Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 56 (2006), No. 2, 389–402

Persistent URL: http://dml.cz/dmlcz/128074

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Czechoslovak Mathematical Journal, 56 (131) (2006), 389–402

ON SOME CONSTRUCTIONS OF ALGEBRAIC OBJECTS

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(Received August 19, 2003)

Abstract. Mono-unary algebras may be used to construct homomorphisms, subalgebras, and direct products of algebras of an arbitrary type.

Keywords: algebra, homomorphism, subalgebra, direct product, variety, mono-unary algebra

MSC 2000: 03C05, 08A60, 08A30, 08A35, 08B99

1. Motivation

In [6], we have presented a construction of all homomorphisms of an algebra into a similar algebra. The main instrument of this construction is a mono-unary algebra: The given algebras $\mathbf{A}$, $\mathbf{A}'$ are replaced by mono-unary algebras $\mathbf{M}$, $\mathbf{M}'$ and there exists a bijection of the set of all homomorphisms of the algebra $\mathbf{A}$ into $\mathbf{A}'$ onto the set of homomorphisms of the algebra $\mathbf{M}$ onto $\mathbf{M}'$ where these homomorphisms have a particular form. The significance of this construction consists in the fact that the construction of all homomorphisms of a mono-unary algebra into an algebra of the same type is known; see [3], [4], [5]. Hence, we find all homomorphisms of the algebra $\mathbf{M}$ into $\mathbf{M}'$, reject all of them that are not of the particular form and find, to any homomorphism of the particular form, the corresponding homomorphism of the algebra $\mathbf{A}$ into $\mathbf{A}'$.

In the present paper, we demonstrate that mono-unary algebras may be used also to construct subalgebras and direct products of arbitrary algebras.

2. Mono-unary algebras

In what follows we denote by $\mathbb{N}$ the set of all natural numbers, i.e. the set of all nonnegative integers.

We now repeat the fundamental information concerning mono-unary algebras (see [3], [4], [5]).

A *mono-unary algebra* is a nonempty set A with a unary operation, i.e. with a mapping o of the set A into itself. We will denote it by $\mathbf{A} = (A, o)$. If n is a natural number, then the nth iteration of o will be denoted by o^n. For $x, y \in A$, we put $(x, y) \in e$ if there exist natural numbers m, n such that $o^m(x) = o^n(y)$. It is easy to see that e is an equivalence on the set A; if $B \in A/e$, then the restriction $o\lceil B$ of o to B is a unary operation on B, which means that $(B, o\lceil B)$ is a subalgebra of $\mathbf{A}$. It will be called a *component* of $\mathbf{A}$. It is easy to see that the algebra $\mathbf{A}$ is a union of its components, i.e., if $\mathbf{B}_i = (B_i, o_i)$ $(i \in I)$ are all components of the algebra $\mathbf{A} = (A, o)$, then $A = \bigcup\limits_{i \in I} A_i$, $o = \bigcup\limits_{i \in I} o_i$. A mono-unary algebra $\mathbf{A}$ is said to be *connected* if it has exactly one component.

Let $\mathbf{A} = (A, o)$ be a mono-unary algebra. An element $x \in A$ is said to have *property* (p) if there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of elements in A such that $x_0 = x$ and $o(x_{n+1}) = x_n$ for any $n \in \mathbb{N}$. We denote by B_∞ the set of all elements in A that have property (p). We put $B_0 = \{x \in A;\ o^{-1}(x) = \emptyset\}$; if $\alpha > 0$ is an ordinal and the set B_λ has been defined for any $\lambda < \alpha$, we put

$$B_\alpha = \Big\{x \in (A - B_\infty) - \bigcup_{\lambda < \alpha} B_\lambda;\ o^{-1}(x) \subseteq \bigcup_{\lambda < \alpha} B_\lambda\Big\}.$$

There exists a least ordinal ϑ such that $B_\vartheta = \emptyset$. Then $A = B_\infty \cup \bigcup\limits_{\lambda < \vartheta} B_\lambda$ with disjoint summands, i.e. the sets on the right side are mutually disjoint. We put $S(x) = \alpha$ if $x \in B_\alpha$ where either $\alpha = \infty$ or $\alpha < \vartheta$; the symbol $S(x)$ is said to be the *grade* of the element $x \in A$. The value ∞ will be regarded as greater than any ordinal.

An element x of a connected mono-unary algebra $\mathbf{A} = (A, o)$ is said to be *cyclic* if there exists an integer $n \in \mathbb{N}$, $n > 0$ such that $o^n(x) = x$. The set of all cyclic elements is called the *cycle* of $\mathbf{A}$ and is denoted by Z. It is easy to see that Z is a finite set; we denote its cardinality by R.

Hence, we have defined the objects S, Z, R for a connected mono-unary algebra $\mathbf{A}$; if it is necessary to stress that these objects are defined for the algebra $\mathbf{A}$, we write $S_{\mathbf{A}}$, $Z_{\mathbf{A}}$, $R_{\mathbf{A}}$ for S, Z, R, respectively.

Let $\mathbf{A} = (A, o)$ be a connected mono-unary algebra and $x_0 \in A$ an element. We put $P^{(0)}(x_0) = \{o^n(x_0);\ n \in \mathbb{N}\}$, $P^{(i+1)}(x_0) = o^{-1}(P^{(i)}(x_0)) - \bigcup\limits_{0 \leqslant k \leqslant i} P^{(k)}(x_0)$ for any integer $i \in \mathbb{N}$. It is easy to see that $A = \bigcup\limits_{i \in \mathbb{N}} P^{(i)}(x_0)$ with disjoint summands.

Let $\mathbf{A} = (A, o)$, $\mathbf{A}' = (A', o')$ be connected mono-unary algebras. The algebra $\mathbf{A}'$ is said to be *admissible* to $\mathbf{A}$ if one of the following conditions is satisfied:

(i) $R_{\mathbf{A}'} \neq 0$ and $R_{\mathbf{A}'}$ divides $R_{\mathbf{A}}$.

(ii) $R_{\mathbf{A}'} = 0 = R_{\mathbf{A}}$ and there exist elements $x_0 \in A$, $x'_0 \in A'$ such that $S_{\mathbf{A}}(o^n(x_0)) \leqslant S_{\mathbf{A}'}((o')^n(x'_0))$ holds for any $n \in \mathbb{N}$.

If (ii) is satisfied, we have $x_0 \in A$, $x'_0 \in A'$ such that $S_{\mathbf{A}}(o^n(x_0)) \leqslant S_{\mathbf{A}'}((o')^n(x'_0))$ holds for any $n \in \mathbb{N}$; the pair (x_0, x'_0) will be called *generating*. If (i) is satisfied, we choose $x_0 \in A$, $x'_0 \in Z_{\mathbf{A}'}$ arbitrarily; clearly, $S_{\mathbf{A}}(o^n(x_0)) \leqslant S_{\mathbf{A}'}((o')^n(x'_0))$ holds for any $n \in \mathbb{N}$; also this pair (x_0, x'_0) will be called *generating*.

These concepts can be used when we construct homomorphisms of mono-unary algebras. We mention that a homomorphism of a mono-unary algebra $\mathbf{A} = (A, o)$ into a mono-unary algebra $\mathbf{A}' = (A', o')$ is a mapping h of the set A into A' such that $h(o(x)) = o'(h(x))$ holds for any $x \in A$.

Construction 1.

