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TRACELESS COMPONENT OF THE CONFORMAL CURVATURE TENSOR IN KÄHLER MANIFOLD

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Dedicated to Professor Shigeru Ishihara on the occasion of his 82nd birthday

Abstract. We investigate the traceless component of the conformal curvature tensor defined by (2.1) in Kähler manifolds of dimension $\geqslant 4$, and show that the traceless component is invariant under concircular change. In particular, we determine Kähler manifolds with vanishing traceless component and improve some theorems (for example, [4, pp. 313–317]) concerning the conformal curvature tensor and the spectrum of the Laplacian acting on p $(0 \leqslant p \leqslant 2)$-forms on the manifold by using the traceless component.

Keywords: Kähler manifold, conformal tensor field, trace decomposition, concircular transformation, spectrum

MSC 2000: 53C

1. Introduction

Recently, in his paper [3], Krupka has investigated the so-called *trace decomposition problem* and proved that a tensor

$$A = (A^{a_1,a_2,\ldots,a_p}_{b_1,b_2,\ldots,b_q})$$

of type (p,q) $(p \leqslant q)$ can always be expressed as the sum of a traceless term and a linear combination of the Kronecker δ-tensors, with traceless coefficients. In particular, he has provided the following Theorem K as explicit decomposition formula of a

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tensor of type $(1,3)$ and derived the Weyl projective, and Weyl conformal curvature tensors by using this analysis.

Theorem K ([3]). *Let $A = (A^a_{dcb})$ be a tensor of type $(1,3)$ in an n-dimensional $(n \geqslant 3)$ Riemannian manifold. Then there exist a unique traceless system $\mathring{A}^* = (\mathring{A}^{*a}_{dcb})$ and unique systems $C = (C_{cb}), D = (D_{cb}), E = (E_{cb})$ such that*

$$A^a_{dcb} = \mathring{A}^{*a}_{dcb} + \delta^a_d C_{cb} + \delta^a_c D_{db} + \delta^a_b E_{dc}.$$

These systems are defined by

$$\begin{aligned}
C_{cb} &= \frac{n(n^2-3)A^t_{tcb} - (n^2-2)A^t_{ctb} + nA^t_{cbt} - 2A^t_{tbc} + nA^t_{btc} - (n^2-2)A^t_{bct}}{(n^2-1)(n^2-4)},\\
D_{cb} &= \frac{-(n^2-2)A^t_{tcb} + n(n^2-3)A^t_{ctb} - (n^2-2)A^t_{cbt} + nA^t_{tbc} - 2A^t_{btc} + nA^t_{bct}}{(n^2-1)(n^2-4)},\\
E_{cb} &= \frac{nA^t_{tcb} - (n^2-2)A^t_{ctb} + n(n^2-3)A^t_{cbt} - (n^2-2)A^t_{tbc} + nA^t_{btc} - 2A^t_{bct}}{(n^2-1)(n^2-4)}.
\end{aligned}$$

Here and in the sequel we use the Einstein convention with respect to the index system $\{a, b, c, d, s, t, \ldots\}$.

In this paper we investigate the traceless component of the conformal curvature tensor (for definition, see (2.1)) in Kähler manifolds of dimension $\geqslant 4$ by using Krupka's analysis ([3]), and show that the traceless component is invariant under concircular change. In particular, as applications of the traceless component, we determine Kähler manifolds with vanishing traceless component and improve some theorems (for example, [4, pp. 313–317]) concerning conformal curvature tensor and the spectrum of the Laplacian acting on p $(0 \leqslant p \leqslant 2)$-forms on the manifold.

2. Preliminaries

Let M be a Kähler manifold of real dimension $n (= 2m)$ and (J, g) its Kähler structure. That is, g is a Riemannian metric and J a complex structure on M such that

$$J^t_c J^s_b g_{ts} = g_{cb}, \nabla_c J^a_b = 0,$$

where g_{cb} and J^a_b are the local components of g and J, respectively, and ∇_c denotes the operator of covariant differentiation with respect to g. We denote local components of the curvature tensor R, those of the Ricci tensor R_1 and the scalar curvature of M by R^a_{dcb}, R_{cb} and s, respectively. It is well known that $J_{cb} = J^t_c g_{tb}$ is skew-symmetric with respect to the indices c and b.

In 1990, Kitahara, Matsuo and Pak ([2]) have introduced the so-called *conformal curvature tensor* C whose local components are given by

$$
\begin{aligned}
(2.1) \qquad C^a_{dcb} = R^a_{dcb} &+ \frac{1}{n}(R^a_d g_{cb} - R^a_c g_{db} + \delta^a_d R_{cb} - \delta^a_c R_{db} \\
&- S^a_d J_{cb} + S^a_c J_{db} - J^a_d S_{cb} + J^a_c S_{db} + 2S_{dc} J^a_b + 2J_{dc} S^a_b) \\
&+ \frac{(n+4)s}{n^2(n+2)}(J^a_d J_{cb} - J^a_c J_{db} - 2J_{dc} J^a_b) - \frac{(3n+4)s}{n^2(n+2)}(\delta^a_d g_{cb} - \delta^a_c g_{db}),
\end{aligned}
$$

where $S_{cb} = -R_{ct} J^t_b = -S_{bc}$ and $S^a_c = S_{ct} g^{ta}$. We can easily see that the conformal curvature tensor satisfies the following properties:

$$
(2.2) \qquad C^t_{tcb} = \frac{2(n-4)}{n} R_{cb} - \frac{2(n-4)s}{n^2} g_{cb}, \quad C^a_{dcb} = -C^a_{cdb}, \quad C^t_{dct} = 0.
$$

$$
(2.3) \qquad \|C\|^2 = \|R\|^2 - \frac{32}{n^2}\|R_1\|^2 - \frac{8(n^2-4n-8)}{n^3(n+20)} s^2,
$$

where $\|T\|$ denotes the norm of a tensor T with respect to g (see also [4]).

The conformal curvature tensor is invariant under conformal change, provided $n \geqslant 4$, and has a very useful property, namely, a Kähler manifold with vanishing conformal curvature tensor is of constant holomorphic sectional curvature, provided $n \geqslant 6$ (for more details, see [2, pp. 13–14, Theorems A and B]).

