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PERIMETER PRESERVER OF MATRICES OVER SEMIFIELDS

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Abstract. For a rank-1 matrix $A = \mathbf{a}\mathbf{b}^t$, we define the *perimeter* of A as the number of nonzero entries in both $\mathbf{a}$ and $\mathbf{b}$. We characterize the linear operators which preserve the rank and perimeter of rank-1 matrices over semifields. That is, a linear operator T preserves the rank and perimeter of rank-1 matrices over semifields if and only if it has the form $T(A) = UAV$, or $T(A) = UA^tV$ with some invertible matrices U and V.

Keywords: linear operator, rank, dominate, perimeter, (U, V)-operator

MSC 2000: 15A03, 15A04, 15A23

1. Introduction and preliminaries

On the study of linear operators that preserve rank of matrices over several semirings, there are many papers ([1]–[3]). Beasley and Pullman [1] characterized the linear operators preserving the rank of Boolean matrices. We consider those linear operators that preserve the perimeter of the rank-1 matrices over *semifields*, which is the nonnegative parts of fields.

Let $\mathscr{M}_{m,n}(\mathbb{F}_+)$ denote the set of all $m \times n$ matrices with entries in $\mathbb{F}_+$, the set of nonnegative part of any field $\mathbb{F}$. Addition, multiplication by scalars, and the product of matrices are also defined as if $\mathbb{F}_+$ were a field. Throughout this paper, we shall adopt the convention that $m \leqslant n$ unless otherwise specified.

The *rank* or *factor rank*, $r(A)$, of a nonzero matrix $A \in \mathscr{M}_{m,n}(\mathbb{F}_+)$ is defined as the least integer k for which there exist $m \times k$ and $k \times n$ matrices B and C with $A = BC$. The rank of a zero matrix is zero. It is well known that $r(A)$ is the least k such that A is the sum of k matrices of rank 1 (see [2], [3]).

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Let $\Delta_{m,n} = \{(i,j)\colon\ 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$, and E_{ij} be the $m \times n$ matrix whose (i,j)th entry is 1 and whose other entries are all 0, and $\mathbb{E}_{m,n} = \{E_{ij}\colon\ (i,j) \in \Delta_{m,n}\}$. We call E_{ij} a *cell*.

The *Boolean algebra* consists of the set $\mathbb{B} = \{0,1\}$ equipped with two binary operations, addition and multiplication. The operations are defined as usual except that $1+1=1$.

If $A = [a_{ij}]$ is any matrix in $\mathscr{M}_{m,n}(\mathbb{F}_+)$, we define $A^* = [a_{ij}{}^*]$ to be the $m \times n$ Boolean matrix whose (i,j)th entry is 1 if and only if $a_{ij} \neq 0$. Then * maps $\mathscr{M}_{m,n}(\mathbb{F}_+)$ onto $\mathscr{M}_{m,n}(\mathbb{B})$, and preserves matrix addition, product, and multiplication by scalars. That is, * is a homomorphism.

It follows that

$$(A+B)^* = A^* + B^* \text{ and } (BC)^* = B^*C^* \tag{1.1}$$

for all $A, B \in \mathscr{M}_{m,n}(\mathbb{F}_+)$ and all $C \in \mathscr{M}_{n,r}(\mathbb{F}_+)$.

An $n \times n$ matrix A over $\mathbb{F}_+$ is said to be *invertible* if there exist an $n \times n$ matrix B over $\mathbb{F}_+$ such that $AB = BA = I_n$, where I_n is the $n \times n$ identity matrix. It is well known that a square matrix A over $\mathbb{F}_+$ is invertible if and only if some permutation of its rows is a diagonal matrix all of whose diagonal entries are nonzero in $\mathbb{F}_+$ (see [2]).

If A and B are in $\mathscr{M}_{m,n}(\mathbb{F}_+)$, we say A *dominates* B (written $B \leqslant A$ or $A \geqslant B$) if $a_{ij} = 0$ implies $b_{ij} = 0$ for all i, j.

For example, if

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix},$$

are matrices in $\mathscr{M}_{2,2}(\mathbb{R}_+)$, then we have $A \leqslant B$ and $B \leqslant A$, but $A \neq B$.

Also we can easily obtain that $A \geqslant B$ if and only if $A + B = A$ for all $A, B \in \mathscr{M}_{m,n}(\mathbb{B})$.