Let $\mathbf{A} = (A, o)$, $\mathbf{A}' = (A', o')$ be connected mono-unary algebras where $\mathbf{A}'$ is admissible to $\mathbf{A}$. Let (x_0, x'_0) be a generating pair of elements.

Put $h(o^n(x_0)) = (o')^n(x'_0)$ for any $n \in \mathbb{N}$, which defines a mapping of the set $P^{(0)}(x_0)$ into A' such that $S_{\mathbf{A}}(x) \leqslant S_{\mathbf{A}'}(h(x))$ holds for any $x \in P^{(0)}(x_0)$.

Let $n \in \mathbb{N}$ and suppose that $h(x)$ has been defined for any $x \in \bigcup\limits_{0 \leqslant k \leqslant n} P^{(k)}(x_0)$ in such a way that $S_{\mathbf{A}}(x) \leqslant S_{\mathbf{A}'}(h(x))$.

If $y \in P^{(n+1)}(x_0)$ is arbitrary, then $o(y) = x \in P^{(n)}(x_0)$ and $x' = h(x) \in A'$ has been defined in such a way that $S_{\mathbf{A}}(x) \leqslant S_{\mathbf{A}'}(x')$. Then there exists $y' \in (o')^{-1}(x')$ such that $S_{\mathbf{A}}(y) \leqslant S_{\mathbf{A}'}(y')$. We put $h(y) = y'$. In this way, h is extended to the set $\bigcup\limits_{0 \leqslant k \leqslant n+1} P^{(k)}(x_0)$ in such a way that $S_{\mathbf{A}}(t) \leqslant S_{\mathbf{A}'}(h(t))$ holds for any $t \in \bigcup\limits_{0 \leqslant k \leqslant n+1} P^{(k)}(x_0)$.

By induction, this mapping can be extended to the set A.

The constructed mapping is a homomorphism of the algebra $\mathbf{A}$ into $\mathbf{A}'$ and any homomorphism of the algebra $\mathbf{A}$ into $\mathbf{A}'$ can be constructed in this way.

It can be proved that no homomorphism of a connected mono-unary algebra $\mathbf{A}$ into a connected mono-unary algebra $\mathbf{A}'$ exists if $\mathbf{A}'$ is not admissible to $\mathbf{A}$.

Construction 2.

Let $\mathbf{A} = (A, o)$, $\mathbf{A}' = (A', o')$ be mono-unary algebras. Denote by $\{(A_i, o_i);\ i \in I\}$ the system of all components of $\mathbf{A}$, by $\{(A'_j, o'_j);\ j \in J\}$ the system of all components of $\mathbf{A}'$.

Let H be a mapping assigning to any component (A_i, o_i) of (A, o) an admissible component (A'_j, o'_j) of (A', o').

Let h_i be a mapping of the set A_i into A'_j constructed by Construction 1 *for the algebras (A_i, o_i), $(A'_j, o'_j) = H((A_i, o_i))$ and for a generating pair. Construct h_i for any $i \in I$. Define $h = \bigcup_{i \in I} h_i$.*

Then h is a homomorphism of the algebra $\mathbf{A}$ into $\mathbf{A}'$ and any homomorphism of the algebra $\mathbf{A}$ into $\mathbf{A}'$ may be constructed in this way.

Hence, we are able to construct all homomorphisms of a mono-unary algebra into an algebra of the same type.

In what follows, we will meet mono-unary algebras with some nullary operations that are called *constants*. An algebra of this type will be denoted by $(A, (c_\iota)_{1 \leqslant \iota < \alpha}, o)$ where (A, o) is a mono-unary algebra, α an ordinal and $c_\iota \in A$ holds for any ι with $1 \leqslant \iota < \alpha$.

If $(A, (c_\iota)_{1 \leqslant \iota < \alpha}, o)$, $(A', (c'_\iota)_{1 \leqslant \iota < \alpha}, o')$ are algebras of this type, then h is a homomorphism of the first algebra into the latter, if it is a homomorphism of the mono-unary algebra (A, o) into (A', o') and if $h(c_\iota) = c'_\iota$ holds for any ι with $1 \leqslant \iota < \alpha$. Hence, our constructions make it possible to construct all homomorphisms of the first algebra into the latter: We construct all homomorphisms of the algebra (A, o) into (A', o') and then we reject any constructed mapping h for which there exists ι such that $1 \leqslant \iota < \alpha$, $h(c_\iota) \neq c'_\iota$.

It is very simple to recognize subalgebras of a mono-unary algebra with some constants. If M is the carrier of a subalgebra, then any constant is an element of M and $a \in M$ implies $o^n(a) \in M$ for any nonnegative integer n.

Also the construction of a direct product of some mono-unary algebras with some constants is simple. Let $\mathbf{A}_\kappa = (A_\kappa, (c_{\kappa\iota})_{1 \leqslant \iota < \alpha}, o_\kappa)$ be a mono-unary algebra with some constants for any ordinal κ with the property $1 \leqslant \kappa < \delta$ where $\delta > 1$ is an ordinal. Then the *direct product* $\mathbf{X}_{1 \leqslant \kappa < \delta} \mathbf{A}_\kappa$ of these algebras is a mono-unary algebra with the carrier $\mathbf{X}_{1 \leqslant \kappa < \delta} A_\kappa$, which is the cartesian product of the sets A_κ. Hence any element of this carrier is of the form $(a_\kappa)_{1 \leqslant \kappa < \delta}$ with $a_\kappa \in A_\kappa$. Any constant can be expressed in the form $(c_{\kappa\iota})_{1 \leqslant \kappa < \delta}$ where $1 \leqslant \iota < \alpha$. The unary operation o assigns the value $(o_\kappa(a_\kappa))_{1 \leqslant \kappa < \delta}$ to any element $(a_\kappa)_{1 \leqslant \kappa < \delta}$ of the carrier.

3. Homomorphisms

Basic information on algebras can be found, e.g., in [1] and [2].

Let A be a nonempty set, $\alpha \geqslant 1$, β ordinals; we suppose that β is infinite. Let $c_\iota \in A$ hold for any ordinal ι with $1 \leqslant \iota < \alpha$; furthermore, for any ordinal ι such that $1 \leqslant \iota < \beta$, let o_ι be an operation of arity $r(o_\iota) \geqslant 1$ on the set A. Hence, A may be considered to be the carrier of an algebra where any c_ι $(1 \leqslant \iota < \alpha)$ is a *nullary* operation (= constant) and any o_ι $(1 \leqslant \iota < \beta)$ a *nonnullary* operation. This algebra will be denoted by

$$\mathbf{A} = (A, (c_\iota)_{1\leqslant\iota<\alpha}, (o_\iota)_{1\leqslant\iota<\beta});$$

it has a particular property: The family $(o_\iota)_{1\leqslant\iota<\beta}$ of nonnullary operations is infinite. An algebra of this form will be called a *robust* algebra or, shortly, an *R-algebra* in what follows.

We may suppose, without loss of generality, that $1 \leqslant \iota < \iota' < \beta$ implies $r(o_\iota) \leqslant r(o_{\iota'})$. If $\alpha = 1$, we write $(A, (o_\iota)_{1\leqslant\iota<\beta})$ for $(A, (c_\iota)_{1\leqslant\iota<1}, (o_\iota)_{1\leqslant\iota<\beta})$.