3. The trace decomposition of the conformal curvature tensor

In this section, we consider Krupka's trace decomposition of the conformal curvature tensor appearing in (2.1).

By means of Theorem K, there exist a unique traceless system $\overset{*}{C} = (\overset{*}{C}{}^a_{dcb})$ and unique systems $C = (C_{cb}), D = (D_{db}), E = (E_{dc})$ such that

$$
(3.1) \qquad C^a_{dcb} = \overset{*}{C}{}^a_{dcb} + \delta^a_d C_{cb} + \delta^a_c D_{db} + \delta^a_b E_{dc}.
$$

These systems are given by

$$
\begin{aligned}
C_{cb} &= \frac{n(n^2-3)C^t_{tcb} - (n^2-2)C^t_{ctb} + nC^t_{cbt} - 2C^t_{tbc} + nC^t_{btc} - (n^2-2)C^t_{bct}}{(n^2-1)(n^2-4)}, \\
D_{cb} &= \frac{-(n^2-2)C^t_{tcb} + n(n^2-3)C^t_{ctb} - (n^2-2)C^t_{cbt} + nC^t_{tbc} - 2C^t_{btc} + nC^t_{bct}}{(n^2-1)(n^2-4)}, \\
E_{cb} &= \frac{nC^t_{tcb} - (n^2-2)C^t_{ctb} + n(n^2-3)C^t_{cbt} - (n^2-2)C^t_{tbc} + nC^t_{btc} - 2C^t_{bct}}{(n^2-1)(n^2-4)}.
\end{aligned}
$$

It is clear from (2.2) that

$$C_{cb} = \frac{1}{n-1} C^t_{tcb}, \quad D_{cb} = -\frac{1}{n-1} C^t_{tcb}, \quad E_{cb} = 0,$$

which together with (2.2) and (3.1) implies

$$C^a_{dcb} = \overset{*}{C}{}^a_{dcb} + \frac{2(n-4)}{n(n-1)}(\delta^a_d R_{cb} - \delta^a_c R_{db}) - \frac{2(n-4)s}{n^2(n-1)}(\delta^a_d g_{cb} - \delta^a_c g_{db}). \tag{3.2}$$

Inserting (3.2) back into (2.1), we have

$$\begin{aligned} \overset{*}{C}{}^a_{dcb} = R^a_{dcb} &+ \frac{1}{n}(R^a_d g_{cb} - R^a_c g_{db} + \delta^a_d R_{cb} - \delta^a_c R_{db} \\ &- S^a_d J_{cb} + S^a_c J_{db} - J^a_d S_{cb} + J^a_c S_{db} + 2S_{dc} J^a_b + 2J_{dc} S^a_b) \\ &+ \frac{(n+4)s}{n^2(n+2)}(J^a_d J_{cb} - J^a_c J_{db} - 2J_{dc} J^a_b) \\ &- \frac{(n^2+5n+12)s}{n^2(n-1)(n+2)}(\delta^a_d g_{cb} - \delta^a_c g_{db}) - \frac{2(n-4)}{n(n-1)}(\delta^a_d R_{cb} - \delta^a_c R_{db}). \end{aligned} \tag{3.3}$$

It is clear from (3.3) that

$$\|\overset{*}{C}\|^2 = \|R\|^2 - \frac{8(n^2-4n+12)}{n^2(n-1)}\|R_1\|^2 - \frac{8(n^2-4n-24)}{n^3(n-1)(n+2)} s^2. \tag{3.4}$$

Theorem 3.1. *The traceless component of the conformal curvature tensor on a Kähler manifold is invariant under concircular change, provided $n \geqslant 4$.*

P r o o f. We consider a conformal change of the Riemannian metrics g_{ba} and $'g_{ba}$ as follows:

$$'g_{ba} = e^{2\varrho} g_{ba}$$

for a smooth function ϱ. It is well known (cf. [6]) that the curvature tensors R and $'R$ corresponding to g and $'g$ are related by

$$'R^a_{dcb} = R^a_{dcb} + \varrho_{db}\delta^a_c - \varrho_{cb}\delta^a_d + g_{db}\varrho^a_c - g_{cb}\varrho^a_d,$$

where ϱ_a denotes the local components of the gradient vector of ϱ and

$$\varrho_{ba} = \nabla_b \varrho_a - \varrho_b \varrho_a + \frac{1}{2}\varrho_t \varrho^t g_{ba}, \quad \varrho^a = \varrho_t g^{at} \quad \varrho^a_b = \varrho_{bt} g^{ta}.$$

Hence we have

$$'R_{ba} = R_{ba} - (n-2)\varrho_{ba} - \varrho^t_t g_{ba}, \quad 'se^{2\varrho} = s - 2(n-1)\varrho^t_t, \tag{3.5}$$

where $'R_{ba}$ and $'s$ denote the Ricci tensor and the scalar curvature corresponding to $'g$, respectively, and $\varrho_t^t = \varrho_{ba} g^{ba}$.

On the other hand, it follows from (3.2) that the trace decomposition of the conformal curvature tensor $'C_{dcb}^a$ corresponding to $'g$ is given by

$$'C_{dcb}^a = '\overset{*}{C}{}_{dcb}^a + \frac{2(n-4)}{n(n-1)}(\delta_d^a {}'R_{cb} - \delta_c^a {}'R_{db}) - \frac{2(n-4)'s}{n^2(n-1)}(\delta_d^a {}'g_{cb} - \delta_c^a {}'g_{db}),$$

from which, using (3.5) and taking account of the fact that C_{dcb}^a is invariant under the conformal change, provided $n \geqslant 4$, we can easily obtain

$$\begin{aligned}\overset{*}{C}{}_{dcb}^a = {}'\overset{*}{C}{}_{dcb}^a &- \frac{2(n-2)(n-4)}{n(n-1)}(\delta_d^a \varrho_{cb} - \delta_c^a \varrho_{db}) \\ &+ \frac{2(n-2)(n-4)}{n^2(n-1)}\varrho_t^t(\delta_d^a g_{cb} - \delta_c^a g_{db}).\end{aligned} \tag{3.6}$$

