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TRAVEL GROUPOIDS

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Abstract. In this paper, by a *travel groupoid* is meant an ordered pair $(V, *)$ such that V is a nonempty set and $*$ is a binary operation on V satisfying the following two conditions for all $u, v \in V$:

$$(u * v) * u = u;$$
$$\text{if } (u * v) * v = u, \text{ then } u = v.$$

Let $(V, *)$ be a travel groupoid. It is easy to show that if $x, y \in V$, then $x * y = y$ if and only if $y * x = x$. We say that $(V, *)$ is on a (finite or infinite) graph G if $V(G) = V$ and

$$E(G) = \{\{u, v\} \colon u, v \in V \text{ and } u \neq u * v = v\}.$$

Clearly, every travel groupoid is on exactly one graph. In this paper, some properties of travel groupoids on graphs are studied.

Keywords: travel groupoid, graph, path, geodetic graph

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By a graph we mean here a (finite or infinite) undirected graph with no multiple edges or loops. We will use the terminology of the book [1] but we extend it also to infinite graphs here. By a geodetic graph we mean a connected graph G such that there exists exactly one shortest $u - v$ path in G for all $u, v \in V(G)$.

The letters $h - n$ will serve for denoting non-negative integers.

1. Travel groupoids

By a *travel groupoid* we will mean an ordered pair $(V, *)$ such that V is a nonempty set and $*$ is a binary operation on V satisfiyng the following axioms (t1) and (t2):

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(t1) $(u * v) * u = u$ (for all $u, v \in V$);

(t2) if $(u * v) * v = u$, then $u = v$ (for all $u, v \in V$).

We say that a travel groupoid $(V, *)$ is *finite* if V is finite. If $(V, *)$ is a travel groupoid, then we say that $*$ is a travel operation on V. Special kinds of travel groupoids (or travel operations) were introduced in [2] and [3].

The idea of the proof of the next proposition can be found in the proof of Lemma 2 in [3].

Proposition 1. *If $(V, *)$ is a travel groupoid, then $x * x = x$ (for each $x \in V$).*

Proof. Let $x \in V$. As follows from (t1), $(x * x) * x = x$. This implies that $((x * x) * x) * x = x * x$. By (t2), $x * x = x$, which completes the proof. □

Proposition 2. *Let $(V, *)$ be a travel groupoid. Then*

$$(1) \qquad x * y = y \quad \text{if and only if } y * x = x \quad (\text{for all } x, y \in V),$$

$$(2) \qquad x * y = x \quad \text{if and only if } x = y \quad (\text{for all } x, y \in V)$$

and

$$(3) \qquad x * (x * y) = x * y \quad (\text{for all } x, y \in V).$$

Proof. Clearly, (1) follows from (t1) and (2) follows from Proposition 1 and (t2).

Consider arbitrary $x, y \in V$. By (t1), $(x * y) * x = x$. As follows from (1), $x * (x * y) = x * y$. Thus (3) holds. □

Let $(V, *)$ be a travel groupoid, and let G be a graph. We say that $(V, *)$ is *on* G or that G *has* $(V, *)$ if $V(G) = V$ and

$$E(G) = \{\{u, v\}\colon\ u, v \in V \text{ and } u \neq u * v = v\}.$$

As follows from Proposition 2, every travel groupoid is on exactly one graph.

Proposition 3. *Let $(V, *)$ be a travel groupoid on a graph G, let $u, v \in V$ and $u \neq v$. Then u and $u * v$ are adjacent vertices of G.*

Proof. The proposition follows from (1) and (3). □

Let $(V,*)$ be a travel groupoid. If $u,v \in V$, then we define

$$u *^0 v = u \tag{4}$$

and

$$u *^{i+1} v = (u *^i v) * v \text{ for every } i \geqslant 0.$$

It is clear that if $j,k \geqslant 0$, then $(u *^j v) *^k v = u *^{j+k} v$.

Proposition 4. *Let $(V,*)$ be a travel groupoid, let $u,v \in V$, and let $k \geqslant 1$. If $u *^k v \neq v$, then $u *^{k-1} v \neq v$ and the elements*

$$u *^{k-1} v, \quad u *^k v, \quad \textit{and} \quad u *^{k+1} v$$

are pairwise distinct.

Proof. Let $u *^k v \neq v$. If $u *^{k-1} v = v$, then Proposition 1 implies that $u *^k v = v$; a contradiction. Thus $u *^{k-1} v \neq v$. Recall that $u *^k v = (u *^{k-1} v) * v$. If $u^k v = u *^{k-1} v$, then, by virtue of (2), $u *^{k-1} v = v$; a contradiction. Thus $u *^k v \neq u *^{k-1} v$. If $u *^k v = u *^{k+1} v$, then it follows from (2) that $u *^k v = v$; a contradiction. Thus $u *^k v \neq u *^{k+1} v$. If $u *^{k+1} v = u *^{k-1} v$, then, as follows from (t2), $u *^{k-1} v = v$; a contradiction. Thus $u *^{k-1} \neq u *^{k+1} v$, which completes the proof. □

Remark 1. Let $(V,*)$ be a travel groupoid, and let $u,v \in V$. If there exists $i \geqslant 0$ such that $u *^i v = v$, then, by virtue of (2), $u *^{i+1} v = v$. This implies that there exists at most one $k \geqslant 1$ such that $u *^{k-1} v \neq v$ and $u *^k v = v$.