Suppose that $\mathbf{A}' = (A', (c'_\iota)_{1\leqslant\iota<\alpha}, (o'_\iota)_{1\leqslant\iota<\beta})$ is a similar R-algebra, i.e., $r(o_\iota) = r(o'_\iota)$ holds for any ordinal ι with $1 \leqslant \iota < \beta$. We will write simply $r(\iota)$ for $r(o_\iota)$ in what follows. A mapping h of the set A into A' is said to be a *homomorphism* of the algebra $\mathbf{A}$ into $\mathbf{A}'$ if $h(c_\iota) = c'_\iota$ for any ordinal ι with $1 \leqslant \iota < \alpha$ and $h(o_\iota(x_1, \ldots, x_{r(\iota)})) = o'_\iota(h(x_1), \ldots, h(x_{r(\iota)}))$ holds for any ordinal ι with the property $1 \leqslant \iota < \beta$ and for any elements $x_1, \ldots, x_{r(\iota)}$ in A.

Let β be an ordinal. In what follows, the following symbol will be useful: We denote by β^- the set of ordinals $\{\iota;\ 1 \leqslant \iota < \beta\}$. If A is a set, then the set of all sequences $(a_\iota)_{1\leqslant\iota<\beta}$, where $a_\iota \in A$ for any ordinal ι with the property $1 \leqslant \iota < \beta$, is denoted by A^{β^-}. Furthermore, if $c \in A$, then $(c)_{1\leqslant\iota<\beta}$ denotes the sequence of the type β^- where any member equals c.

We now assign a mono-unary algebra with some nullary operations to any R-algebra of the above described type: Let $\mathbf{A} = (A, (c_\iota)_{1\leqslant\iota<\alpha}, (o_\iota)_{1\leqslant\iota<\beta})$ be an R-algebra as above. We put

$$\mathbf{U}(\mathbf{A}) = (A^{\beta^-}, ((c_\iota)_{1\leqslant\gamma<\beta})_{1\leqslant\iota<\alpha}, \mathbf{u}[(o_\iota)_{1\leqslant\iota<\beta}]);$$

here $(c_\iota)_{1\leqslant\gamma<\beta}$ denotes the sequence of the type β^- where any member equals c_ι according to the above introduced agreement concerning sequences whose all members are equal. Furthermore, we define

$$(*) \qquad \mathbf{u}[(o_\iota)_{1\leqslant\iota<\beta}]((x_\iota)_{1\leqslant\iota<\beta}) = (o_\iota(x_1, \ldots, x_{r(\iota)}))_{1\leqslant\iota<\beta}$$

for any sequence $(x_\iota)_{1\leqslant\iota<\beta}$ of the type β^- formed of elements of the set A. The symbol $\mathbf{u}[(o_\iota)_{1\leqslant\iota<\beta}]$ of this unary operation reflects the fact that the operation is constructed on the basis of the operations o_ι where $1 \leqslant \iota < \beta$.

Hence, $\mathbf{U}(\mathbf{A})$ is a mono-unary algebra with the unary operation $\mathbf{u}[(o_\iota)_{1\leqslant\iota<\beta}]$ and with some nullary operations.

A mapping f of the set A^{β^-} into $(A')^{\beta^-}$ is said to be *decomposable* if there exists a mapping h of the set A into A' such that $f((x_\iota)_{1\leqslant\iota<\beta}) = (h(x_\iota))_{1\leqslant\iota<\beta}$ for any element $(x_\iota)_{1\leqslant\iota<\beta} \in A^{\beta^-}$. In such a case, we put $f = h^{\times\beta^-}$. Clearly, h is a surjection of the set A onto A' if and only if $h^{\times\beta^-}$ is a surjection of the set A^{β^-} onto $(A')^{\beta^-}$.

Theorem 1. *Let* $\mathbf{A} = (A, (c_\iota)_{1\leqslant\iota<\alpha}, (o_\iota)_{1\leqslant\iota<\beta})$, $\mathbf{A}' = (A', (c'_\iota)_{1\leqslant\iota<\alpha}, (o'_\iota)_{1\leqslant\iota<\beta})$ *be similar R-algebras, h a mapping of the set A into A′. Then the following assertions are equivalent.*

(i) *h is a homomorphism of the algebra* $\mathbf{A}$ *into* $\mathbf{A}'$.

(ii) $h^{\times\beta^-}$ *is a homomorphism of the algebra* $\mathbf{U}(\mathbf{A})$ *into* $\mathbf{U}(\mathbf{A}')$.

Proof. If (i) holds, then $h^{\times\beta^-}$ is a mapping of the set A^{β^-} into $(A')^{\beta^-}$. Since $h(c_\iota) = c'_\iota$ for any ι with $1 \leqslant \iota < \alpha$, we obtain $h^{\times\beta^-}((c_\iota)_{1\leqslant\gamma<\beta}) = (h(c_\iota))_{1\leqslant\gamma<\beta} = (c'_\iota)_{1\leqslant\gamma<\beta}$. Furthermore, for any $(x_\iota)_{1\leqslant\iota<\beta} \in A^{\beta^-}$, we obtain

$$\begin{aligned} h^{\times\beta^-}(\mathbf{u}[(o_\iota)_{1\leqslant\iota<\beta}]((x_\iota)_{1\leqslant\iota<\beta})) &= h^{\times\beta^-}((o_\iota(x_1,\ldots,x_{r(\iota)}))_{1\leqslant\iota<\beta}) \\ &= (h(o_\iota(x_1,\ldots,x_{r(\iota)})))_{1\leqslant\iota<\beta} \\ &= (o'_\iota(h(x_1),\ldots,h(x_{r(\iota)})))_{1\leqslant\iota<\beta} \\ &= \mathbf{u}[(o'_\iota)_{1\leqslant\iota<\beta}]((h(x_\iota))_{1\leqslant\iota<\beta}) \\ &= \mathbf{u}[(o'_\iota)_{1\leqslant\iota<\beta}](h^{\times\beta^-}((x_\iota)_{1\leqslant\iota<\beta})). \end{aligned}$$

Hence $h^{\times\beta^-}$ is a homomorphism of the mono-unary algebra with some constants $(A^{\beta^-}, ((c_\iota)_{1\leqslant\gamma<\beta})_{1\leqslant\iota<\alpha}, \mathbf{u}[(o_\iota)_{1\leqslant\iota<\beta}])$ into $((A')^{\beta^-}, ((c'_\iota)_{1\leqslant\gamma<\beta})_{1\leqslant\iota<\alpha}, \mathbf{u}[(o'_\iota)_{1\leqslant\iota<\beta}])$. Thus (ii) holds.

If (ii) holds, there exists a mapping h of the set A into A' such that

$$h^{\times\beta^-}((c_\iota)_{1\leqslant\gamma<\beta}) = (c'_\iota)_{1\leqslant\gamma<\beta}$$

for any ordinal ι with $1 \leqslant \iota < \alpha$, which means $h(c_\iota) = c'_\iota$ for any ι with the property $1 \leqslant \iota < \alpha$. Furthermore, for any $(x_\iota)_{1\leqslant\iota<\beta} \in A^{\beta^-}$, we obtain

$$\begin{aligned} (h(o_\iota(x_1,\ldots,x_{r(\iota)})))_{1\leqslant\iota<\beta} &= h^{\times\beta^-}((o_\iota(x_1,\ldots,x_{r(\iota)}))_{1\leqslant\iota<\beta}) \\ &= h^{\times\beta^-}(\mathbf{u}[(o_\iota)_{1\leqslant\iota<\beta}]((x_\iota)_{1\leqslant\iota<\beta})) \\ &= \mathbf{u}[(o'_\iota)_{1\leqslant\iota<\beta}](h^{\times\beta^-}((x_\iota)_{1\leqslant\iota<\beta})) \\ &= \mathbf{u}[(o'_\iota)_{1\leqslant\iota<\beta}]((h(x_\iota))_{1\leqslant\iota<\beta}) \\ &= (o'_\iota(h(x_1),\ldots,h(x_{r(\iota)})))_{1\leqslant\iota<\beta}. \end{aligned}$$

It follows that $o'_\iota(h(x_1),\ldots,h(x_{r(\iota)})) = h(o_\iota(x_1,\ldots,x_{r(\iota)}))$ for any ordinal ι with the property $1 \leqslant \iota < \beta$ and any elements $x_1,\ldots,x_{r(\iota)}$ in A. Thus, h is a homomorphism of the algebra $\mathbf{A}$ into $\mathbf{A}'$ and (i) holds. □

As a consequence, we obtain

Construction 3.