Lowercase, boldface letters will represent column vectors, all vectors $\mathbf{u}$ are column vectors ($\mathbf{u}^t$ is a row vector) for $\mathbf{u} \in \mathbb{F}_+{}^m [= \mathscr{M}_{m,1}(\mathbb{F}_+)]$.

It is easy to verify that the rank of $A \in \mathscr{M}_{m,n}(\mathbb{F}_+)$ is 1 if and only if there exist nonzero vectors $\mathbf{a} \in \mathscr{M}_{m,1}(\mathbb{F}_+)$ and $\mathbf{b} \in \mathscr{M}_{n,1}(\mathbb{F}_+)$ such that $A = \mathbf{a}\mathbf{b}^t$. We call $\mathbf{a}$ the *left factor*, and $\mathbf{b}$ the *right factor* of A. But these vectors $\mathbf{a}$ and $\mathbf{b}$ are not uniquely determined by A.

For example,

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} [1 \quad 2] = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} [2 \quad 4] = \dots.$$

For any vector $\mathbf{u} \in \mathscr{M}_{m,1}(\mathbb{F}_+)$, we define $|\mathbf{u}|$ to be the number of nonzero entries in $\mathbf{u}$.

Lemma 1.1. *For any factorization* $\mathbf{a}\mathbf{b}^t$ *of an* $m \times n$ *rank*-1 *matrix* A *over* $\mathbb{F}_+$, $|\mathbf{a}|$ *and* $|\mathbf{b}|$ *are uniquely determined by* A.

Proof. Consider the $m \times n$ Boolean matrix $A^* = [a_{ij}{}^*]$. By (1.1), $A^* = \mathbf{a}^*(\mathbf{b}^*)^t$ is the rank-1 matrix. It is easy to show that $|\mathbf{a}^*|$ and $|\mathbf{b}^*|$ are uniquely determined by A^*. Therefore $|\mathbf{a}|$ and $|\mathbf{b}|$ are uniquely determined by A. □

Let A be any rank-1 matrix in $\mathscr{M}_{m,n}(\mathbb{F}_+)$. We define the *perimeter* of A, $P(A)$, as $|\mathbf{a}| + |\mathbf{b}|$ for arbitrary factorization $A = \mathbf{a}\mathbf{b}^t$. Even though the factorizations of A are not unique, Lemma 1.1 shows that the perimeter of A is unique, and that $P(A) = P(A^*)$.

Proposition 1.2. *If* A, B *and* $A + B$ *are rank*-1 *matrices in* $\mathscr{M}_{m,n}(\mathbb{F}_+)$, *then* $P(A + B) < P(A) + P(B)$.

Proof. Since $P(A) = P(A^*)$, it is sufficient to consider $A, B, A+B \in \mathscr{M}_{m,n}(\mathbb{B})$. Let $A = \mathbf{a}\mathbf{x}^t, B = \mathbf{b}\mathbf{y}^t$ and $A + B = \mathbf{c}\mathbf{z}^t$ be any factorizations of A, B and $A + B$. Then we have for all i, j

$$a_i\mathbf{x} + b_i\mathbf{y} = c_i\mathbf{z} \tag{1.2}$$

and

$$x_j\mathbf{a} + y_j\mathbf{b} = z_j\mathbf{c}. \tag{1.3}$$

If $B \leqslant A$, then we have $A + B = A$. Thus we obtain that

$$P(A + B) = P(A) < P(A) + P(B)$$

because $P(B) \neq 0$, as required.

Similar argument shows that if $A \leqslant B$, then $P(A + B) < P(A) + P(B)$. So we can assume that $A \nleqslant B$ and $B \nleqslant A$. We consider three cases.

Case 1) $\mathbf{a} \nleqslant \mathbf{b}$ and $\mathbf{b} \nleqslant \mathbf{a}$. The equation (1.2) implies that $a_i\mathbf{x} = c_i\mathbf{z}$ and $b_j\mathbf{y} = c_j\mathbf{z}$ for some nonzero $a_i, c_i, b_j, c_j \in \mathbb{B}$ so that $\mathbf{x} = \mathbf{y} = \mathbf{z}$. Thus we have the following

$$P(A + B) = P\left((\mathbf{a} + \mathbf{b})\mathbf{z}^t\right) = |\mathbf{a} + \mathbf{b}| + |\mathbf{z}| < (|\mathbf{a}| + |\mathbf{z}|) + (|\mathbf{b}| + |\mathbf{z}|) = P(A) + P(B)$$

as required.