The following theorem motivates the terms "travel groupoids" and "travel operations".

Theorem 1. *Let $(V,*)$ be a travel groupoid on a graph G, let $u,v \in V$, and let $k \geqslant 1$. Assume that $u *^{k-1} v \neq v$. Then the sequence*

$$u *^0 v, \ldots, u *^{k-1} v, u *^k v \tag{5}$$

is a walk in G. Moreover, if $u *^k v = v$, then the sequence (5) is an $u - v$ path in G.

Proof. Since $u *^{k-1} \neq v$, it follows from Proposition 4 that $u *^h v \neq v$ for each h, $0 \leqslant h \leqslant k-1$. By the definition, $u *^{h+1} v = (u *^h v) * v$ for each h, $0 \leqslant h \leqslant k-1$. Thus, by virtue of Proposition 3, the sequence (5) is a walk in G.

Let $u *^k v = v$. Assume that there exist i and j, $1 \leqslant i < j \leqslant k$, such that $u *^i v = u *^j v$. By virtue of Proposition 4, $j < k$. Thus $v = u *^k v = (u *^j v) *^{k-j} v = (u *^i v) *^{k-j} v = u *^{k-(j-i)} v \neq v$; a contradiction. We see that the vertices $u *^0 v, \ldots, u *^{k-1} v, u *^k v$ are pairwise distinct. Hence the sequence (5) is an $u - v$ path in G, which completes the proof. □

Let G be a geodetic graph, and let d denote the distance function of G. Put $V = V(G)$. It is not difficult to see that if $u, v \in V$ and $u \neq v$, then there exists exactly one vertex $A_G(u, v)$ such that

$$d(u, A_G(u, v)) = 1 \quad \text{and} \quad d(A_G(u, v), v) = d(u, v) - 1.$$

Define a binary operation $*$ on V as follows:

$$x * y = A_G(x, y) \quad \text{if } x \neq y$$

and

$$x * y = x \quad \text{if } x = y$$

for all $x, y \in V$. We will say that $(V, *)$ is the *proper* groupoid of G.

It is clear that the proper groupoid of a geodetic graph G is a travel groupoid on G. Thus every geodetic graph has at least one travel groupoid.

Obviously, every tree is a geodetic graph. (Note that the proper groupoid of a finite tree was characterized in [3]).

Proposition 5. *Every finite tree has exactly one travel groupoid.*

Proof. Consider an arbitrary finite tree T. Put $V = V(T)$. Let $(V, *)$ be the proper groupoid of T. Suppose, to the contrary, that there exists a travel groupoid $(V, \circ)$ of T such that $(V, \circ)$ is different from $(V, *)$. Then there exist $u, v \in V$ such that $u \circ v \neq u * v$. By Proposition 3, both vertices $u * v$ and $u \circ v$ are adjacent to u in T. Since $u \circ v \neq u * v$, we see that the vertices $u \circ v$ and $u * v$ belong to distinct components of $T - u$. Recall that $(V, *)$ is the proper groupoid of T. This implies that the vertices $u * v$ and v belong to the same component of $T - u$. Since T contains no cycle, then, by virtue of (t1) and (t2), the vertices

$$u \circ v,\ u \circ^2 v,\ u \circ^3 v,\ \ldots$$

are pairwise distinct, which contradicts the fact that V is finite. Thus the proposition is proved. □

2. Simple travel groupoids

We say that a travel groupoid $(V,*)$ is *simple* if it satisfies the following axiom

(t3) if $v*u\neq u$, then $u*(v*u)=u*v$ (for all $u,v\in V$).

Note that the travel groupoids discussed in [2] are simple.

Remark 2. Let $(V,*)$ be a simple travel groupoid, and let $u,v\in V$ such that $v*u\neq u$. By (1), $u*v\neq v$. Thus, by (t3), $u*(v*u)=u*v$ and $v*(u*v)=v*u$.

The next remark gives an example of a travel groupoid which is not simple.

Remark 3. Let D be a directed cycle with $|V(D)|=2n$, where $n\geqslant 2$. Put $V=V(D)$. Clearly, for every $u\in V$ there exists exactly one vertex, say the vertex u', such that (u,u') is a directed edge in D. Let C denote the underlying graph of D. Obviously, C is a cycle of length $2n$. Let d denote the distance function of C. We denote by $*$ the binary operation on V defined as follows for all $v,w\in V$:

$v*w=v$ if $d(v,w)=0$;

$v*w=v'$ if $d(v,w)=n$;

$v*w$ is the only vertex t of C with the property that $d(v,t)=1$ and $d(t,w)=d(v,w)-1$ if $0<d(v,w)<n$.

It is obvious that $(V,*)$ is a travel groupoid. Consider arbitrary $x,y\in V$ such that $d(x,y)=n$. Then $d(x*y,x*(y*x))=2$. Thus $(V,*)$ is not simple.