Hence, if the conformal change is concircular, that is, if

$$\varrho_{ba} = \frac{1}{n}\varrho_t^t g_{ba},$$

then (3.6) yields $\overset{*}{C}{}_{dcb}^a = {}'\overset{*}{C}{}_{dcb}^a$, which means that $\overset{*}{C}{}_{dcb}^a$ is invariant under the concircular change. □

Next we prove

Theorem 3.2. *A Kähler manifold of real dimension $n \geqslant 4$ is of constant holomorphic sectional curvature if and only if the manifold is Einstein and the traceless component of the conformal curvature tensor vanishes everywhere.*

Proof. It is clear from (3.3) that $\overset{*}{C}{}_{dcb}^a = 0$ implies

$$\begin{aligned}R_{dcb}^a = &-\frac{1}{n}(R_d^a g_{cb} - R_c^a g_{db} + \delta_d^a R_{cb} - \delta_c^a R_{db} - S_d^a J_{cb} + S_c^a J_{db} \\ &- J_d^a S_{cb} + J_c^a S_{db} + 2S_{dc}J_b^a + 2J_{dc}S_b^a) \\ &- \frac{(n+4)s}{n^2(n+2)}(J_d^a J_{cb} - J_c^a J_{db} - 2J_{dc}J_b^a) \\ &+ \frac{(n^2+5n+12)s}{n^2(n-1)(n+2)}(\delta_d^a g_{cb} - \delta_c^a g_{db}) + \frac{2(n-4)}{n(n-1)}(\delta_d^a R_{cb} - \delta_c^a R_{db}).\end{aligned}$$

If the manifold is Einstein, we have $R_{cb} = \frac{s}{n} g_{cb}$ and $S_{cb} = \frac{s}{n} J_{cb}$. Inserting those equations back into the above euqation, we obtain

$$R_{dcb}^a = \frac{s}{n(n+2)}\{\delta_d^a g_{cb} - g_{db}\delta_c^a + J_{cb}J_d^a - J_{db}J_c^a - 2J_{dc}J_b^a\},$$

which means that the manifold is of constant holomorphic sectional curvature. Conversely, if the manifold is of constant holomorphic sectional curvature, then $R_{ba} = \frac{s}{n} g_{ba}$ and consequently $S_{ba} = \frac{s}{n} J_{ba}$. Substituting those equations into (3.3), we can see that $\overset{*}{C}{}^a_{dcb} = 0$. □

4. Kähler manifolds with $\overset{*}{C} = 0$

Let M be an n-dimensional Kähler manifold with Kähler structure (J, g). Then the the following relations hold on M:

$$\begin{aligned} R^a_{dct} J^t_b &= R^t_{dcb} J^a_t, \quad R^a_{tcb} J^t_d = -R^a_{dtb} J^t_c, \\ J^t_b R^a_t &= R^t_b J^a_t, \quad J^t_c R_{tb} = -R_{ct} J^t_b, \quad J^t_c J^s_b R_{ts} = R_{cb}, \\ \nabla_t R^t_{dcb} &= \nabla_d R_{cb} - \nabla_c R_{db}, \quad \nabla_b s = 2\nabla_t R^t_b. \end{aligned} \tag{4.1}$$

Thus the tensor S_{cb} satisfies

$$\begin{gathered} S_{cb} = -\frac{1}{2} J^{ts} R_{cbts} = J^{ts} R_{tcbs}, \quad J^t_c S_{tb} = -S_{ct} J^t_b = -R_{cb} \\ \nabla_a S_{cb} = J^t_c \nabla_a R_{tb}, \quad \nabla_t S^t_c = \frac{1}{2} J^t_c \nabla_t s, \quad J^t_d \nabla_t S_{cb} = J^t_d J^s_c \nabla_t R_{sb} \end{gathered} \tag{4.2}$$

(cf. [4], [11]).

On the other hand, since the differential form $S = \frac{1}{2} S_{cb}\, \mathrm{d}x^c \wedge \mathrm{d}x^b$ is closed (for details, see [10], p.72), it follows that

$$J^e_t \nabla_e S_{sa} = \nabla_a R_{st} - \nabla_s R_{at},$$

from which, transvecting with $J^t_c J^s_b$, we have

$$\nabla_c R_{ba} = J^t_c J^s_b (\nabla_a R_{st} - \nabla_s R_{at}). \tag{4.3}$$

Differentiating (3.3) covariantly and using (4.1) and (4.2), we can easily obtain

$$\begin{aligned} \nabla_e \overset{*}{C}{}^e_{dcb} = {} & \frac{n^3 - n^2 - 12n - 24}{2n^2(n+2)(n-1)} \{(\nabla_d s) g_{cb} - (\nabla_c s) g_{db}\} \\ & - \frac{(n^2 - 8)}{2n^2(n+2)} (\nabla_e s) \{J_{cb} J^e_d - J_{db} J^e_c - 2 J^e_b J_{dc}\} \\ & + \frac{n^2 - 5n + 10}{n(n-1)} \{\nabla_d R_{cb} - \nabla_c R_{db}\}. \end{aligned} \tag{4.4}$$

Thus we have

Theorem 4.1. *On a Kähler manifold of real dimension $n \geqslant 6$ with $\nabla \overset{*}{C} = 0$, the Ricci tensor R_{cb} is parallel, that is,*

$$\nabla_c R_{ba} = 0,$$

P r o o f. Transvecting $J_t^c J_s^b$ to (4.4) with $\nabla \overset{*}{C} = 0$ and taking account of (4.3), we have

$$0 = \frac{n^3 - n^2 - 12n - 24}{2n^2(n+2)(n-1)}\{(\nabla_d s) g_{ts} - (\nabla_a s) J_t^a J_{sd}\} - \frac{(n^2-8)}{2n^2(n+2)}(\nabla_a s)\{J_{ts} J_d^a + g_{ds}\delta_t^a + 2\delta_s^a g_{dt}\} + \frac{n^2 - 5n + 10}{n(n-1)}\nabla_s R_{td},$$

from which, transvecting with g^{ts} and using (4.1), we can easily obtain

$$\frac{(n-2)(n-4)}{n(n-1)}(\nabla_d s) = 0. \tag{4.5}$$

Combining (4.4) with $\nabla \overset{*}{C} = 0$ and (4.5) give

$$\nabla_d R_{cb} - \nabla_c R_{db} = 0,$$

provided $n \geqslant 6$, which together with (4.3) implies our result. □

Theorem 4.2. *A Kähler manifold of real dimension $n \geqslant 6$ is of constant holomorphic sectional curvature if and only if the traceless component of the conformal curvature tensor vanishes everywhere.*