Case 2) $\mathbf{a} \leqslant \mathbf{b}$. Then $\mathbf{x} \nleqslant \mathbf{y}$. Thus (1.3) implies that $x_j\mathbf{a} = z_j\mathbf{c}$ for some nonzero $x_j, z_j \in \mathbb{B}$ and $\mathbf{b} \leqslant \mathbf{c}$. Therefore $\mathbf{a} = \mathbf{b} = \mathbf{c}$ and we have

$$P(A + B) = P\left(\mathbf{c}(\mathbf{x} + \mathbf{y})^t\right) = |\mathbf{c}| + |\mathbf{x} + \mathbf{y}| < (|\mathbf{c}| + |\mathbf{x}|) + (|\mathbf{c}| + |\mathbf{y}|) = P(A) + P(B)$$

as required.

Case 3) $\mathbf{b} \leqslant \mathbf{a}$. It is similar to the Case 2).

□

A mapping $T\colon \mathscr{M}_{m,n}(\mathbb{F}_+) \to \mathscr{M}_{m,n}(\mathbb{F}_+)$ is called a *linear operator* if $T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$ for all $A, B \in \mathscr{M}_{m,n}(\mathbb{F}_+)$ and for all $\alpha, \beta \in \mathbb{F}_+$.

In this paper, we characterize the linear operators that preserve the rank and the perimeter of every rank-1 matrix over semifields. These are motivated by analogous results for the linear operators which preserve all ranks in $\mathscr{M}_{m,n}(\mathbb{F}_+)$. However, we obtain results and proofs in the view of the perimeter analog.

2. Perimeter preservers of matrices over semifields

In this section, we will characterize the linear operators that preserve the perimeter of every rank-1 matrix in $\mathscr{M}_{m,n}(\mathbb{F}_+)$. We also find some characterizations of the perimeter preservers.

Let T be a linear operator on $\mathscr{M}_{m,n}(\mathbb{F}_+)$. Then we say that

(1) T is a (U,V)*-operator* if there exist invertible matrices $U \in \mathscr{M}_{m,m}(\mathbb{F}_+)$ and $V \in \mathscr{M}_{n,n}(\mathbb{F}_+)$ such that $T(A) = UAV$ for all A in $\mathscr{M}_{m,n}(\mathbb{F}_+)$, or $m = n$ and $T(A) = UA^tV$ for all A in $\mathscr{M}_{m,n}(\mathbb{F}_+)$.

(2) T *preserves rank* 1 if $r(T(A)) = 1$ whenever $r(A) = 1$ for all $A \in \mathscr{M}_{m,n}(\mathbb{F}_+)$.

(3) T *preserves perimeter* k of rank-1 matrices if $P(T(A)) = k$ whenever $P(A) = k$ for all $A \in \mathscr{M}_{m,n}(\mathbb{F}_+)$ with $r(A) = 1$.

Proposition 2.1. *If T is a (U,V)-operator on $\mathscr{M}_{m,n}(\mathbb{F}_+)$, then T preserves both rank and perimeter of rank-*1 *matrices.*

Proof. Since T is a (U,V)-operator, there exist invertible matrices $U \in \mathscr{M}_{m,m}(\mathbb{F}_+)$ and $V \in \mathscr{M}_{n,n}(\mathbb{F}_+)$ such that either $T(A) = UAV$, or $m = n$ and $T(A) = UA^tV$ for all A in $\mathscr{M}_{m,n}(\mathbb{F}_+)$. Let A be a matrix in $\mathscr{M}_{m,n}(\mathbb{F}_+)$ with $r(A) = 1$ and $A = \mathbf{a}\mathbf{b}^t$ be any factorization of A with $P(A) = |\mathbf{a}| + |\mathbf{b}|$. For the case $T(A) = UAV$,

$$T(A) = UAV = (U\mathbf{a})(\mathbf{b}^tV) = (U\mathbf{a})(V^t\mathbf{b})^t.$$

Thus we have

$$r\left(T(A)\right) = r\left((U\mathbf{a})(V^t\mathbf{b})^t\right) = 1,$$

and

$$P\left(T(A)\right) = |U\mathbf{a}| + |V^t\mathbf{b}| = |\mathbf{a}| + |\mathbf{b}| = P(A).$$

For the case $T(A) = UA^tV$, we can show that $r\left(T(A)\right) = 1$ and $P\left(T(A)\right) = |\mathbf{a}| + |\mathbf{b}|$ by the similar method as above.

