ for all $x,y \in V(S)$;

(tc3) if $(x,y) \in \mathcal{A}_S$, then $xySz$ or $yxSz$ for all $x,y,z \in V(S)$;

(tc4) $xySz$ if and only if $xySy$ and there exists $t \in V(S)$ such that $(y,t) \in \mathcal{A}_S$, $ytSx$, and $tySz$ for all $x,y,z \in V(S)$.

Remark 2. It is obvious that all the axioms (sp1), (sp2), (sp3), (tl1) (tl2), (tc1), (tc2), (tc3), and (tc4) can be formulated in the language of the first-order logic.

Lemma 5. *Let S be a tree-controlled signpost system. Then $S^{\mathcal{A}}$ is a tree-like signpost system.*

Proof. Consider arbitrary $u,v,w \in V(S)$. Assume that $uvS^{\mathcal{A}}w$. Then $(u,v) \in \mathcal{A}_S$ and $uvSw$. Since S satisfies axiom (tc1), we have $(v,u) \in \mathcal{A}_S$. Since S satisfies axioms (sp1) and (sp2), we have $vuSu$ and $\neg(vuSw)$. This means that $vuS^{\mathcal{A}}u$ and $\neg(vuS^{\mathcal{A}}w)$. Hence $S_{\mathcal{A}}$ satisfies axioms (sp1) and (sp2). Let $u \neq v$. Since S satisfies axiom (tc2), there exists exactly one $t \in V(S)$ such that $utSv$ and $(u,t) \in \mathcal{A}_S$. This implies that $S^{\mathcal{A}}$ satisfies axioms (sp3) and (tl1). Assume that $uvS^{\mathcal{A}}v$. Then $(u,v) \in \mathcal{A}_S$ and $uvSv$. Since S satisfies axioms (sp1) and (tc3), we see that $(v,u) \in \mathcal{A}_S$ and, moreover, $uvSw$ or $vuSw$. Thus $uvS^{\mathcal{A}}w$ or $vuS^{\mathcal{A}}w$. Hence $S^{\mathcal{A}}$ satisfies axiom (tl2) and therefore $S^{\mathcal{A}}$ is a tree-like signpost system, which completes the proof. □

Recall that if P is a ct-pair, then (by Proposition 1), the guide to P is a signpost system. Moreover, if S is a tree-controlled signpost system, then $S^{\mathcal{A}}$ is a signpost system as well. The next theorem is the main result of this paper.

Theorem 2. *Let G and H be graphs, and let S be a signpost system. Then the following two statements are equivalent:*

(I) *(G,H) is a* ct-*pair and S is the guide to (G,H);*

(II) *S is tree-controlled, G is the underlying graph of S, and H is the underlying graph of $S^{\mathcal{A}}$.*

Proof. (I) → (II): Let (G,H) be a ct-pair and let S be the guide to (G,H). Then H is a spanning tree of G. Obviously, $V(G) = V(H) = V(S)$. Consider arbitrary $u,v,w \in V(S)$.

It follows from the definition of the guide to a ct-pair that $uv \in E(G)$ if and only if $uvSv$. Thus G is the underlying graph of S.

Recall that H is a tree. It is clear that $(u, v) \in \mathcal{A}_S$ if and only if $uv \in E(H)$. Hence S satisfies axiom (tc1). Moreover, it is easy to see that $uvS^{\mathcal{A}}w$ if and only if $uv \in E(H)$ and v belongs to the $u - w$ path in H. This implies that $S^{\mathcal{A}}$ is the step system of H. Theorem 1 implies that $S^{\mathcal{A}}$ is a tree-like signpost system and H is the underlying graph of $S^{\mathcal{A}}$. Hence S satisfies axioms (tc2) and (tc3). It remains to prove that S satisfies axiom (tc4).