P r o o f. Taking the symmetric part of the tensor $\overset{*}{C}_{dcba} = \overset{*}{C}{}_{dcb}^{e} g_{ea}$ with respect to the indices b and a, we can obtain

$$\overset{*}{C}_{dcba} + \overset{*}{C}_{dcab} = \frac{2(n-4)}{n(n-1)}(g_{da}R_{cb} + g_{db}R_{ca} - g_{ca}R_{db} - g_{cb}R_{da})$$

because of $R_{dcba} = R_{dcb}^{e} g_{ea} = -R_{dcab}$. Hence $\overset{*}{C}{}_{dcb}^{a} = 0$ implies

$$\frac{n-4}{n(n-1)}(g_{da}R_{cb} + g_{db}R_{ca} - g_{ca}R_{db} - g_{cb}R_{da}) = 0,$$

and consequently

$$\frac{n-4}{n(n-1)}(nR_{cb} - s g_{cb}) = 0,$$

which yields our assertion. □

5. Spectrum of the Laplacian and traceless component of the conformal curvature tensor

Let M be a compact Kähler manifold of real dimension n and denote by Δ the Laplacian acting on p-forms on M, $0 \leqslant p \leqslant n$. Then we have the spectrum for each p:

$$\mathrm{Spec}^p(M,g) = \{0 \leqslant \lambda_{0,p} \leqslant \lambda_{1,p} \leqslant \lambda_{2,p} \leqslant \ldots \uparrow +\infty\},$$

where each eigenvalue $\lambda_{\alpha,p}$ is repeated as many as times as its multiplicity indicates. Furthermore, the Minakshisundaram-Pleijel-Gaffney's formula for $\mathrm{Spec}^p(M,g)$ is given by

$$\sum_{\alpha=0}^{\infty} \exp(-\lambda_{\alpha,p} t) \sim (4\pi t)^{-\frac{1}{2}n} \sum_{\alpha=0}^{\infty} a_{\alpha,p} t^{\alpha} \quad \text{as} \quad t \to 0^+,$$

where the constants $A_{\alpha,p}$ are spectral invariants. In particular, for $p = 0$, we have

$$a_{0,0} = \int_M \mathrm{d}M = \mathrm{Vol}(M,g), \tag{5.1}$$

$$a_{1,0} = \frac{1}{6} \int_M s \, \mathrm{d}M, \tag{5.2}$$

$$a_{2,0} = \frac{1}{360} \int_M \{2\|R\|^2 - 2\|R_1\|^2 + 5s^2\} \, \mathrm{d}M, \tag{5.3}$$

where $\mathrm{d}M$ denotes the natural volume element of (M,g) (cf. [1]). For $p = 1$, we have

$$a_{0,1} = n\mathrm{Vol}(M,g), \tag{5.4}$$

$$a_{1,1} = \frac{n-6}{6} \int_M s \, \mathrm{d}M, \tag{5.5}$$

$$a_{2,1} = \frac{1}{360} \int_M \{2(n-15)\|R\|^2 - 2(n-90)\|R_1\|^2 + 5(n-12)s^2\} \, \mathrm{d}M \tag{5.6}$$

(cf. [7]). For $p = 2$, we have

$$a_{0,2} = \frac{n(n-1)}{2} \mathrm{Vol}(M,g), \tag{5.7}$$

$$a_{1,2} = \frac{n^2 - 13n + 24}{12} \int_M s \, \mathrm{d}M, \tag{5.8}$$

$$\begin{aligned} a_{2,2} = \frac{1}{720} \int_M \{&2(n^2 - 31n + 240)\|R\|^2 \\ &- 2(n^2 - 181n + 1080)\|R_1\|^2 + 5(n^2 - 25n + 120)s^2\} \, \mathrm{d}M \end{aligned} \tag{5.9}$$

(cf. [5], [8], [9]).

We next recall the following lemma provided by Tanno([7]) for later use.

Lemma 5.1 ([7]). *Let (M,g) and (M',g') be compact orientable Riemannian manifolds with* $\mathrm{Vol}(M,g)=\mathrm{Vol}(M',g')$ *and* $\int_M s\,\mathrm{d}M=\int_{M'} s'\,\mathrm{d}M'$. *If* $s'=\text{constant}$, *then* $\int_M s^2\,\mathrm{d}M \geqslant \int_{M'} s'^2\,\mathrm{d}M'$ *with equality if and only if* $s=\text{constant}=s'$.

A straightforward computation using (3.4) yields

$$a_{2,0}=\frac{1}{180}\int_M\{\|\overset{*}{C}\|^2-b_0(n)\|Q\|^2\}\,\mathrm{d}M+\frac{c_0(n)}{360}\int s^2\,\mathrm{d}M, \tag{5.10}$$

where Q is a tensor of type $(0,2)$ defined by $Q=R_1-\frac{s}{n}g$, and

$$b_0(n)=\frac{(n^3-9n^2+32n-96)}{n^2(n-1)}<0 \quad \text{for} \quad n=2,4,6;$$
$$c_0(n)=\frac{5n^2+8n+12}{n(n+2)}>0.$$

Thus we have

Theorem 5.2. *Let M and M' be compact Kähler manifolds. Assume that* $\mathrm{Spec}^0 M=\mathrm{Spec}^0 M'$. *Then* $\dim M=\dim M'=n$, *and*

(a) *for $n=4,6$, M is of constant holomorphic sectional curvature if and only if M' is, and* $s'=\text{constant}=s$;

(b) *when M and M' are Einstein and $n\geqslant 4$, M is of constant holomorphic sectional curvature if and only if M' is, and $s'=s$.*

Proof. Our assumption $\mathrm{Spec}^0 M=\mathrm{Spec}^0 M'$ implies $a_{0,0}=a'_{0,0}$ and $a_{1,0}=a'_{1,0}$. Hence (5.1) and (5.2) yield

$$\mathrm{Vol}(M)=\mathrm{Vol}(M'),\qquad \int_M s\,\mathrm{d}M=\int_{M'} s'\,\mathrm{d}M'. \tag{5.11}$$