Let $\mathbf{A} = (A, (c_\iota)_{1\leqslant\iota<\alpha}, (o_\iota)_{1\leqslant\iota<\beta})$, $\mathbf{A}' = (A', (c'_\iota)_{1\leqslant\iota<\alpha}, (o'_\iota)_{1\leqslant\iota<\beta})$ *be similar R-algebras.*

Construct the algebras $\mathbf{U}(\mathbf{A})$, $\mathbf{U}(\mathbf{A}')$.

Construct all homomorphisms of the algebra $\mathbf{U}(\mathbf{A})$ *into* $\mathbf{U}(\mathbf{A}')$ *according to Section* 2.

Preserve all homomorphisms that are decomposable.

For any decomposable homomorphism $f = h^{\times\beta^-}$ *of the algebra* $\mathbf{U}(\mathbf{A})$ *into* $\mathbf{U}(\mathbf{A}')$ *construct the mapping* h.

Then h *is a homomorphism of the algebra* $\mathbf{A}$ *into* $\mathbf{A}'$ *and any homomorphism of* $\mathbf{A}$ *into* $\mathbf{A}'$ *may be constructed in this way.*

Remark 1. In [6], similar constructions were described for algebras that are not robust. The constructions of [6] are effective for finite algebras.

Example 1. Let $A = \mathbb{N} - \{0\}$. For any $i \in \mathbb{N} - \{0\}$ define the unary operation $o_i(x) = i$ for any $x \in A$. We denote the least infinite ordinal by ω_0. Then $(A, (o_i)_{1 \leqslant i < \omega_0})$ is an R-algebra with infinitely many unary operations. We find all endomorphisms of this algebra using the above presented construction.

The unary operation $\mathbf{u}[(o_i)_{1 \leqslant i < \omega_0}]$ is defined as follows:

$$\mathbf{u}[(o_i)_{1 \leqslant i < \omega_0}]((x_i)_{1 \leqslant i < \omega_0}) = (o_i(x_1))_{1 \leqslant i < \omega_0} = (i)_{1 \leqslant i < \omega_0} \text{ for any } (x_i)_{1 \leqslant i < \omega_0} \in A^{\omega_0^-}.$$

This means that the mono-unary algebra $(A^{\omega_0^-}, \mathbf{u}[(o_i)_{1 \leqslant i < \omega_0}])$ has exactly one component and this component has a cycle; this cycle is formed by the element $(i)_{1 \leqslant i < \omega_0}$. It follows that $f((i)_{1 \leqslant i < \omega_0}) = (i)_{1 \leqslant i < \omega_0}$ holds for any endomorphism f of the last algebra. If f is decomposable, there exists a mapping h of A into itself such that $f = h^{\times\beta^-}$; therefore $(i)_{1 \leqslant i < \omega_0} = f((i)_{1 \leqslant i < \omega_0}) = (h(i))_{1 \leqslant i < \omega_0}$, which entails $h(i) = i$ for any i with $1 \leqslant i < \omega_0$. Hence $f((x_i)_{1 \leqslant i < \omega_0}) = (h(x_i))_{1 \leqslant i < \omega_0} = (x_i)_{1 \leqslant i < \omega_0}$ for any element $(x_i)_{1 \leqslant i < \omega_0} \in A^{\omega_0^-}$, which implies that the identity mapping on the set $A^{\omega_0^-}$ is the only decomposable endomorphism of the algebra $(A^{\omega_0^-}, \mathbf{u}[(o_i)_{1 \leqslant i < \omega_0^-}])$ and that the identity mapping on A is the only endomorphism of the algebra $(A, (o_i)_{1 \leqslant i < \omega_0})$.

Lemma 1. *Let* $\mathbf{A}$, $\mathbf{A}'$ *be similar R-algebras. If* $\mathbf{U}(\mathbf{A}) = \mathbf{U}(\mathbf{A}')$, *then* $\mathbf{A} = \mathbf{A}'$.

Proof. Put $\mathbf{A} = (A, (c_\iota)_{1 \leqslant \iota < \alpha}, (o_\iota)_{1 \leqslant \iota < \beta})$, $\mathbf{A}' = (A', (c'_\iota)_{1 \leqslant \iota < \alpha}, (o'_\iota)_{1 \leqslant \iota < \beta})$. If $\mathbf{A} \neq \mathbf{A}'$, then the following cases can occur:

(1) $A \neq A'$. Then, clearly, $A^{\beta^-} \neq (A')^{\beta^-}$ and, therefore, $\mathbf{U}(\mathbf{A}) \neq \mathbf{U}(\mathbf{A}')$.

(2) $A = A'$ and there exists an ordinal ι_0 such that $1 \leqslant \iota_0 < \alpha$, $c_{\iota_0} \neq c'_{\iota_0}$. Then $(c_{\iota_0})_{1 \leqslant \iota < \beta} \neq (c'_{\iota_0})_{1 \leqslant \iota < \beta}$ and, hence, $\mathbf{U}(\mathbf{A}) \neq \mathbf{U}(\mathbf{A}')$.

(3) $A = A'$, $c_\iota = c'_\iota$ for any ordinal ι with the property $1 \leqslant \iota < \alpha$ and there exist an ordinal ι_0 such that $1 \leqslant \iota_0 < \beta$ and some elements $x_1, \ldots, x_{r(\iota_0)}$ in the set $A = A'$ such that $o_{\iota_0}(x_1, \ldots, x_{r(\iota_0)}) \neq o'_{\iota_0}(x_1, \ldots, x_{r(\iota_0)})$. If we put $x_\gamma = x_{r(\iota_0)}$

for any ordinal γ with the property $r(\iota_0) < \gamma < \beta$, then $\mathbf{u}[(o_\iota)_{1\leqslant\iota<\beta}]((x_\iota)_{1\leqslant\iota<\beta}) \neq \mathbf{u}[(o'_\iota)_{1\leqslant\iota<\beta}]((x_\iota)_{1\leqslant\iota<\beta})$ because the ι_0th coordinates of these elements are different. It follows that $\mathbf{U}(\mathbf{A}) \neq \mathbf{U}(\mathbf{A}')$.

Hence $\mathbf{A} \neq \mathbf{A}'$ implies $\mathbf{U}(\mathbf{A}) \neq \mathbf{U}(\mathbf{A}')$ and, therefore, $\mathbf{U}(\mathbf{A}) = \mathbf{U}(\mathbf{A}')$ entails $\mathbf{A} = \mathbf{A}'$. □

Example 2. Let $\mathbf{A}$ be an R-algebra.