Lemma 1. *Let $(V,*)$ be a simple travel groupoid, let $u,v,w\in V$, and let $k\geqslant 1$. Assume that $u*^{k-1}w\neq w$ and*

$$v*u=w. \tag{6}$$

Then

$$u*^{i}v=u*^{i}w \tag{7}$$

and

$$v*(u*^{i}w)=w \tag{8}$$

for each i, $0\leqslant i\leqslant k$, and

$$u*^{k-1}v\neq v.$$

Proof. We will first prove that (7) and (8) hold for each i, $0\leqslant i\leqslant k$. We proceed by induction on i. Let first $i=0$. Obviously, $u*^{0}v=u=u*^{0}w$. By (6),

$v * (u *^0 w) = w$. Let now $1 \leqslant i \leqslant k$. By the induction hypothesis,

$$u *^{i-1} v = u *^{i-1} w \tag{9}$$

and

$$v * (u *^{i-1} w) = w. \tag{10}$$

Since $u *^{k-1} w \neq w$, Proposition 4 implies that $u *^{i-1} w \neq w$. By virtue of (10), $v * (u *^{i-1} w) = w \neq u *^{i-1} w$. As follows from (t3) and Remark 2,

$$(u *^{i-1} w) * v = (u *^{i-1} w) * (v * (u *^{i-1} w)) \tag{11}$$

and

$$v * (u *^{i-1} w) = v * ((u *^{i-1} w) * v). \tag{12}$$

Obviously, $u *^i v = (u *^{i-1} v) * v$. It follows from (9), (11) and (10) that $(u *^{i-1} v) * v = (u *^{i-1} w) * v = (u *^{i-1} w) * (v * (u *^{i-1} w)) = (u *^{i-1} w) * w$. Thus $u *^i v = u *^i w$ and (7) holds.

Next, as follows from (7), (9), (12) and (10), $v * (u *^i w) = v * (u *^i v) = v * ((u *^{i-1} v) * v) = v * ((u *^{i-1} w) * v) = v * (u *^{i-1} w) = w$. Thus (8) holds.

We want to prove now that $u *^{k-1} v \neq v$. Suppose, to the contrary, that $u *^{k-1} v = v$. By (7), $u *^{k-1} v = u *^{k-1} w$, and thus $w \neq u *^{k-1} w = v$. As follows from (8) and Proposition 1, $w = v * (u *^{k-1} w) = v * v = v$, which completes the proof. □

Proposition 6. *Let $(V, *)$ be a simple travel groupoid, and let $k \geqslant 1$. If $x, y \in V$, $x *^{k-1} y \neq y$, and $x *^k y = y$, then $y *^{k-1} x \neq x$ and*

$$y *^j x = x *^{k-j} y \tag{13}$$

for each j, $0 \leqslant j \leqslant k$.

Proof. We proceed by induction on k.

Let first $k = 1$. Consider arbitrary $x, y \in V$ such that $x *^0 y \neq y$ and $x *^1 y = y$. Then $x \neq y$. We have $y *^0 x \neq x$. Obviously, (13) holds for $j = 0$. As follows from (1), (13) holds also for $j = 1$.

Let now $k \geqslant 2$. Consider arbitrary $x, y \in V$ such that $x *^{k-1} y \neq y$ and $x *^k y = y$. Since $x *^{k-1} y \neq y$, it follows from (2) that $x \neq y$. Put $z = x * y$. Then $z *^{k-2} y \neq y$ and $z *^{k-1} y = y$. By the induction hypothesis, $y *^{k-2} z \neq z$ and

$$y *^j z = z *^{(k-1)-j} y = (x * y) *^{(k-1)-j} y = x *^{k-j} y \tag{14}$$

for each j, $0 \leqslant j \leqslant k-1$. Since $y *^{k-2} z \neq z$ and $x * y = z$, Lemma 1 implies that

$$y *^j x = y *^j z \tag{15}$$

for each j, $0 \leqslant j \leqslant k-1$. Combining (14) and (15) for each j, $0 \leqslant j \leqslant k-1$, we get (13) for each j, $0 \leqslant j \leqslant k-1$ This means that $y *^{k-1} x = x * y$. Since $x \neq y$, (2) implies that $y *^{k-1} x \neq x$. Moreover, by (t1), $y *^k x = (y *^{k-1} x) * x = (x * y) * x = x$. Hence (13) holds also for $j = k$, which completes the proof. □

Corollary 1. *Let $(T, *)$ be a simple travel groupoid, let $u, v \in V$, and let $k \geqslant 1$ If $u *^k v = v$, then $v *^k u = u$.*

Proof. The case of $u = v$ is obvious. Let $u \neq v$. Then $u *^0 v \neq v$. Since $u *^k v = v$, we see that there exists i, $1 \leqslant i \leqslant k$, such that $u *^{i-1} v \neq v$ and $u *^i v = v$. By virtue of Proposition 6, $v *^i u = u *^0 v = u$ and therefore

$$v *^k u = (v *^i u) *^{k-i} u = u *^{k-i} u = u,$$

which completes the proof. □

Theorem 2. *Let $(V, *)$ be a simple travel groupoid on a graph G, let $u, v \in V$ and let $k \geqslant 1$. Assume that $u *^{k-1} v \neq v$ and $u *^k v = v$. Then the sequence*

$$v *^0 u, \ldots, v *^{k-1} u, v *^k u$$

is a $v - u$ path in G.