Moreover, since $a_{2,0}=a'_{2,0}$, it follows from (5.10) that

$$\begin{aligned}\int_M\{\|\overset{*}{C}\|^2-b_0(n)\|Q\|^2\}\,\mathrm{d}M&+\frac{c_0(n)}{2}\int s^2\,\mathrm{d}M\\ &=\int_{M'}\{\|\overset{*}{C}{}'\|^2-b_0(n)\|Q'\|^2\}\,\mathrm{d}M+\frac{c_0(n)}{2}\int s'^2\,\mathrm{d}M'.\end{aligned} \tag{5.12}$$

(a) For $n=4,6$, if M' is of constant holomorphic sectional curvature, then $\overset{*}{C}{}'=0$ and $Q'=0$ and consequently (5.12) gives

$$\int_M\{\|\overset{*}{C}\|^2-b_0(n)\|Q\|^2\}\,\mathrm{d}M+\frac{c_0(n)}{2}\Big(\int s^2\,\mathrm{d}M-\int s'^2\,\mathrm{d}M'\Big)=0.$$

Since $s' =$ constant, Lemma 5.1 implies $\int s^2 \,\mathrm{d}M \geqslant \int s'^2 \,\mathrm{d}M'$ and consequently $\overset{*}{C} = 0$ and $Q = 0$. By means of Theorem 3.2 M is of constant holomorphic sectional curvature.

(b) If $Q = Q' = 0$, then s and s' are both constants for $n \geqslant 4$. Thus (5.11) gives $s = s'$, which together with (5.12) implies

$$\int_M \|\overset{*}{C}\|^2 \,\mathrm{d}M = \int_{M'} \|\overset{*}{C}'\|^2 \,\mathrm{d}M'.$$

Hence we have our assertions. □

We next consider the case of $p = 1$. In this case it follows from (3.4) and (5.6) that

$$a_{2,1} = \frac{1}{360} \int_M [2(n-15)\|\overset{*}{C}\|^2 - 2b_1(n)\|Q\|^2 + c_1(n)s^2] \,\mathrm{d}M, \tag{5.13}$$

where

$$b_1(n) = \frac{n^4 - 99n^3 + 242n^2 - 576n + 1440}{n^2(n-1)} < 0 \quad \text{for } 2 < n < 97;$$
$$c_1(n) = \frac{5n^3 - 5n^2 + 72n + 120}{n(n+2)} > 0 \quad \text{for} \quad n = 2 \text{ or } 10 \leqslant n.$$

Thus we have

Theorem 5.3. *Let M and M' be compact Kähler manifolds. Assume that* $\mathrm{Spec}^1 M = \mathrm{Spec}^1 M'$. *Then* $\dim M = \dim M' = n$, *and*

(a) *for* $16 \leqslant n \leqslant 96$, *$M$ is of constant holomorphic sectional curvature if and only if M' is, and* $s' =$ constant $= s$;

(b) *when M and M' are Einstein, and $n \geqslant 4$ and $n \neq 6$, M is of constant holomorphic sectional curvature if and only if M' is, and $s' = s$.*

Proof. Our assumption $\mathrm{Spec}^1 M = \mathrm{Spec}^1 M'$ implies $a_{0,1} = a'_{0,1}$ and $a_{1,1} = a'_{1,1}$. Hence (5.4) and (5.5) yield

$$\mathrm{Vol}(M) = \mathrm{Vol}(M'), \quad \int_M s \,\mathrm{d}M = \int_{M'} s' \,\mathrm{d}M', \tag{5.14}$$

provided $n \neq 6$. Moreover, since $a_{2,1} = a'_{2,1}$, it follows from (5.13) that

$$\begin{aligned} &\int_M \{(n-15)\|\overset{*}{C}\|^2 - b_1(n)\|Q\|^2\} \,\mathrm{d}M + \frac{c_1(n)}{2} \int s^2 \,\mathrm{d}M \\ &\quad = \int_{M'} \{(n-15)\|\overset{*}{C}'\|^2 - b_1(n)\|Q'\|^2\} \,\mathrm{d}M + \frac{c_1(n)}{2} \int s'^2 \,\mathrm{d}M'. \end{aligned} \tag{5.15}$$

(a) For $16 \leqslant n \leqslant 96$, if M' is of constant holomorphic sectional curvature, then $\overset{*}{C}{}' = 0$ and $Q' = 0$ and consequently (5.15) gives

$$\int_M \{(n-15)\|\overset{*}{C}\|^2 - b_1(n)\|Q\|^2\}\,\mathrm{d}M + \frac{c_1(n)}{2}\Big(\int s^2\,\mathrm{d}M - \int s'^2\,\mathrm{d}M'\Big) = 0.$$

Since $s' = $ constant, Lemma 5.1 implies $\int s^2\,\mathrm{d}M \geqslant \int s'^2\,\mathrm{d}M'$ and consequently $\overset{*}{C} = 0$ and $Q = 0$. By means of Theorem 3.2 M is of constant holomorphic sectional curvature.

(b) When $Q = Q' = 0$ and $n \geqslant 4$, s and s' are both constants. Thus (5.14) gives $s = s'$, which together with (5.15) implies

$$\int_M \|\overset{*}{C}\|^2\,\mathrm{d}M = \int_{M'} \|\overset{*}{C}{}'\|^2\,\mathrm{d}M'.$$

Hence we have our assertions. □

Theorem 5.4. *Let M and M' be compact Kähler manifolds. Assume that* $\mathrm{Spec}^0 M = \mathrm{Spec}^0 M'$ *and* $\mathrm{Spec}^1 M = \mathrm{Spec}^1 M'$. *Then* $\dim M = \dim M' = n$, *and*

(a) *for $n \geqslant 4$, M is of constant holomorphic sectional curvature if and only if M' is, and $s' = $ constant $= s$;*

(b) *for $n \geqslant 4$, M is Einstein if and only if M' is, and $s' = s$.*

Proof. Our assumption $\mathrm{Spec}^0 M = \mathrm{Spec}^0 M'$ implies $a_{0,0} = a'_{0,0}$ and $a_{1,0} = a'_{1,0}$. Hence (5.1) and (5.2) yield

$$\mathrm{Vol}(M) = \mathrm{Vol}(M'), \qquad \int_M s\,\mathrm{d}M = \int_{M'} s'\,\mathrm{d}M'. \tag{5.16}$$