Therefore a (U,V)-operator preserves the rank and the perimeter of rank-1 matrices over $\mathbb{F}_+$. $\square$

For a rank-1 matrix A over $\mathbb{F}_+$, we note that $P(A) = 2$ if and only if it is nonzero scalar multiple of a cell. We say that A is a *row (column) matrix* if A has a nonzero entries only in one row (column). Then we have the following Lemma.

Lemma 2.2. *Let T be a linear operator on $\mathscr{M}_{m,n}(\mathbb{F}_+)$. If T preserves rank and perimeter 2 of rank-1 matrices, then the following statements hold:*

(1) *T maps a cell into a nonzero scalar multiple of a cell.*

(2) *T maps a row (or a column) of a matrix into a row or a column (if $m = n$) with scalar multiplication.*

Proof. (1) Since T has preserves perimeter 2, T maps a cell into nonzero scalar multiple of a cell. (2) If not, then there exist two distinct cells E_{ij}, E_{ih} in some ith row such that $T(E_{ij})$ and $T(E_{ih})$ lie in two different rows and different columns. Then the rank of $E_{ij} + E_{ih}$ is 1 but that of $T(E_{ij} + E_{ih}) = T(E_{ij}) + T(E_{ih})$ is 2, a contradiction. $\square$

The following is an example of a linear operator that preserves rank and perimeter 2 of rank-1 matrices, but the operator does not preserve perimeter p ($\geqslant 3$) and is not a (U, V)-operator.

Example 2.3. Let $T\colon \mathscr{M}_{n,n}(\mathbb{F}_+) \to \mathscr{M}_{n,n}(\mathbb{F}_+)$ be defined by

$$T(A) = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\right)E_{kl}$$

for all $A = [a_{ij}] \in \mathscr{M}_{n,n}(\mathbb{F}_+)$, where E_{kl} is a fixed cell. Then it is easy to verify that T is a linear operator and preserves rank and perimeter 2 of rank-1 matrices. But T does not preserve perimeter p ($\geqslant 3$). For, let A be a rank-1 matrix with perimeter p ($\geqslant 3$). Then $T(A) = \alpha E_{kl}$ for some nonzero scalar $\alpha \in \mathbb{F}_+$. Therefore $T(A)$ has rank 1 and perimeter 2.

Moreover, T is not a (U, V)-operator. For, let $B = [b_{ij}] \in \mathscr{M}_{n,n}(\mathbb{F}_+)$ with $b_{ij} = 1$ for all i, j. Then $T(B) = n^2 E_{kl}$. So we cannot find invertible matrices $U, V \in \mathscr{M}_{n,n}(\mathbb{F}_+)$ such that $T(B) = UBV$. This shows that T is not a (U, V)-operator.

Let $R_i = \{E_{ij}\colon 1 \leqslant j \leqslant n\}$, $C_j = \{E_{ij}\colon 1 \leqslant i \leqslant m\}$, $\mathscr{R} = \{R_i\colon 1 \leqslant i \leqslant m\}$ and $\mathscr{C} = \{C_j\colon 1 \leqslant j \leqslant n\}$. For a linear operator T on $\mathscr{M}_{m,n}(\mathbb{F}_+)$, define $T^*(A) = [T(A)]^*$ for all A in $\mathscr{M}_{m,n}(\mathbb{F}_+)$. Let $T^*(R_i) = \{T^*(E_{ij})\colon 1 \leqslant j \leqslant n\}$ for all $i = 1, \ldots, m$ and $T^*(C_j) = \{T^*(E_{ij})\colon 1 \leqslant i \leqslant m\}$ for all $j = 1, \ldots, n$.