Proof. Combining Theorem 1 and Proposition 6, we get the theorem. □

The next two lemmas will be used in Section 3.

Lemma 2. *Let $(V, *)$ be a simple travel groupoid, let $u, v \in V$, and let $j \geqslant 1$. Assume that $v *^j u \neq u$. Then $u * (v *^j u) = u * v$.*

Proof. We proceed by induction on j. The case of $j = 1$ immediatelly follows from the definition of a simple travel groupoid. Let $j \geqslant 2$. Since $v *^j u \neq u$, it follows from Proposition 4 that $v *^{j-1} u \neq u$. By the induction hypothesis,

$$u * (v *^{j-1} u) = u * v. \tag{16}$$

Obviously, $v *^j u = (v *^{j-1} u) * u$. Since $(v *^{j-1} u) * u \neq u$, (t3) implies that

$$u * ((v *^{j-1} u) * u) = u * (v *^{j-1} u). \tag{17}$$

Combining (16) and (17), we get $u * (v *^j u) = u * ((v *^{j-1} u) * u) = u * v$, which completes the proof. □

Lemma 3. *Let* $(V,*)$ *be a simple travel groupoid, let* $u,v\in V$*, and let* $k\geqslant 2$*. Assume that* $u*^{k-1}v\neq v$ *and* $u*^{k}v=v$*. Then*

$$u*^{i}(v*u)=u*^{i}v$$

for each i*,* $0\leqslant i\leqslant k-1$*.*

P r o o f. Proposition 6 implies that $v*^{k-1}u\neq u$ and $v*^{k}u=u$. Put $w=v*u$. Then $w*^{k-2}u\neq u$ and $w*^{k-1}u=u$. By Proposition 6 again, $u*^{k-2}w\neq w$. Lemma 1 implies that $u*^{i}v=u*^{i}w=u*^{i}(v*u)$ for each i, $0\leqslant i\leqslant k-1$, which completes the proof. □

Remark 4. Let $(V,*)$ be a simple travel groupoid on a finite graph G. It was proved in [2] and [4] that G is a geodetic graph and $(V,*)$ is its proper groupoid if and only if G is connected and $(V,*)$ satisfies the following axiom

(tg) if $w*v=v$ and $u*v\neq u*w$, then $w*(u*v)=v$ (for all $u,v,w\in V$).

The assumption that G is connected can not be deleted. There exists a simple travel goupoid satisfying (tg) on a finite disconnected graph (see Remark 2 in [2]).

3. Non-confusing travel groupoids

Let $(V,*)$ be a travel groupoid, and let $u,v\in V$ such that $u\neq v$. By (2), $u*v\neq u$ and by (t2), $u*^{2}v\neq u$. If there exists $i\geqslant 3$ such that $u*^{i}v=u$, then we say that the ordered pair (u,v) is a *confusing pair* in $(V,*)$.

The next lemma will be used the in the next section.

Lemma 4. *Let* $(V,*)$ *be a travel groupoid, let* $u,v\in V$*,* $u\neq v$*, and let* $i\geqslant 3$ *such that* $u*^{i}v=u$*. Then there exists* j*,* $3\leqslant j\leqslant i$*, such that* $u*^{j}v=u$ *and the elements*

$$u*^{0}v,\dots,u*^{j-2}v,\quad\text{and}\quad u*^{j-1}v$$

are pairwise distinct.

P r o o f. Since $u\neq v$, (t2) implies that there exists j, $3\leqslant j\leqslant i$ such that $u*^{j}v=u$ and all the elements

$$u*^{1}v,u*^{2}v,\dots,u*^{j-1}v$$

are different from u. Assume that there exist k and m, $1\leqslant k<m\leqslant j-1$, such that $u*^{k}v=u*^{m}v$. Then $m\geqslant k+2$. It is clear that

$$u*^{n}v\in\{u*^{k}v,u*^{k+1}v,\dots,u*^{m-1}v\}\quad\text{for all}\ n\geqslant m,$$

and therefore $u*^{i}v\neq u$, which is a contradiction. Thus the lemma is proved. □

Remark 5. Let $(V, *)$ be a travel groupoid on a finite graph G. It is clear that if G is not connected, then $(V, *)$ has a confusing pair.

We say that a travel groupoid $(V, *)$ is *non-confusing* if there exists no confusing pair in $(V, *)$.