Moreover, the assumptions $\mathrm{Spec}^0 M = \mathrm{Spec}^0 M'$ and $\mathrm{Spec}^1 M = \mathrm{Spec}^1 M'$ give $a_{2,0} = a'_{2,0}$ and $a_{2,1} = a'_{2,1}$, from which together with (5.3) and (5.6), we have

$$\int_M (5\|R\|^2 + 13s^2)\,\mathrm{d}M = \int_{M'} (5\|R'\|^2 + 13s'^2)\,\mathrm{d}M, \tag{5.17}$$

$$\int_M (10\|R_1\|^2 + s^2)\,\mathrm{d}M = \int_{M'} (10\|R_1'\|^2 + s'^2)\,\mathrm{d}M. \tag{5.18}$$

(a) It follows from (3.4) and (5.17) that

$$\begin{aligned}\int_M &\Big\{5\|\overset{*}{C}\|^2 + \frac{40(n^2-4n+12)}{n^2(n-1)}\|R_1\|^2 + c_{0,1}s^2\Big\}\,\mathrm{d}M\\ &= \int_{M'} \Big\{5\|\overset{*}{C}{}'\|^2 + \frac{40(n^2-4n+12)}{n^2(n-1)}\|R_1'\|^2 + c_{0,1}s'^2\Big\}\,\mathrm{d}M'\end{aligned}$$

where

$$c_{0,1} = \frac{13n^4 + 13n^3 + 14n^2 - 160n + 480}{n^2(n-1)(n+2)}.$$

Taking account of (5.18), the above equation reduces to

$$(5.19) \qquad \int_M 5\|\overset{*}{C}\|^2 \,\mathrm{d}M - \int_{M'} 5\|\overset{*}{C}{}'\|^2 \,\mathrm{d}M' + d_{0,1}\Big(\int_M s^2 \,\mathrm{d}M - \int_{M'} s'^2 \,\mathrm{d}M'\Big) = 0,$$

where

$$d_{0,1} = \frac{13n^4 + 9n^3 + 22n^2 - 176n + 384}{n^2(n-1)(n+2)},$$

which is positive for $n \geqslant 2$. On the other hand, since $\|Q\|^2 = \|R_1\|^2 - \frac{s^2}{n}$, (5.18) reduces to

$$(5.20) \qquad \int_M \|Q\|^2 \,\mathrm{d}M - \int_{M'} \|Q'\|^2 \,\mathrm{d}M' + \frac{n+10}{10n}\Big(\int_M s^2 \,\mathrm{d}M - \int_{M'} s'^2 \,\mathrm{d}M'\Big) = 0.$$

Thus, if M' is of constant holomorphic sectional curvature, then it follows from Theorem 3.2 that $\overset{*}{C}{}' = 0$ and $Q' = 0$. Hence s' is constant for $n \geqslant 4$, and consequently Lemma 5.1, (4.16), (4.19) and (4.20) lead to $\overset{*}{C} = 0$, $Q = 0$ and $s = \text{constant} = s'$. Therefore M is also of constant holomorphic sectional curvature.

(b) is easily obtained from (5.20). □

Finally we consider the case of $p = 2$. In this case it follows from (3.4) and (5.9) that

$$(5.21) \qquad a_{2,2} = \frac{1}{720}\int_M \{2(n-15)(n-16)\|\overset{*}{C}\|^2 - 2b_2(n)\|Q\|^2 + c_2(n)s^2\}\,\mathrm{d}M$$

where

$$b_2(n) = \frac{1}{n^2(n-1)}(n^5 - 190n^4 + 1541n^3 - 4088n^2 + 10656n - 23040) < 0 \quad \text{for } n = 2 \quad \text{or} \quad 6 \leqslant n \leqslant 180;$$

$$c_2(n) = \frac{1}{n(n+2)}(5n^4 - 117n^3 + 724n^2 - 732n - 480) > 0 \quad \text{for } 2 \leqslant n \leqslant 8 \quad \text{or} \quad n \geqslant 14.$$

Thus we have

Theorem 5.5. *Let M and M' be compact Kähler manifolds. Assume that* $\mathrm{Spec}^2 M = \mathrm{Spec}^2 M'$. *Then* $\dim M = \dim M' = n$, *and*

(a) *for $n = 6, 8, 14$ or $18 \leqslant n \leqslant 180$, M is of constant holomorphic sectional curvature if and only if M' is, and $s' = \text{constant} = s$;*

(b) *for $n = 16$, M is Einstein if and only if M' is, and $s' = s$;*

(c) *when M and M' are Einstein and $n \geqslant 4$ and $n \neq 16$, M is of constant holomorphic sectional curvature if and only if M' is, and $s' = s$.*

Proof. Our assumption $\mathrm{Spec}^2 M = \mathrm{Spec}^2 M'$ implies $a_{0,2} = a'_{0,2}$ and $a_{1,2} = a'_{1,2}$. Hence (5.7) and (5.8) yield

$$\mathrm{Vol}(M) = \mathrm{Vol}(M'), \quad \int_M s\,\mathrm{d}M = \int_{M'} s'\,\mathrm{d}M'. \tag{5.22}$$

Moreover, since $a_{2,2} = a'_{2,2}$, it follows from (5.21) that

$$\begin{aligned} &\int_M \{(n-15)(n-16)\|\overset{*}{C}\|^2 - b_2(n)\|Q\|^2\}\,\mathrm{d}M + \frac{c_2(n)}{2}\int s^2\,\mathrm{d}M \\ &= \int_{M'} \{(n-15)(n-16)\|\overset{*}{C}{}'\|^2 - b_2(n)\|Q'\|^2\}\,\mathrm{d}M + \frac{c_2(n)}{2}\int s'^2\,\mathrm{d}M'. \end{aligned} \tag{5.23}$$

(a) For $n = 6, 8, 14$ or $18 \leqslant n \leqslant 180$, if M' is of constant holomorphic sectional curvature, then $\overset{*}{C}{}' = 0$ and $Q' = 0$ and consequently (5.23) gives

$$\int_M \{(n-15)(n-16)\|\overset{*}{C}\|^2 - b_2(n)\|Q\|^2\}\,\mathrm{d}M + \frac{c_2(n)}{2}\Big(\int s^2\,\mathrm{d}M - \int s'^2\,\mathrm{d}M'\Big) = 0.$$

Since $s' = \text{constant}$, Lemma 5.1 implies $\int s^2\,\mathrm{d}M \geqslant \int s'^2\,\mathrm{d}M'$ and consequently $\overset{*}{C} = 0$ and $Q = 0$. By means of Theorem 3.2 M is of constant holomorphic sectional curvature.