Lemma 2.4. *Let T be a linear operator on $\mathscr{M}_{m,n}(\mathbb{F}_+)$. Suppose that T preserves rank and perimeters 2 and p (for some $p \geqslant 3$) of every rank-1 matrix. Then*

(1) *T maps two distinct cells in a row (or a column) into a nonzero scalar multiple of two distinct cells in a row or in a column;*

(2) *in the case of $m = n$, if T maps some R_i into a row (or column) matrix then T maps every row matrix into a row (or column) matrix, and if T maps some C_j into a row (column) matrix then T maps every column matrix into a row (column) matrix.*

Proof. (1) Let E_{ij} and E_{ih} be two distinct cells in an ith row. Suppose $T(E_{ij}) = \alpha E_{rl}$ and $T(E_{ih}) = \beta E_{rl}$ for some nonzero scalars $\alpha, \beta \in \mathbb{F}_+$. Then T maps the ith row of a matrix A into rth row or lth column by Lemma 2.2. Without loss of generality, we assume the former. Thus for any rank-1 matrix A with perimeter p ($\geqslant 3$) which dominates $E_{ij} + E_{ih}$, we can show that $T(A)$ has perimeter at most $p - 1$, a contradiction.

(2) If not, then there exist rows R_i and R_j such that $T^*(R_i) \subseteq R_r$ and $T^*(R_j) \subseteq C_s$ for some r, s. Consider a rank-1 matrix $D = E_{ip} + E_{iq} + E_{jp} + E_{jq}$ with $p \neq q$. Then we have

$$T(D) = T(E_{ip} + E_{iq}) + T(E_{jp} + E_{jq}) = (\alpha_1 E_{rp'} + \alpha_2 E_{rq'}) + (\beta_1 E_{p''s} + \beta_2 E_{q''s})$$

for some $p' \neq q'$ and $p'' \neq q''$ and some nonzero scalars $\alpha_i, \beta_i \in \mathbb{F}_+$ by (1). Therefore $r\,(T(D)) \neq 1$ and T does not preserve rank 1, a contradiction. □

Now we have an interesting example:

Example 2.5. Let $m \geqslant 3$ and $n \geqslant 4$. Define a linear operator $T\colon \mathscr{M}_{m,n}(\mathbb{F}_+) \to \mathscr{M}_{m,n}(\mathbb{F}_+)$ by $T(A) = [b_{ij}]$ for all $A = [a_{ij}] \in \mathscr{M}_{m,n}(\mathbb{F}_+)$, where

$$b_{ij} = \begin{cases} \sum\limits_{k=1}^{m} a_{kt} & \text{if } i = 1, \\ 0 & \text{if } i \geqslant 2, \end{cases}$$

with $t \equiv k + (j-1) \pmod{n}$ and $1 \leqslant t \leqslant n$. Then T maps each row and each column into the first row with some scalar multiplication. And T preserves both rank and perimeters $2, 3$ and $n+1$ of rank-1 matrices. But T does not preserve perimeters k ($k \geqslant 4$ and $k \neq n+1$) of rank-1 matrices: For if A has perimeter k, then we can choose a $2 \times (k-2)$ submatrix of A with perimeter k which is mapped to k distinct cells in the first row of $T(A)$. Thus $T(A)$ has perimeter $k+1$. Therefore T does not preserve perimeter k of rank-1 matrices.

For a linear operator T on $\mathscr{M}_{m,n}(\mathbb{F}_+)$ preserving rank and perimeter 2 of rank-1 matrices, we define the corresponding mapping $T'\colon\ \Delta_{m,n} \to \Delta_{m,n}$ by $T'(i,j) = (k,l)$ whenever $T(E_{ij}) = b_{ij}E_{kl}$ for some nonzero scalar $b_{ij} \in \mathbb{F}_+$. Then T' is well-defined by Lemma 2.2–(1).

Lemma 2.6. *Let T be a linear operator defined by $T(A) = \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} a_{ij}u_{ij}(E_{T'(i,j)})$ for some function $T'\colon\ \Delta_{m,n} \to \Delta_{m,n}$ and for some nonzero scalar $u_{ij} \in \mathbb{F}_+$, $i = 1,\dots,m$, $j = 1,\dots,n$. If T preserves both rank and perimeters 2 and k (for some $k \geqslant 4$, $k \neq n+1$) of rank-1 matrices, then the corresponding map T' is a bijection on $\Delta_{m,n}$.*