If $\mathbf{U}(\mathbf{A})$ is known, then $\mathbf{A} = (A, (c_\iota)_{1\leqslant\iota<\alpha}, (o_\iota)_{1\leqslant\iota<\beta})$ can be constructed as follows:

Let $(a_\iota)_{1\leqslant\iota<\beta} \in A^{\beta^-}$ be arbitrary. Then

$$A = \{x_1;\ (x_\iota)_{1\leqslant\iota<\beta} \in A^{\beta^-},\ x_\iota = a_\iota \ \text{ for } \ \iota > 1\}.$$

A constant of the algebra $\mathbf{U}(\mathbf{A})$ is a sequence of the type β^- where all members have the same value c_ι for some ordinal ι with $1 \leqslant \iota < \alpha$; hence, all constants c_ι can be easily constructed.

If ι_0 is an ordinal with the property $1 \leqslant \iota_0 < \beta$ and $x_1, \ldots, x_{r(\iota_0)} \in A$ are arbitrary, then $o_{\iota_0}(x_1, \ldots, x_{r(\iota_0)})$ is the ι_0th member of the sequence $\mathbf{u}[(o_\iota)_{1\leqslant\iota<\beta}]((x_\iota)_{1\leqslant\iota<\beta})$ where $x_\gamma = x_{r(\iota_0)}$ for any ordinal γ with the property $r(\iota_0) < \gamma < \beta$.

4. Subalgebras

Theorem 2. *Let $\mathbf{A} = (A, (c_\iota)_{1\leqslant\iota<\alpha}, (o_\iota)_{1\leqslant\iota<\beta})$ be an R-algebra, $\emptyset \neq M \subseteq A$ a set. Then the following two assertions are equivalent.*

(i) *M is the carrier of a subalgebra of $\mathbf{A}$.*

(ii) *M^{β^-} is the carrier of a subalgebra of the algebra $\mathbf{U}(\mathbf{A})$.*

P r o o f. If (i) holds, $c_\iota \in M$ for any ι with the property $1 \leqslant \iota < \alpha$, which implies $(c_\iota)_{1\leqslant\gamma<\beta} \in M^{\beta^-}$ for any ordinal ι with $1 \leqslant \iota < \alpha$. Furthermore, if $(x_\iota)_{1\leqslant\iota<\beta} \in M^{\beta^-}$, then $o_\iota(x_1, \ldots, x_{r(\iota)}) \in M$ for any ordinal ι with the property $1 \leqslant \iota < \beta$, which implies $\mathbf{u}[((o_\iota)_{1\leqslant\iota<\beta}]((x_\iota)_{1\leqslant\iota<\beta}) = (o_\iota(x_1, \ldots, x_{r(\iota)}))_{1\leqslant\iota<\beta} \in M^{\beta^-}$. Hence M^{β^-} is the carrier of a subalgebra of the algebra $(A^{\beta^-}, ((c_\iota)_{1\leqslant\gamma<\beta})_{1\leqslant\iota<\alpha}, \mathbf{u}[(o_\iota)_{1\leqslant\iota<\beta}]) = \mathbf{U}(\mathbf{A})$ and (ii) holds.

If (ii) is satisfied and $1 \leqslant \iota < \alpha$, then $(c_\iota)_{1\leqslant\gamma<\beta} \in M^{\beta^-}$, which implies $c_\iota \in M$. Let ι be an ordinal such that $1 \leqslant \iota < \beta$ and $x_1, \ldots, x_{r(\iota)}$ arbitrary elements in M. Put $x_\gamma = x_{r(\iota)}$ for any ordinal γ with the property $r(\iota) < \gamma < \beta$. Then $(x_\gamma)_{1\leqslant\gamma<\beta} \in M^{\beta^-}$ and, therefore, $(o_\gamma(x_1, \ldots, x_{r(\gamma)}))_{1\leqslant\gamma<\beta} = \mathbf{u}[(o_\gamma)_{1\leqslant\gamma<\beta}]((x_\gamma)_{1\leqslant\gamma<\beta}) \in M^{\beta^-}$, which entails $o_\iota(x_1, \ldots, x_{r(\iota)}) \in M$. Thus M is the carrier of a subalgebra of the algebra $(A, (c_\iota)_{1\leqslant\iota<\alpha}, (o_\iota)_{1\leqslant\iota<\beta}) = \mathbf{A}$ and (i) holds. □

Example 3. Consider the R-algebra $(A,(o_i)_{1\leqslant i<\omega_0})$ presented in Example 1. We have proved that the element $(i)_{1\leqslant i<\omega_0}$ forms the only cycle in the algebra $(A^{\omega_0^-},\mathbf{u}[(o_i)_{1\leqslant i<\omega_0}])$ and that $\mathbf{u}[(o_i)_{1\leqslant i<\omega_0}]((x_i)_{1\leqslant i<\omega_0}) = (i)_{1\leqslant i<\omega_0}$ holds for any $(x_i)_{1\leqslant i<\omega_0} \in A^{\omega_0^-}$.

Let $\emptyset \neq M \subseteq A$ be the carrier of a subalgebra of the algebra $(A,(o_i)_{1\leqslant i<\omega_0})$. Let $m \in M$ be arbitrary. Put $x_i = m$ for any i with $1 \leqslant i < \omega_0$. Then $(x_i)_{1\leqslant i<\omega_0} \in M^{\omega_0^-}$ and, therefore, $(i)_{1\leqslant i<\omega_0} = \mathbf{u}[(o_i)_{1\leqslant i<\omega_0}]((x_i)_{1\leqslant i<\omega_0}) \in M^{\omega_0^-}$ because $M^{\omega_0^-}$ is the carrier of a subalgebra of the algebra $(A^{\omega_0^-},\mathbf{u}[(o_i)_{1\leqslant i<\omega_0}])$. It follows that $i \in M$ for any $i \in A = \mathbb{N} - \{0\}$. Hence $M = A$ and, therefore, the algebra $(A,(o_i)_{1\leqslant i<\omega_0})$ has no proper subalgebra.

As a consequence of the above considerations, we obtain

Construction 4.

Let $\mathbf{A} = (A,(c_\iota)_{1\leqslant\iota<\alpha},(o_\iota)_{1\leqslant\iota<\beta})$ *be an* R*-algebra.*

Construct the algebra $\mathbf{U}(\mathbf{A})$*.*

Construct all subalgebras of the algebra $\mathbf{U}(\mathbf{A})$*.*

Preserve all subalgebras whose carrier is of the form M^{β^-} *for some* $M \neq \emptyset$*.*

Then any constructed set M *is the carrier of a subalgebra of* $\mathbf{A}$ *and any subalgebra of* $\mathbf{A}$ *may be constructed in this way.*

5. Direct products

In what follows, we define direct products of algebras.