(b) is trivial.

(c) When $Q = Q' = 0$ and $n \geqslant 4$, s and s' are both constants. Thus (5.22) gives $s = s'$, which together with (5.23) implies

$$\int_M \|\overset{*}{C}\|^2\,\mathrm{d}M = \int_{M'} \|\overset{*}{C}{}'\|^2\,\mathrm{d}M',$$

provided $n \neq 16$. Hence we have our assertions. $\square$

Theorem 5.6. *Let M and M' be compact Kähler manifolds. Assume that* $\mathrm{Spec}^0 M = \mathrm{Spec}^0 M'$ *and* $\mathrm{Spec}^2 M = \mathrm{Spec}^2 M'$. *Then* $\dim M = \dim M' = n$, *and*

(a) *for $n \geqslant 4$, M is of constant holomorphic sectional curvature if and only if M' is, and $s' = \text{constant} = s$.*

(b) *for $n = 4$ or $n \geqslant 14$, M is Einstein if and only if M' is, and $s' = s$.*

Proof. Our assumption $\mathrm{Spec}^0 M = \mathrm{Spec}^0 M'$ yields $a_{0,0} = a'_{0,0}$ and $a_{1,0} = a'_{1,0}$. Hence it follows from (5.1) and (5.2) that

$$\mathrm{Vol}(M) = \mathrm{Vol}(M'), \quad \int_M s\,\mathrm{d}M = \int_{M'} s'\,\mathrm{d}M'. \tag{5.24}$$

Moreover, the assumptions $\mathrm{Spec}^0 M = \mathrm{Spec}^0 M'$ and $\mathrm{Spec}^2 M = \mathrm{Spec}^2 M'$ give $a_{2,0} = a'_{2,0}$ and $a_{2,2} = a'_{2,2}$, from which together with (5.3) and (5.6), we have

$$\begin{aligned} &\int_M \{(5n-28)\|R\|^2 + (13n-80)s^2\}\,\mathrm{d}M \\ &\qquad = \int_{M'} \{(5n-28)\|R'\|^2 + (13n-80){s'}^2\}\,\mathrm{d}M, \end{aligned} \tag{5.25}$$

$$\begin{aligned} &\int_M \{2(5n-28)\|R_1\|^2 + (n-20)s^2\}\,\mathrm{d}M \\ &\qquad = \int_{M'} 2(5n-28)\|R'_1\|^2 + (n-20){s'}^2\}\,\mathrm{d}M. \end{aligned} \tag{5.26}$$

(a) It follows from (3.4) and (5.25) that

$$\begin{aligned} &\int_M \Big\{(5n-28)\|\overset{*}{C}\|^2 + \frac{8(5n-28)(n^2-4n+12)}{n^2(n-1)}\|R_1\|^2 + c_{0,2}s^2\Big\}\,\mathrm{d}M \\ &\quad = \int_{M'} \Big\{(5n-28)\|\overset{*}{C}{}'\|^2 + \frac{8(5n-28)(n^2-4n+12)}{n^2(n-1)}\|R'_1\|^2 + c_{0,2}{s'}^2\Big\}\,\mathrm{d}M' \end{aligned}$$

where

$$c_{0,2} = \frac{13n^5 - 67n^4 - 66n^3 - 224n^2 - 64n + 5376}{n^2(n-1)(n+2)}.$$

Taking account of (5.26), the above equation reduces to

$$\begin{aligned} &\int_M (5n-28)\|\overset{*}{C}\|^2\,\mathrm{d}M - \int_{M'} (5n-28)\|\overset{*}{C}{}'\|^2\,\mathrm{d}M' \\ &\qquad + d_{0,2}\Big(\int_M s^2\,\mathrm{d}M - \int_{M'} {s'}^2\,\mathrm{d}M'\Big) = 0, \end{aligned} \tag{5.27}$$

where

$$d_{0,2} = \frac{13n^5 - 71n^4 + 22n^3 - 400n^2 + 160n + 7296}{n^2(n-1)(n+2)},$$

which is positive for $n \geqslant 6$ and negative for $n = 4$. On the other hand, since $\|Q\|^2 = \|R_1\|^2 - s^2/n$, (5.26) reduces to

$$
\begin{aligned}
(5.28) \qquad & \int_M 2(5n-28)\|Q\|^2 \,\mathrm{d}M - \int_{M'} 2(5n-28)\|Q'\|^2 \,\mathrm{d}M' \\
& + \frac{(n-14)(n+4)}{n}\Big(\int_M s^2 \,\mathrm{d}M - \int_{M'} s'^2 \,\mathrm{d}M'\Big) = 0.
\end{aligned}
$$

We first consider the case of $n = 4$. In this case, if M' is of constant holomorphic sectional curvature, then it follows from Theorem 3.2 that $\overset{*}{C}{}' = 0$ and $Q' = 0$. Hence s' is constant for $n = 4$, and consequently Lemma 5.1, (5.24), (5.27) and (5.28) imply that $\overset{*}{C} = 0, Q = 0$ and $s = \text{constant} = s'$. Therefore M is also of constant holomorphic sectional curvature. In the case of $6 \leqslant n \leqslant 12$, we fail to derive our assertion in this way, and so we have to find another method. In fact, using (2.3), we can rewrite (5.25) in the form

$$
\begin{aligned}
& \int_M \Big\{(5n-28)\|C\|^2 + \frac{32(5n-28)}{n^2}\|R_1\|^2 + e_{0,2}s^2\Big\} \,\mathrm{d}M \\
& = \int_M \Big\{(5n-28)\|C'\|^2 + \frac{32(5n-28)}{n^2}\|R_1'\|^2 + e_{0,2}s'^2\Big\} \,\mathrm{d}M',
\end{aligned}
$$