Proposition 7. *Let $(V, *)$ be a finite non-confusing travel groupoid, and let $u, v \in V$ and $u \neq v$. Then there exists exactly one $k \geqslant 1$ such that $u *^{k-1} v \neq v$ and $u *^k v = v$.*

Proof. Define

$$u_i = u *^i v \quad \text{for all } i \geqslant 0.$$

Suppose, to the contrary, that

$$u_i \neq v \quad \text{for all } i \geqslant 0.$$

Since V is finite, there exist j and m, $0 \leqslant j < m$, such that $u_m = u_j$. We have

$$u_m = u_j *^{m-j} v = u_j.$$

Thus $m - j \geqslant 3$ and (u_j, v) is a confusing pair in $(V, *)$, which is a contradiction. We have prove that there exists $k \geqslant 1$ such that $u *^{k-1} v \neq v$ and $u *^k v = v$. By Remark 1, k is defined uniquely. Thus the theorem is proved. □

Theorem 3. *Let $(V, *)$ be a finite travel groupoid on a graph G. Then $(V, *)$ is non-confusing if and only if the following statement holds for all distinct $u, v \in V$:*

there exists $k \geqslant 1$ such that the sequence

$$u *^0 v, \ldots, u *^{k-1} v, u *^k v$$

is an $u - v$ path in G.

Proof. Combining Theorem 1 and Proposition 7, we obtain the theorem. □

The next remark gives an example of a simple travel groupoid on a finite geodetic graph. This travel groupoid has a confusing pair.

Remark 6. Let $m, n \geqslant 3$ be odd, and let $u_0, u_1, \ldots, u_{m-1}, v, w_0, w_1, \ldots, w_{n-1}$ are pairwise distinct elements. Put

$$U = \{u_0, u_1, \ldots, u_{m-1}, v\} \quad \text{and} \quad W = \{w_0, w_1 \ldots, w_{n-1}, v\}.$$

Obviously, $U \cap W = \{v\}$. Define $u_m = u_0$ and $w_n = w_0$. Let G_U be the graph with $V(G_U) = U$ and

$$E(G_U) = \{u_0u_1, \ldots, u_{m-2}u_{m-1}, u_{m-1}u_m, u_0v\}.$$

Moreover, let G_W be the graph with $V(G_W) = W$ and

$$E(G_W) = \{w_0w_1, \ldots, w_{n-2}w_{n-1}, w_{n-1}w_n, w_0v\}.$$

Since m and n are odd, we see that both G_U and G_W are geodetic graphs. At the end of Section 1, the mapping A_G was defined for a geodetic graph G. In the same way, we define the mappings A_{G_U} and A_{G_W} for the geodetic graphs G_U and G_W respectively.

Put $V = U \cup W$. We denote by $*$ the binary operation on V defined for all $x, y \in V$ as follows:

$x * y = x$ if $x = y$;
$x * y = A_{G_U}(x, y)$ if $x, y \in U$ and $x \neq y$;
$x * y = A_{G_W}(x, y)$ if $x, y \in W$ and $x \neq y$;
$x * y = u_i$ if $x = u_{i-1}$ and $y \in W \setminus \{v\}$ for each i, $0 \leqslant i \leqslant m-1$;
$x * y = w_j$ if $x = w_{j-1}$ and $y \in U \setminus \{v\}$ for each j, $0 \leqslant j \leqslant n-1$.

It is easy to see that $(V, *)$ is a simple travel groupoid. The ordered pair (u_0, w_0) is an example of a confusing pair in $(V, *)$. Let G_0 denote the graph of $(V, *)$. It is easy to see that G is a geodetic graph.

Proposition 8. *Let $(V, *)$ be a finite simple non-confusing travel groupoid, let $u, v \in V$. Then $(u *^i v) *^i u = u$ for each $i \geqslant 0$.*

P r o o f. The case of $u = v$ follows immediately from Proposition 1. Assume that $u \neq v$. According to Proposition 7, there exists $k \geqslant 1$ such that $u *^{k-1} v \neq v$ and $u *^k v = v$. Recall that $(V, *)$ is simple. If $i \geqslant k$, then $u *^i v = v$ and, by virtue of Corollary 1, $(u *^i v) *^i u = v *^i u = u$. Let $i < k$. By Proposition 6, $v *^k u = u$ and $v *^{k-i} u = u *^i v$. Thus $(u *^i v) *^i u = (v *^{k-i} u) *^i u = v *^k u = u$, which completes the proof. $\square$

Proposition 9. *Let $(V, *)$ be a finite simple non-confusing travel groupoid, and let $u, v, w \in V$ such that $u \neq v$. Assume that there exists $k \geqslant 1$ such that $u *^{k-1} w \neq v$ and $u *^k w = v$. Then $u *^{k-1} v \neq v$ and $u *^k v = v$.*

P r o o f. We see that $u \neq w$ (otherwise, $u *^k w = u \neq v$; a contradiction). By Proposition 7, there exists exactly one $m \geqslant 1$ such that $u *^{m-1} w \neq w$ and $u *^m w = w$.

If $k > m$, then $u *^{k-1} w = u *^k w$; a contradiction. Thus $k \leqslant m$. As follows from Proposition 6, $w *^{m-1} u \neq u$ and $w *^{m-j} u = u *^j w$ for each j, $0 \leqslant j \leqslant m$. Thus $w *^m u = u$. Since $u *^k w = v$, we get $v = w *^{m-k} u$. Hence

$$v *^{k-1} u = (w *^{m-k} u) *^{k-1} u \neq u \quad \text{and} \quad v *^k u = (w *^{m-k} u) *^k u = u.$$

If we apply Proposition 6 again, we get $u *^{k-1} v \neq v$ and $u *^k v = v$, which completes the proof. □

Let $(V, *)$ be a finite travel groupoid on a graph G, and let $x, y \in V$. Clearly, x and y are distinct and non-adjacent vertices of G if and only if $x * y \neq y$.