Let $\mathbf{A}_\kappa = (A_\kappa, o_\kappa)$ be an algebra with one operation o_κ of arity $r \geqslant 1$ for any ordinal κ with the property $1 \leqslant \kappa < \delta$ where $\delta > 1$ is an ordinal. Then the *direct product* $\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{A}_\kappa$ of these algebras is a similar algebra whose carrier equals $\mathbf{X}_{1\leqslant\kappa<\delta}A_\kappa$ and its operation $\mathbf{x}[(o_\kappa)_{1\leqslant\kappa<\delta}]$ is defined as follows:

$$(**)\qquad \begin{aligned}\mathbf{x}[(o_\kappa)_{1\leqslant\kappa<\delta}]&((a_{\kappa 1})_{1\leqslant\kappa<\delta},\dots,(a_{\kappa r})_{1\leqslant\kappa<\delta})\\ &= \mathbf{x}[(o_\kappa)_{1\leqslant\kappa<\delta}](((a_{\kappa i})_{1\leqslant\kappa<\delta})_{1\leqslant i\leqslant r})\\ &= (o_\kappa(a_{\kappa 1},\dots,a_{\kappa r}))_{1\leqslant\kappa<\delta}\end{aligned}$$

for any elements $(a_{\kappa 1})_{1\leqslant\kappa<\delta},\dots,(a_{\kappa r})_{1\leqslant\kappa<\delta}$ of the set $\mathbf{X}_{1\leqslant\kappa<\delta}A_\kappa$. The symbol $\mathbf{x}[(o_\kappa)_{1\leqslant\kappa<\delta}]$ for this operation reflects the fact that this operation is constructed on the basis of the operations o_κ where $1 \leqslant \kappa < \delta$.

If $r = 1$, we obtain

$$(***)\qquad \mathbf{x}[(o_\kappa)_{1\leqslant\kappa<\delta}]((a_\kappa)_{1\leqslant\kappa<\delta}) = (o_\kappa(a_\kappa))_{1\leqslant\kappa<\delta}$$

for any element $(a_\kappa)_{1\leqslant\kappa<\delta} \in \mathbf{X}_{1\leqslant\kappa<\delta}A_\kappa$. This is in accord with the definition quoted in Section 2.

The operator $\mathbf{x}$ can be used to express the structure of the algebra $\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{A}_\kappa$ where $\mathbf{A}_\kappa = (A_\kappa, (o_{\kappa\iota})_{1\leqslant\iota<\beta})$ and $o_{\kappa\iota}$ is an operation of arity $r(\iota) \geqslant 1$ on the set A_κ for any ordinal ι with the property $1 \leqslant \iota < \beta$. We put

$$\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{A}_\kappa = (\mathbf{X}_{1\leqslant\kappa<\delta}A_\kappa, (\mathbf{x}[(o_{\kappa\iota})_{1\leqslant\kappa<\delta}])_{1\leqslant\iota<\beta}).$$

For the general case $\mathbf{A}_\kappa = (A_\kappa, (c_{\kappa\iota})_{1\leqslant\iota<\alpha}, (o_{\kappa\iota})_{1\leqslant\iota<\beta})$ where any $c_{\kappa\iota}$ is a constant in A_κ for any ordinal ι with $1 \leqslant \iota < \alpha$, the product $\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{A}_\kappa$ will be defined by

$$\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{A}_\kappa = (\mathbf{X}_{1\leqslant\kappa<\delta}A_\kappa, ((c_{\kappa\iota})_{1\leqslant\kappa<\delta})_{1\leqslant\iota<\alpha}, (\mathbf{x}[(o_{\kappa\iota})_{1\leqslant\kappa<\delta}])_{1\leqslant\iota<\beta}).$$

Let $1 \leqslant \kappa < \delta$, $1 \leqslant \iota < \beta$ be ordinals, suppose $a_{\kappa\iota} \in A_\kappa$ for any κ and ι satisfying these conditions. For any $((a_{\kappa\iota})_{1\leqslant\iota<\beta})_{1\leqslant\kappa<\delta}$ put

$$b(((a_{\kappa\iota})_{1\leqslant\iota<\beta})_{1\leqslant\kappa<\delta}) = ((a_{\kappa\iota})_{1\leqslant\kappa<\delta})_{1\leqslant\iota<\beta}.$$

Clearly, the elements $a_{\kappa\iota}$ with $1 \leqslant \kappa < \delta$, $1 \leqslant \iota < \beta$ represent an infinite matrix. Furthermore, $((a_{\kappa\iota})_{1\leqslant\iota<\beta})_{1\leqslant\kappa<\delta}$ means that lines of this matrix were formed first and then a column of these lines was built up; $((a_{\kappa\iota})_{1\leqslant\kappa<\delta})_{1\leqslant\iota<\beta}$ represents a reverse procedure: first all columns were formed and then a line of these columns. Hence $((a_{\kappa\iota})_{1\leqslant\iota<\beta})_{1\leqslant\kappa<\delta}$, $((a_{\kappa\iota})_{1\leqslant\kappa<\delta})_{1\leqslant\iota<\beta}$ represent the same matrix in two different ways. It follows that b is a bijection of the set $\mathbf{X}_{1\leqslant\kappa<\delta}A_\kappa^{\beta^-}$ onto $(\mathbf{X}_{1\leqslant\kappa<\delta}A_\kappa)^{\beta^-}$.

Theorem 3. *Let $\mathbf{A}_\kappa = (A_\kappa, (c_{\kappa\iota})_{1\leqslant\iota<\alpha}, (o_{\kappa\iota})_{1\leqslant\iota<\beta})$ be similar R-algebras for any ordinal κ with the property $1 \leqslant \kappa < \delta$ where $\delta > 1$ is an ordinal. Then the mapping b is an isomorphism of the algebra $\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{U}(\mathbf{A}_\kappa)$ onto $\mathbf{U}(\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{A}_\kappa)$.*

Proof. (1) It follows from the definitions that $\mathbf{U}(\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{A}_\kappa)$ is an algebra with the carrier $(\mathbf{X}_{1\leqslant\kappa<\delta}A_\kappa)^{\beta^-}$, i.e., the elements of the carrier are sequences of the type β^- of sequences of the type δ^-; any of them can be expressed in the form $((a_{\kappa\iota})_{1\leqslant\kappa<\delta})_{1\leqslant\iota<\beta}$ where $a_{\kappa\iota} \in A_\kappa$ for any ordinals κ, ι with the property $1 \leqslant \kappa < \delta$, $1 \leqslant \iota < \beta$.

Any constant of this algebra is of the form $((c_{\kappa\iota})_{1\leqslant\kappa<\delta})_{1\leqslant\gamma<\beta}$ where $1 \leqslant \iota < \alpha$.

The unary operation of the algebra $\mathbf{U}(\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{A}_\kappa)$ assigns the value

$$\mathbf{u}[(\mathbf{x}[(o_{\kappa\iota})_{1\leqslant\kappa<\delta}])_{1\leqslant\iota<\beta}](((a_{\kappa\iota})_{1\leqslant\kappa<\delta})_{1\leqslant\iota<\beta}) = ((o_{\kappa\iota}(a_{\kappa 1}, \ldots, a_{\kappa r(\iota)}))_{1\leqslant\kappa<\delta})_{1\leqslant\iota<\beta}$$

to any element $((a_{\kappa\iota})_{1\leqslant\kappa<\delta})_{1\leqslant\iota<\beta} \in (\mathbf{X}_{1\leqslant\kappa<\delta}A_\kappa)^{\beta^-}$.

Indeed, if we replace o_ι by $\mathbf{x}[(o_{\kappa\iota})_{1\leqslant\kappa<\delta}]$ and x_ι by $(a_{\kappa\iota})_{1\leqslant\kappa<\delta}$ in (*), we obtain

$$\begin{aligned}&\mathbf{u}[(\mathbf{x}[(o_{\kappa\iota})_{1\leqslant\kappa<\delta}])_{1\leqslant\iota<\beta}](((a_{\kappa\iota})_{1\leqslant\kappa<\delta})_{1\leqslant\iota<\beta})\\ &= (\mathbf{x}[(o_{\kappa\iota})_{1\leqslant\kappa<\delta}]((a_{\kappa 1})_{1\leqslant\kappa<\delta},\dots,(a_{\kappa r(\iota)})_{1\leqslant\kappa<\delta}))_{1\leqslant\iota<\beta}.\end{aligned}$$

The last expression equals $((o_{\kappa\iota}(a_{\kappa 1},\dots,a_{\kappa r(\iota)}))_{1\leqslant\kappa<\delta})_{1\leqslant\iota<\beta}$ by (**).