Proof. By Lemma 2.2, $T(E_{ij}) = b_{ij}E_{rl}$ for some $(r,l) \in \Delta_{m,n}$ and some nonzero scalar $b_{ij} \in \mathbb{F}_+$. Without loss of generality, we may assume that T maps the ith row of a matrix into the rth row. Suppose $T'(i,j) = T'(p,q)$ for some distinct pairs $(i,j),(p,q) \in \Delta_{m,n}$. By the definition of T', we have $T(E_{ij}) = b_{ij}E_{rl}$ and $T(E_{pq}) = b_{pq}E_{rl}$ for some nonzero scalars $b_{ij}, b_{pq} \in \mathbb{F}_+$. If $i = p$ or $j = q$, then we have contradictions by Lemma 2.4–(1). So let $i \neq p$ and $j \neq q$. If $k = n+k' \geqslant n+2$, consider a matrix

$$D = \sum_{s=1}^{n} E_{is} + \sum_{t=1}^{n} E_{pt} + \sum_{h=1}^{k'-2}\sum_{g=1}^{n} E_{hg}$$

with rank 1 and perimeter $n+k' = k$. Then T maps the ith and pth row of D into nonzero scalar multiple of the rth row by Lemma 2.4. Thus the perimeter of $T(D)$ is less than $n+k' = k$, a contradiction.

If $4 \leqslant k \leqslant n$, we will show that we can choose a $2\times(k-2)$ submatrix from the ith and pth row whose image under T has a $1\times k$ submatrix in the rth row as follows: Since $T(E_{ij}) = b_{ij}E_{rl}$ and $T(E_{pq}) = b_{pq}E_{rl}$, T maps the ith row and the pth row into the rth row. But T maps distinct cells in each row (or column) to distinct cells by Lemma 2.4. Now, choose E_{ij}, E_{pj} but do not choose E_{iq}, E_{pq}. Since there is a cell E_{ph} ($h \neq j,q$) in the pth row such that $T'(p,h) = T'(i,q)$ but $T'(i,h) \neq T'(p,j)$, we choose a 2×2 submatrix $E_{ij}+E_{ih}+E_{pj}+E_{ph}$ whose image under T is a 1×4 submatrix in the rth row. And we can choose a cell E_{ps} ($s \neq q,j,h$) such that $T'(i,s) \neq T'(p,j), T'(p,q), T'(p,h)$. Then we have a 2×3 submatrix $E_{ij}+E_{ih}+E_{is}+E_{pj}+E_{ph}+E_{ps}$ whose image under T is a 1×5 submatrix in the rth row. Similarly, we can choose a $2\times(k-2)$ submatrix whose image under T is an $1\times k$ submatrix in the rth row. This shows that T does not preserve the perimeter k of a rank-1 matrix, a contradiction.

Hence $T'(i,j) \neq T'(p,q)$ for any two distinct pairs $(i,j),(p,q) \in \Delta_{m,n}$. Therefore T' is a bijection. $\square$

We obtain the following characterization theorem for linear operators preserving the rank and the perimeter of rank-1 matrices over semifields.

Theorem 2.7. *Let T be a linear operator on $\mathscr{M}_{m,n}(\mathbb{F}_+)$. Then the following are equivalent:*
(1) *T is a (U,V)-operator;*
(2) *T preserves both rank and perimeter of rank-1 matrices;*
(3) *T preserves both rank and perimeters 2 and k (for some $k \geqslant 4$, $k \neq n+1$) of rank-1 matrices.*

P r o o f. (1) implies (2) by Proposition 2.1. It is obvious that (2) implies (3). We now show that (3) implies (1). Assume (3). Then the corresponding mapping $T'\colon \Delta_{m,n} \to \Delta_{m,n}$ is a bijection by Lemma 2.6.

By Lemma 2.4, there are two cases; (a) T^* maps $\mathscr{R}$ onto $\mathscr{R}$ and maps $\mathscr{C}$ onto $\mathscr{C}$ or (b) T^* maps $\mathscr{R}$ onto $\mathscr{C}$ and $\mathscr{C}$ onto $\mathscr{R}$.

Case a). We note that $T^*(R_i) = R_{\sigma(i)}$ and $T^*(C_j) = C_{\tau(j)}$ for all i, j, where σ and τ are permutations of $\{1,\dots,m\}$ and $\{1,\dots,n\}$, respectively. Let P and Q be the permutation matrices corresponding to σ and τ, respectively. Then for any $E_{ij} \in \mathbb{E}_{m,n}$, we can write $T(E_{ij}) = b_{ij}E_{\sigma(i)\tau(j)}$ for some nonzero scalar $b_{ij} \in \mathbb{F}_+$. Now we claim that for all $i, l \in \{1,\dots,m\}$ and all $j, r \in \{1,\dots,n\}$,

$$\frac{b_{ij}}{b_{ir}} = \frac{b_{lj}}{b_{lr}}.$$

Consider a matrix $E = E_{ij} + E_{ir} + E_{lj} + E_{lr}$ with rank 1. Then we have

$$T(E) = b_{ij}E_{\sigma(i)\tau(j)} + b_{ir}E_{\sigma(i)\tau(r)} + b_{lj}E_{\sigma(l)\tau(j)} + b_{lk}E_{\sigma(l)\tau(r)}.$$