Let $(V, *)$ be a simple non-confusing travel groupoid, and let $x, y \in V$ such that $x * y \neq y$. By virtue of Proposition 7, there exists exactly one $k \geqslant 2$ such that

$$x *^{k-1} y \neq y \quad \text{and} \quad x *^k y = y. \tag{18}$$

As follows from Proposition 6,

$$y *^{k-1} x \neq x \quad \text{and} \quad y *^k x = x. \tag{19}$$

Put

$$y = x' \quad \text{and} \quad x = y'. \tag{20}$$

Consider arbitrary $u, v \in V$ such that $u \in \{x, y\}$. Assume that there exists $j \geqslant 1$ such that $u *^{j-1} v \neq u'$ and $u *^j v = u'$. By virtue of Remark 1 and Proposition 9, $j = k$. Moreover, Proposition 9 implies that

$$\text{if } u *^k v = u', \text{ then } u *^{k-1} v \neq u'. \tag{21}$$

By the *xy-strengthening* of $*$ on V we mean the binary operation $\circ$ on V defined for all $u, v \in V$ as follows:

$u \circ v = u *^k v$ if $u \in \{x, y\}$ and $u *^k v = u'$;

$u \circ v = u * v$ otherwise.

This means that $w \circ w = w$ for every $w \in V$.

Lemma 5. *Let* $(V,*)$ *be a finite simple non-confusing travel groupoid on a graph* G*, let* $x, y \in V$ *such that* $x * y \neq y$*, and let* $\circ$ *be the* xy*-strengthening of* $*$ *on* V*. Then* $(V,\circ)$ *is a simple non-confusing travel groupoid on* $G + xy$*.*

P r o o f. There exists exactly one $k \geqslant 2$ such that (18) holds. Moreover, we have (19). Use the convention (20).

We first show that $(V,\circ)$ satisfies the axioms (t1), (t2), and (t3) and that $(V,\circ)$ is non-confusing. Consider arbitrary $r, s \in V$.

Verification of (t1). Put $t = (r \circ s) \circ r$. We will show that $t = r$. If $r = s$, then $r \circ s = r$ and therefore $t = r \circ r = r$. Assume that $r \neq s$. Then there exists $i \in \{1, k\}$ such that $r \circ s = r *^i s$. Since $r \neq s$ and $(V,*)$ is non-confusing, we get $r *^i s \neq r$. There exists $j \in \{1, k\}$ such that $t = (r *^i s) \circ r = (r *^i s) *^j r$. If $i = 1$, then (t1) and Proposition 1 imply that $t = r$. Assume that $i = k$. Then $r \in \{x, y\}$ and $r *^i s = r'$. This implies that $j = k$. We get $t = (r *^k s) *^k r = r$ again.

Verification of (t2). Obviously, there exist $i, j \in \{1, k\}$ such that $(r \circ s) \circ s = (r *^i s) *^j s = r *^{i+j} s$. Let $(r \circ s) \circ s = r$. Since $(V,*)$ is non-confusing, we get $r = s$.

We see that $(V,\circ)$ is a travel groupoid.

Verification of (t3). Assume that $s \circ r \neq r$. As follows from (2), $s \neq r$. We will prove that $r \circ (s \circ r) = r \circ s$. If $r, s \in \{x, y\}$, then $s *^k r = r$ and therefore $s \circ r = r$, which is a contradiction. Thus

(22) $$\text{at most one of } r \text{ and } s \text{ belongs to } \{x, y\}.$$

Since $(V,*)$ is non-confusing, Proposition 7 implies that there exists $m \geqslant 1$ such that $r *^{m-1} s \neq s$ and $r *^m s = s$. Recall that $k \geqslant 2$. Since $(V,*)$ is simple, it follows from Lemma 3 that

(23) $$\text{if } k < m, \text{ then } r *^k (s * r) = r *^k s.$$

Recall that $s \circ r \neq r$. Since $s \circ r = s *^k r$ or $s * r$, Remark 1 implies that $s * r \neq r$. By (t3),

$$r * (s * r) = r * s.$$

Let first $r \in \{x, y\}$ and $r *^k s = r'$. Then $r \circ s = r *^k s = r'$. By (22), $s \neq r'$. Then $k < m$ and $s \circ r = s * r$. It follows from (23) that $r *^k (s * r) = r'$ and therefore

$$r \circ (s \circ r) = r \circ (s * r) = r *^k (s * r) = r *^k s = r' = r \circ s.$$

Let now $r \in \{x, y\}$ and $r *^k s \neq r'$. Then $r \circ s = r * s$. By (22), $s \neq r'$ and thus $s \circ r = s * r$. Assume that $r *^k (s * r) = r'$. Then $r \circ (s * r) = r *^k (s * r)$. As follows from (21), $r *^{k-1} (s * r) \neq r'$. By virtue of Lemma 1, $r *^k s = r *^k (s * r) = r'$, which is a contradiction. Thus $r \circ (s * r) = r * (s * r)$ and therefore $r \circ (s \circ r) = r * s = r \circ s$.