where

$$
e_{0,2} = \frac{13n^5 - 54n^4 - 120n^3 - 384n^2 + 576n + 1792}{2n^3(n+2)}.
$$

This equality together with (5.26) implies

$$
\begin{aligned}
(5.29) \qquad & \int_M (5n-28)\|C\|^2 \,\mathrm{d}M - \int_{M'} (5n-28)\|C'\|^2 \,\mathrm{d}M' \\
& + f_{0,2}\Big(\int_M s^2 \,\mathrm{d}M - \int_{M'} s'^2\Big) \,\mathrm{d}M',
\end{aligned}
$$

where

$$
f_{0,2} = \frac{13n^5 - 54n^4 - 136n^3 - 96n^2 + 1216n + 1792}{2n^3(n+2)},
$$

which is positive for $n \geqslant 6$. Hence, in the case of $n \geqslant 6$, Lemma 5.1 and (5.29) imply $C = 0$, which yields our assertion as already mentioned in Section 2.

(b) is easily obtained from (5.28). □

Theorem 5.7. *Let M and M' be compact Kähler manifolds. Assume that* $\mathrm{Spec}^1 M = \mathrm{Spec}^1 M'$ *and* $\mathrm{Spec}^2 M = \mathrm{Spec}^2 M'$. *Then* $\dim M = \dim M' = n$, *and*

(a) *for* $4 \leqslant n \leqslant 14$ *or* $n \geqslant 24$, *M is of constant holomorphic sectional curvature if and only if M' is, and* $s' = \text{constant} = s$.

(b) *for* $n \geqslant 4$, *M is Einstein if and only if M' is. Moreover, in this case* $s' = s$, *provided* $n = 4$ *or* $n \geqslant 10$.

Proof. Our assumption $\mathrm{Spec}^1 M = \mathrm{Spec}^1 M'$ yields $a_{0,1} = a'_{0,1}$ and consequently it follows from (5.4) that $\mathrm{Vol}(M) = \mathrm{Vol}(M')$. Since $\mathrm{Spec}^2 M = \mathrm{Spec}^2 M'$, $a_{1,2} = a'_{1,2}$ yields $\int_M s \, \mathrm{d}M = \int_{M'} s' \, \mathrm{d}M'$. Summing up, we have

$$\mathrm{Vol}(M) = \mathrm{Vol}(M'), \quad \int_M s \, \mathrm{d}M = \int_{M'} s' \, \mathrm{d}M'. \tag{5.30}$$

Moreover, the assumptions $\mathrm{Spec}^1 M = \mathrm{Spec}^1 M'$ and $\mathrm{Spec}^2 M = \mathrm{Spec}^2 M'$ give $a_{2,1} = a'_{2,1}$ and $a_{2,2} = a'_{2,2}$, from which together with (5.6) and (5.9), we have

$$\begin{aligned} &\int_M \{(5n^2 - 51n - 360)\|R\|^2 + (13n^2 - 147n + 360)s^2\} \, \mathrm{d}M \\ &\quad = \int_{M'} \{(5n^2 - 51n - 360)\|R'\|^2 + (13n^2 - 147n + 360){s'}^2\} \, \mathrm{d}M, \end{aligned} \tag{5.31}$$

$$\begin{aligned} &\int_M \{2(5n + 24)\|R_1\|^2 + (n - 24)s^2\} \, \mathrm{d}M \\ &\quad = \int_{M'} 2(5n + 24)\|R'_1\|^2 + (n - 24){s'}^2\} \, \mathrm{d}M. \end{aligned} \tag{5.32}$$

(a) It follows from (3.4) and (5.31) that

$$\begin{aligned} &\int_M \Big\{(5n^2 - 51n - 360)\|\overset{*}{C}\|^2 \\ &\qquad + \frac{8(5n^2 - 51n - 360)(n^2 - 4n + 12)}{n^2(n-1)}\|R_1\|^2 + c_{1,2}s^2\Big\} \, \mathrm{d}M \\ &\quad = \int_{M'} \Big\{(5n^2 - 51n - 360)\|\overset{*}{C}{}'\|^2 \\ &\qquad + \frac{8(5n^2 - 51n - 360)(n^2 - 4n + 12)}{n^2(n-1)}\|R'_1\|^2 + c_{1,2}{s'}^2\Big\} \, \mathrm{d}M' \end{aligned}$$

where

$$c_{1,2} = \frac{13n^6 - 134n^5 + 227n^4 + 86n^3 - 2928n^2 + 21312n + 69120}{n^2(n-1)(n+2)}.$$

Taking account of (5.32), the above equation reduces to

$$
\begin{aligned}
&2(5n^2-51n+360)n^2(n-1)(5n+24)\Big\{\int_M \|\overset{*}{C}\|^2\,\mathrm{d}M-\int_{M'}\|\overset{*}{C}{}'\|^2\,\mathrm{d}M'\Big\}\\
&\qquad + d_{1,2}\Big(\int_M s^2\,\mathrm{d}M-\int_{M'} s'^2\,\mathrm{d}M'\Big)=0,
\end{aligned}
\tag{5.33}
$$

where

$$
d_{1,2} = \frac{40n^7-1341n^6+5598n^5+78541n^4+55210n^3-135312n^2-21312n-69120}{n^2(n-1)(n+2)},
$$

which is positive for $4\leqslant n\leqslant 14$ or $n\geqslant 24$. On the other hand, since $\|Q\|^2=\|R_1\|^2-\frac{s^2}{n}$, (5.32) reduces to

$$
\begin{aligned}
&\int_M \{2(5n+24)\|Q\|^2\,\mathrm{d}M-\int_{M'}2(5n+24)\|Q'\|^2\,\mathrm{d}M'\\
&\qquad+\frac{(n-6)(n-8)}{n}\Big(\int_M s^2\,\mathrm{d}M-\int_{M'}s'^2\,\mathrm{d}M'\Big)=0.
\end{aligned}
\tag{5.34}
$$

Hence, from (5.33) and (5.34) we complete our assertion.

(b) is easily obtained from Lemma 5.1 and (5.34). □

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