Assume that $uvSw$. By Lemma 1, $uvSv$ and $u \neq v$. Moreover, since S is the guide to (G, H), we see that v belongs to the $u - w$ path in H. Combining the fact that $S^{\mathcal{A}}$ is the step system of H with Lemma 4, we see that there exists $t \in V(S)$ such that $vtS^{\mathcal{A}}u$ and $tvS^{\mathcal{A}}w$; therefore $(v, t) \in \mathcal{A}_S$, $vtSu$, and $tvSw$.

Conversely, assume that $uvSv$ and there exists $t \in V(S)$ such that $(v, t) \in \mathcal{A}_S$, $vtSu$, and $tvSw$. Since $uvSv$, we get $u \neq v$. Lemma 4 implies that v belongs to the $u - w$ path in H. Since $uvSv$, we get $uv \in E(G)$. According to the definition of the guide to a ct-pair, $uvSw$.

Hence S satisfies axiom (tc4) and therefore S is tree-controlled.

(II) $\rightarrow$ (I): Let S be tree-controlled, let G be the underlying graph of S, and let H be the underlying graph of $S^{\mathcal{A}}$. Obviously, $V(G) = V(S) = V(S^{\mathcal{A}}) = V(H)$. Consider arbitrary $u, v, w \in V(S)$.

By Lemma 5, $S^{\mathcal{A}}$ is a tree-like signpost system. Recall that H is the underlying graph of $S^{\mathcal{A}}$. Theorem 1 implies that H is a tree and $S^{\mathcal{A}}$ is the step system of H.

Obviously, if $uvS^{\mathcal{A}}v$, then $uvSv$. This implies that H is a factor of G. Since H is a tree, we see that (G, H) is a ct-pair. It remains to prove that S is the guide to (G, H).

Assume that $uv \in E(G)$ and v belongs to the $u - w$ path in H. Since $uv \in E(G)$, we have $u \neq v$. By Lemma 5, there exists $t \in V(H)$ such that

$$tv \in E(H),\ \ t \text{ belongs to the } u - v \text{ path in } H$$
$$\text{and } v \text{ belongs to the } t - w \text{ path in } H.$$

Since H is the underlying graph of $S^{\mathcal{A}}$, we have $vtS^{\mathcal{A}}t$ and thus, by Corollary 2, $(v, t) \in \mathcal{A}_S$. Recall that $S^{\mathcal{A}}$ is the step system of H. We have $vtS^{\mathcal{A}}u$ and $tvS^{\mathcal{A}}w$. This implies that $vtSu$ and $tvSw$. Since $uv \in E(G)$, we have $uvSv$. Thus, by axiom (tc4), $uvSw$.

Conversely, assume that $uvSw$. Since S satisfies axiom (tc4), we see that $uvSv$ and there exists $t \in V(S)$ such that $(v, t) \in A_{\mathcal{S}}$, $vtSu$, and $tvSw$. It is clear that $vtS^{\mathcal{A}}u$ and $tvS^{\mathcal{A}}w$. Since $S^{\mathcal{A}}$ is the step system of H, we see that (4) holds. Recall that $uvSv$. By Lemma 1, $u \neq v$. Lemma 4 implies that v belongs to the $u - w$ path in H. Since $uvSv$ and G is the underlying graph of S, we have $uv \in E(G)$.

Hence S is the guide to (G, H), which completes the proof. □

Proposition 3. *Let S be the guide to a* ct*-pair. Then $S^{\mathcal{A}}$ is a tree-like signpost system.*

P r o o f. By Proposition 1, S is a signpost system. Theorem 2 implies that S is tree-controlled. Hence, by Lemma 5, $S^{\mathcal{A}}$ is a tree-like signpost system. □

The following two corollaries are immediate consequences of Theorem 1.

Corollary 3. *A signpost system S is tree-controlled if and only if there exists a* ct*-pair P such that S is the guide to P.*

Corollary 4. *Let φ denote the mapping from the class of all* ct*-pairs into the class of all signpost systems defined as follows:*

$$\varphi(P) \text{ is the guide to } P \text{ for every ct-pair } P.$$

Then φ is a bijective mapping from the class of all ct*-pairs onto the class of all tree-controlled signpost systems.*

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