Finally, let $r \notin \{x, y\}$. Then $r \circ s = r * s$ and $r \circ (s \circ r) = r * (s \circ r)$. Assume that $s \in \{x, y\}$ and $s *^k r = s'$. Then $s \circ r = s *^k r$. Since $s *^k r \neq r$, Lemma 2 implies that $r \circ (s \circ r) = r * (s *^k r) = r * s = r \circ s$. If

$$s \notin \{x, y\} \quad \text{or} \quad (s \in \{x, y\} \ \text{and} \ s *^k r \neq s'),$$

then $s \circ r = s * r$ and therefore $r \circ (s \circ r) = r * (s * r) = r * s = r \circ s$.

Thus $(V, \circ)$ is simple.

Assume that $r \neq s$ and there exists $i \geqslant 1$ such that $r \circ^i s = r$. Clearly, there exists $m \geqslant i$ such that $r \circ^i s = r *^m s$. We have that $r *^m s = r$, which contradicts the fact that $(V, *)$ is non-confusing. Thus $(V, \circ)$ is non-confusing, too.

Recall that $x \neq y$ and $x \circ y = y$. We can see that $(V, \circ)$ is a simple non-confusing travel groupoid on $G + xy$, which completes the proof of the lemma. □

Theorem 4. *For every finite connected graph G there exists a simple non-confusing travel groupoid on G.*

P r o o f. Put $V = V(G)$ and $\beta(G) = |E(G)| - |V| + 1$. We proceed by induction on $\beta(G)$. Obviously, $\beta(G) \geqslant 0$. Let first $\beta(G) = 0$. Then G is a tree. It is easy to see that its proper groupoid is simple and non-confusing. Let now $\beta(G) \geqslant 1$. Then there exist distinct $x, y \in V$ such that x and y are adjacent in G and $G - xy$ is connected. By the induction hypothesis, there exists a simple non-confusing travel groupoid $(V, *)$ on $G - xy$. Lemma 5 implies that there exists a simple non-confusing groupoid on G, which completes the proof. □

4. Smooth and semi-smooth travel groupoids

We say that a travel groupoid $(V, *)$ is *smooth* if it satisfies the following axiom

(t4) if $u * v = u * w$, then $u * (w * v) = u * v$ (for all $u, v, w \in V$).

Moreover, we say that a travel groupoid $(V, *)$ is *semi-smooth* if it satisfies the following axiom

(t5) if $u * v = u * w$, then $u * (v * w) = u * v$ or $u * ((v * w) * w) = u * v$ (for all $u, v, w \in V$).

Obviously, every smooth travel groupoid is semi-smooth.

Proposition 10. *Every semi-smooth travel groupoid is non-confusing.*

Proof. Let $(V,*)$ be a semi-smooth travel groupoid. Obviously, there exists a graph G such that $(V,*)$ is on G. Suppose, to the contrary, that there exists a confusing pair in $(V,*)$. As follows from Lemma 4, there exist $u, w \in V$ and $k \geqslant 3$ such that $u \neq w$ and $u *^k w = u$, and the vertices $u *^0 w, \dots, u *^{k-2} w$ and $u *^{k-1} w$ are pairwise distinct. Define

$$u_i = u *^i w \quad \text{for } i = 0, 1, \dots, k.$$

Hence $u_0 \neq u_1 \neq u_{k-1} \neq u_k = u_0$. Obviously, $u_0 * u_1 = u_1$ and $u_0 * u_k = u_0 * u_0 \neq u_1$. Moreover, $u_0 * u_{k-1} = u_k * u_{k-1} = (u_{k-1} * w) * u_{k-1}$ and thus, by (t1), $u_0 * u_{k-1} = u_{k-1}$. This implies that there exist j, $0 \leqslant j \leqslant k-2$, such that $u_0 * u_j = u_1$ and $u_0 * u_{j+1} \neq u_1 \neq u_0 * u_{j+2}$. We have $u_0 * w = u_1 = u_0 * u_j$, $u_0 * (u_j * w) = u_0 * u_{j+1} \neq u_0 * u_j$, and $u_0 * ((u_j * w) * w) = u_0 u_{j+2} \neq u_0 * u_j$, which contradicts (t5). Thus the proposition is proved. □

Proposition 11. *Every complete bipartite graph has a simple smooth travel groupoid.*

Proof. Let G be complete bipartite graph. Put $V = V(G)$. There exist nonempty sets U and U' such that $U \cap U' = \emptyset$, $U \cup U' = V$ and the following statement holds for all distinct $v, w \in V$:

v and w are adjacent in G if and only if $|\{v, w\} \cap U| = 1 = |\{v, w\} \cap U'|$.

Recall that U and U' are nonempty. Choose a vertex $u \in U$ and a vertex $u' \in U'$. We denote by $*$ the binary operation on V defined as follows:

$x * y = x$ if $x = y$;
$x * y = y$ if x and y are adjacent in G;
$x * y = u'$ if $x, y \in U$ and $x \neq y$;
$x * y = u$ if $x, y \in U'$ and $x \neq y$.