Since $T(E)$ has rank 1, it follows that $\frac{b_{ij}}{b_{ir}} = \frac{b_{lj}}{b_{lr}}$. Let $C \in \mathscr{M}_{m,m}(\mathbb{F}_+)$ and $D \in \mathscr{M}_{n,n}(\mathbb{F}_+)$ be diagonal matrices such that $c_{11} = 1$, $d_{11} = b_{11}$, $c_{ii} = \frac{b_{i1}}{b_{11}}$, and $d_{jj} = b_{1j}$ for all $i = 2,\dots,m$ and $j = 2,\dots,n$. Then $b_{ij} = c_{ii}d_{jj}$ for all $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$.

Let $A = [a_{ij}]$ be any $m \times n$ matrix in $\mathscr{M}_{m,n}(\mathbb{F}_+)$. Then we have

$$\begin{aligned} T(A) &= T\biggl(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}E_{ij}\biggr) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}T(E_{ij}) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ij}E_{\sigma(i)\tau(j)} = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ii}a_{ij}E_{\sigma(i)\tau(j)}d_{jj} \\ &= CPAQD. \end{aligned}$$

Since $CP = U$ is an $m \times m$ invertible matrix and $QD = V$ is an $n \times n$ invertible matrix, it follows that T is a (U, V)-operator.

Case b). Then $m = n$ and $T^*(R_i) = C_{\sigma(i)}$ and $T^*(C_j) = R_{\tau(j)}$ for all i, j, where σ and τ are permutations of $\{1, \ldots, m\}$. By an argument similar to Case a), we obtain that $T(A)$ is of the form $T(A) = CPA^tQD$. Thus T is a (U, V)-operator. $\square$

We say that a linear operator T on $\mathscr{M}_{m,n}(\mathbb{F}_+)$ *strongly preserves* perimeter k of rank-1 matrices if $P(T(A)) = k$ if and only if $P(A) = k$.

Consider a linear operator T on $\mathscr{M}_{2,2}(\mathbb{F}_+)$ defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a + b + c + d)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then T preserves both rank and perimeter 2 of rank-1 matrices but does not strongly preserve perimeter 2, since $T\left(\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\right) = \begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix}$ with $P\left(\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\right) = 4$ but $P\left(\begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix}\right) = 2$.

Theorem 2.8. *Let T be a linear operator on $\mathscr{M}_{m,n}(\mathbb{F}_+)$. Then T preserves both rank and perimeter of rank-1 matrices if and only if it strongly preserves perimeter 2 and preserves perimeter 3 of rank-1 matrices.*

Proof. Suppose T strongly preserves perimeter 2 and preserves perimeter 3 of rank-1 matrices. Then T maps each row of a matrix into a nonzero scalar multiple of a row or a column (if $m = n$). Since T strongly preserves perimeter 2, T maps each cell onto a nonzero scalar multiple of a cell. This means that the corresponding mapping T' is a bijection. Thus T preserves both rank and perimeter of rank-1 matrices by a similar method as in the proof of Theorem 2.7.

The converse is immediate. $\square$

Theorem 2.9. *Let T be a linear operator on $\mathscr{M}_{m,n}(\mathbb{F}_+)$ that preserves the rank of rank-1 matrices. Then T preserves the perimeter of rank-1 matrices if and only if it strongly preserves perimeter 2 of rank-1 matrices.*

Proof. Suppose T strongly preserves perimeter 2 of rank-1 matrices. Then T maps each cell onto a nonzero scalar multiple of a cell. Thus T' is a bijection. Since T preserves rank 1, it maps a row of a matrix into a row or a column (if $m = n$). Thus T preserves both rank and perimeter of rank-1 matrices by similar methods to the proof of Theorem 2.7.

The converse is immediate. $\square$

Thus we have characterizations of the linear operators that preserve both rank and perimeter of rank-1 matrices over semifields.

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