(2) On the other hand, we have $\mathbf{U}(\mathbf{A}_\kappa) = (A_\kappa^{\beta^-}, ((c_{\kappa\iota})_{1\leqslant\gamma<\beta})_{1\leqslant\iota<\alpha}, \mathbf{u}[(o_{\kappa\iota})_{1\leqslant\iota<\beta}])$ and, therefore, $\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{U}(\mathbf{A}_\kappa)$ is an algebra whose carrier equals the set $\mathbf{X}_{1\leqslant\kappa<\delta}A_\kappa^{\beta^-}$, i.e., the set of all sequences of the type δ^- of the sequences of the type β^-; any of them can be expressed in the form $((a_{\kappa\iota})_{1\leqslant\iota<\beta})_{1\leqslant\kappa<\delta}$ where $a_{\kappa\iota}\in A_\kappa$ for any ordinals κ, ι with the property $1\leqslant\kappa<\delta$, $1\leqslant\iota<\beta$.

Any constant is of the form $((c_{\kappa\iota})_{1\leqslant\gamma<\beta})_{1\leqslant\kappa<\delta}$ where $1\leqslant\iota<\alpha$.

The unary operation $\mathbf{x}[(\mathbf{u}[(o_{\kappa\iota})_{1\leqslant\iota<\beta}])_{1\leqslant\kappa<\delta}]$ applied to the element

$$((a_{\kappa\iota})_{1\leqslant\iota<\beta})_{1\leqslant\kappa<\delta} \quad \text{with } a_{\kappa\iota}\in A_\kappa$$

provides

$$(\mathbf{u}[(o_{\kappa\iota})_{1\leqslant\iota<\beta}]((a_{\kappa\iota})_{1\leqslant\iota<\beta}))_{1\leqslant\kappa<\delta} = ((o_{\kappa\iota}(a_{\kappa 1},\dots,a_{\kappa r(\iota)}))_{1\leqslant\iota<\beta})_{1\leqslant\kappa<\delta}.$$

Indeed, if we replace o_κ by $\mathbf{u}[(o_{\kappa\iota})_{1\leqslant\iota<\beta}]$ and a_κ by $(a_{\kappa\iota})_{1\leqslant\iota<\beta}$ in (***), we obtain

$$\mathbf{x}[(\mathbf{u}[(o_{\kappa\iota})_{1\leqslant\iota<\beta}])_{1\leqslant\kappa<\delta}](((a_{\kappa\iota})_{1\leqslant\iota<\beta})_{1\leqslant\kappa<\delta}) = (\mathbf{u}[(o_{\kappa\iota})_{1\leqslant\iota<\beta}]((a_{\kappa\iota})_{1\leqslant\iota<\beta})_{1\leqslant\kappa<\delta}.$$

The last expression equals $((o_{\kappa\iota}(a_{\kappa 1},\dots,a_{\kappa r(\iota)}))_{1\leqslant\iota<\beta})_{1\leqslant\kappa<\delta}$ by (*).

(3) If we compare the results obtained in (1) and (2), we see that the mapping b is an isomorphism of the algebra $\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{U}(\mathbf{A}_\kappa)$ onto $\mathbf{U}(\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{A}_\kappa)$. □

As a consequence, we obtain

Construction 5.

Let $\mathbf{A}_\kappa = (A_\kappa, (c_{\kappa\iota})_{1\leqslant\iota<\alpha}, (o_{\kappa\iota})_{1\leqslant\iota<\beta})$ *be similar R-algebras for any ordinal* κ *with the property* $1\leqslant\kappa<\delta$ *where* $\delta>1$ *is an ordinal.*

Construct the algebra $\mathbf{U}(\mathbf{A}_\kappa)$ *for any* κ *with* $1\leqslant\kappa<\delta$.

Construct the algebra $\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{U}(\mathbf{A}_\kappa)$ *according to Section* 2.

Construct the algebra $\mathbf{U}(\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{A}_\kappa)$ *using the mapping* b.

Construct the algebra $\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{A}_\kappa$ *using Example* 2.

Example 4. Put $\delta = 3$, $A_1 = A_2 = A = \mathbf{N} - \{0\}$, $\alpha = 1$, $\beta = \omega_0$,

$$o_{1i}(x,y) = x + y + i, \ \ o_{2i}(x,y) = xyi \ \text{ for any } \ x,y \in A$$
$$\text{and any } \ i \ \text{ with } \ 1 \leqslant i < \omega_0,$$
$$\mathbf{A}_1 = (A, (o_{1i})_{1 \leqslant i < \omega_0}), \quad \mathbf{A}_2 = (A, (o_{2i})_{1 \leqslant i < \omega_0}).$$

Then

$$\mathbf{U}(\mathbf{A}_1) = (A^{\omega_0^-}, \mathbf{u}[(o_{1i})_{1 \leqslant i < \omega_0}])$$

where $\mathbf{u}[(o_{1i})_{1 \leqslant i < \omega_0}]((x_i)_{1 \leqslant i < \omega_0}) = (x_1 + x_2 + i)_{1 \leqslant i < \omega_0}$ for any element $(x_i)_{1 \leqslant i < \omega_0} \in A_1^{\omega_0^-}$. Similarly

$$\mathbf{U}(\mathbf{A}_2) = (A^{\omega_0^-}, \mathbf{u}[(o_{2i})_{1 \leqslant i < \omega_0}])$$

where $\mathbf{u}[(o_{2i})_{1 \leqslant i < \omega_0}]((x_i)_{1 \leqslant i < \omega_0}) = (x_1 x_2 i)_{1 \leqslant i < \omega_0}$ for any element $(x_i)_{1 \leqslant i < \omega_0} \in A^{\omega_0^-}$. Clearly, $\mathbf{X}_{1 \leqslant \kappa < 3} \mathbf{U}(\mathbf{A}_\kappa) = (A^{\omega_0^-} \times A^{\omega_0^-}, \mathbf{x}[(\mathbf{u}[(o_{\kappa i})_{1 \leqslant i < \omega_0}])_{1 \leqslant \kappa < 3}])$. An element of the set $A^{\omega_0^-} \times A^{\omega_0^-}$ will be written in the form $((x_i)_{1 \leqslant i < \omega_0}, (y_i)_{1 \leqslant i < \omega_0})$. If we apply the last operation to this element, we obtain the value

$$((o_{1i}(x_1, x_2))_{1 \leqslant i < \omega_0}, (o_{2i}(y_1, y_2))_{1 \leqslant i < \omega_0}) = ((x_1 + x_2 + i)_{1 \leqslant i < \omega_0}, (y_1 y_2 i)_{1 \leqslant i < \omega_0}).$$

Using the mapping b and Theorem 3, we see that $\mathbf{U}(\mathbf{X}_{1 \leqslant \kappa < 3} \mathbf{A}_\kappa) = \mathbf{U}(\mathbf{A}_1 \times \mathbf{A}_2)$ is an algebra with the carrier $(A \times A)^{\omega_0^-}$. Hence any element of this set has the form $((x_i, y_i))_{1 \leqslant i < \omega_0}$ where $x_i, y_i \in A$ holds for any i with the property $1 \leqslant i < \omega_0$. The unary operation of this algebra assigns the value $((x_1 + x_2 + i, y_1 y_2 i))_{1 \leqslant i < \omega_0}$ to this element. It follows that the binary operation $\mathbf{x}[(o_{\kappa i})_{1 \leqslant \kappa < 3}]$ assigns the value $(x_1 + x_2 + i, y_1 y_2 i)$ to any pair $(x_1, y_1) \in A_1 \times A_2$, $(x_2, y_2) \in A_1 \times A_2$ and to any ordinal i with $1 \leqslant i < \omega_0$. Hence, $\mathbf{A}_1 \times \mathbf{A}_2 = (\mathbf{X}_{1 \leqslant \kappa < 3} A_\kappa, (\mathbf{x}[(o_{\kappa i})_{1 \leqslant \kappa < 3}])_{1 \leqslant i < \omega_0})$ is defined using Construction 5.