It can be easily verified that $(V,*)$ satisfies (t1), (t2), (t3), and (t4). Hence $(V,*)$ is a simple smooth travel groupoid. □

Recall that every tree is a geodetic graph and that the proper groupoid of every geodetic graph is a simple travel groupoid.

Proposition 12. *The proper groupoid of every tree is a smooth travel groupoid.*

Proof is easy. □

Note that every complete graph is geodetic. Obviously, the proper groupoid of every complete graph is a smooth travel groupoid.

Theorem 5. *Let G be a geodetic graph of diameter two, and let $(V,*)$ be the proper groupoid on G. Then $(V,*)$ is a smooth travel groupoid.*

Proof. Clearly, $(V,*)$ is a simple travel groupoid such that

$$x *^2 y = y \quad \text{for all } x, y \in V. \tag{24}$$

We will prove that $(V,*)$ is smooth. Suppose, to the contrary, that $(V,*)$ is not smooth. Then there exist $u, v, w \in V$ such that $u * v = u * w$ and

$$u * (v * w) \neq u * v. \tag{25}$$

This implies that $v \neq v * w \neq w$. By (24), $v *^2 w = w$. Recall that $(V,*)$ is simple. Since $v * w \neq w$, Proposition 6 implies that $w * v \neq v$, $w *^2 v = v$, and $w * v = v * w$. Thus the sequence

$$v, v * w = w * v, w$$

is a shortest $v - w$ path in G.

Put $t = u * v$. Then $t = u * w$. As follows from (24), $t * v = v$ and $t * w = w$. By (1), $v * t = t$. If $t = v$, then $u * w = v$ and therefore $u *^2 w = v * w \neq w$; a contradiction. Thus $t \neq v$. Since $v * t = t$, we see that v and t are adjacent in G. Let $t = w$. Since $t * v = v$, we get $w * v = v$; a contradiction. Thus $t \neq w$. Since $t * w = w$, we see that t is adjacent to w. Thus the sequence

$$v, t, w$$

is a shortest $v - w$ path in G.

Assume that $t = v * w$. Using (3), we see that $u * (v * w) = u * (u * v) = u * v$, which contradicts (25). Thus $u * v \neq v * w$. We see that G has two distinct shortest $v - w$ paths in G. This means that G is not a geodetic graph, which is a contradiction. Thus the theorem is proved. □

We pose two questions.

Question 1. Does there exists a geodetic graph G such that the proper groupoid of G is not smooth? (If so, does there exists a geodetic groupoid G such that the proper groupoid of G is not semi-smooth?)

Question 2. Does there exists a connected graph G such that G has no smooth travel groupoid? (If so, does there exists a connected graph G such that G has no semi-smooth travel groupoid?)

5. Graphs with travel groupoids

Recall that, by Theorem 4, every finite connected graph has a simple non-confusing travel groupoid.

Theorem 6. *Let G be a finite graph. Then G has a travel groupoid if and only if G is connected or G is disconnected and no component of G is a tree.*

Proof. Assume that G is connected or G is disconnected and no component of G is a tree. If G is connected, then, by Theorem 4, there exists a travel groupoid on G. Let G be disconnected. Then every component of G contains a cycle. It is easy to see that there exists a mapping f of $V(G)$ into itself such that the following statements hold for every $u \in V(G)$:

$$u \text{ and } f(u) \text{ are adjacent vertices in } G$$

and

$$u \neq f(f(u)).$$

By virtue of Theorem 4, every component F of G has a travel groupoid, say a travel groupoid $(V(F), *_F)$. For all $x, y \in V(G)$, we define

$$x * y = x *_H y \text{ if there exists a component } H \text{ of } G \text{ such that } x, y \in V(H)$$

and

$$x * y = f(x) \text{ if } x \text{ and } y \text{ belong to distinct components of } G.$$

It is easy to see that $(V(G), *)$ satisfies (t1) and (t2). Hence G has a travel groupoid.

Conversely, assume that G is disconnected and at least one component T of G is a tree. Suppose, to the contrary, that G has a travel groupoid, say a travel groupoid $(V, *)$, where $V = V(G)$. Consider $u \in V(T)$ and $v \in V(G) \setminus V(T)$. Since $V(T)$ is finite and T contains no cycle, we see that there exists $k \geqslant 1$ such that $u *^{k+1} v = u *^{k-1} v$. We have $((u *^{k-1} v) * v) * v = u *^{k-1} v$, and thus, by (t2), $u^{k-1}v = v$. Proposition 3 implies that u and v belong to the same component of G, which is a contradion. Thus the theorem is proved. □

Question 3. Does there exist an infinite graph G with no finite components such that G has no travel groupoid?

Remark 7. Let $(V, *)$ be a finite travel groupoid. Put

$$X = \{(u, v, w)\colon\ u, v, w \in V \ \text{ and } \ v = u * w\}.$$

Then (V, X) is a signpost system in the sense of [5]. We say that (V, X) is the signpost system of $(V, *)$.

The signpost systems of travel groupoids create a special subclass of the class of all signpost systems. The terms "simple", "non-confusing" and "smooth" introduced in the present paper for travel groupoids are inspired by the same terms used for signpost systems in [5].

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