6. Varieties

A *variety* of algebras is a class $\mathbf{V}$ of similar algebras that contains, with any algebra, all homomorphic images of this algebra and all isomorphic images of its subalgebras; furthermore, with any family of algebras in $\mathbf{V}$, it contains all isomorphic images of the direct product of these algebras (cf. [1], [2]).

The above obtained results make it easier to construct further members of a given variety $\mathbf{V}$ if some members are known.

Example 5. If $\mathbf{V}$ is a variety of R-algebras and $\mathbf{A} \in \mathbf{V}$, then any homomorphic image of the algebra $\mathbf{A}$ may be obtained by constructing the mono-unary algebra $\mathbf{U}(\mathbf{A})$. If $\mathbf{A}'$ is an algebra similar to $\mathbf{A}$ and is suspected to be a homomorphic image of $\mathbf{A}$, it is sufficient to determine whether there exists a decomposable homomorphism of the mono-unary algebra $\mathbf{U}(\mathbf{A})$ onto $\mathbf{U}(\mathbf{A}')$ or not. If there exists a decomposable homomorphism of $\mathbf{U}(\mathbf{A})$ onto $\mathbf{U}(\mathbf{A}')$, there exists a homomorphism of the algebra $\mathbf{A}$ onto $\mathbf{A}'$.

Similarly, if $(A, (c_\iota)_{1\leqslant\iota<\alpha}, (o_\iota)_{1\leqslant\iota<\beta}) = \mathbf{A} \in \mathbf{V}$ and $\emptyset \neq M \subseteq A$, then M is the carrier of a subalgebra of the algebra $\mathbf{A}$ if and only if M^{β^-} is the carrier of a subalgebra of the mono-unary algebra $\mathbf{U}(\mathbf{A})$, which can be determined on the basis of results of Section 2.

Finally, if $(\mathbf{A}_\kappa)_{1\leqslant\kappa<\delta}$ is a family of algebras in the variety $\mathbf{V}$, then the algebra $\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{A}_\kappa$ can be constructed using Construction 5.

7. Generalization

In the presented considerations we concentrated on constructions concerning R-algebras. We prove that some simple methods make it possible to transfer the results to arbitrary algebras.

Let $\mathbf{A} = (A, (c_\iota)_{1\leqslant\iota<\alpha}, (o_\iota)_{1\leqslant\iota<\beta})$ be an algebra where β is a finite ordinal (i.e. a natural number) with $\beta > 1$. Then we put $o_\iota = o_{\beta-1}$ for any ι with $\beta \leqslant \iota < \omega_0$, $\mathbf{R}(\mathbf{A}) = (A, (c_\iota)_{1\leqslant\iota<\alpha}, (o_\iota)_{1\leqslant\iota<\omega_0})$. Then $\mathbf{R}(\mathbf{A})$ is an R-algebra. Clearly, if $\mathbf{R}(\mathbf{A})$ and β are known, it is easy to reconstruct $\mathbf{A}$.

The following two lemmas hold trivially.

Lemma 2. *Let $\mathbf{A} = (A, (c_\iota)_{1\leqslant\iota<\alpha}, (o_\iota)_{1\leqslant\iota<\beta})$, $\mathbf{A}' = (A', (c'_\iota)_{1\leqslant\iota<\alpha}, (o'_\iota)_{1\leqslant\iota<\beta})$ be similar algebras that are not R-algebras, h a mapping of the set A into A'. Then h is a homomorphism of the algebra $\mathbf{A}$ into $\mathbf{A}'$ if and only if it is a homomorphism of the algebra $\mathbf{R}(\mathbf{A})$ into $\mathbf{R}(\mathbf{A}')$.*

Lemma 3. *Let $\mathbf{A} = (A, (c_\iota)_{1\leqslant\iota<\alpha}, (o_\iota)_{1\leqslant\iota<\beta})$ be an algebra that is not an R-algebra, $\emptyset \neq M \subseteq A$ a set. Then M is the carrier of a subalgebra of the algebra $\mathbf{A}$ if and only if it is the carrier of a subalgebra of $\mathbf{R}(\mathbf{A})$.*

Lemma 4. *Let $\delta > 1$ be an ordinal, $\mathbf{A}_\kappa = (A_\kappa, (c_{\kappa\iota})_{1\leqslant\iota<\alpha}, (o_{\kappa\iota})_{1\leqslant\iota<\beta})$ similar algebras that are not R-algebras for any κ with the property $1 \leqslant \kappa < \delta$. Then $\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{R}(\mathbf{A}_\kappa) = \mathbf{R}(\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{A}_\kappa)$.*

P r o o f. The set $\mathbf{X}_{1\leqslant\kappa<\delta}A_\kappa$ is the carrier of both algebras.

Let ι be an ordinal such that $1 \leqslant \iota < \alpha$. An element of the set $\mathbf{X}_{1\leqslant\kappa<\delta} A_\kappa$ is the ιth constant in the algebra $\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{R}(\mathbf{A}_\kappa)$ if and only if its κth coordinate is the ιth constant in $\mathbf{R}(\mathbf{A}_\kappa)$, i.e. in the algebra $\mathbf{A}_\kappa$ for any ordinal κ with the property $1 \leqslant \kappa < \delta$, which means that it is the ιth constant in $\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{A}_\kappa$ and, therefore, in the algebra $\mathbf{R}(\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{A}_\kappa)$. Thus, both algebras have the same constants.

If ι is an ordinal such that $1 \leqslant \iota < \beta$, then the ιth operations of the algebras $\mathbf{A}_\kappa$ and $\mathbf{R}(\mathbf{A}_\kappa)$ are the same for any ordinal κ with $1 \leqslant \kappa < \delta$ and, therefore, the ιth operations in the algebras $\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{A}_\kappa$ and $\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{R}(\mathbf{A}_\kappa)$ are the same. Since the ιth operations in the algebras $\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{A}_\kappa$ and $\mathbf{R}(\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{A}_\kappa)$ are the same, we obtain that the ιth operations in the algebras $\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{R}(\mathbf{A}_\kappa)$ and $\mathbf{R}(\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{A}_\kappa)$ are the same. In particular, the operations with index $\iota = \beta - 1$ are the same in both algebras, which entails that also operations with any index ι satisfying the condition $\beta \leqslant \iota < \omega_0$ are the same in both algebras.

It follows that the algebras $\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{R}(\mathbf{A}_\kappa)$ and $\mathbf{R}(\mathbf{X}_{1\leqslant\kappa<\delta}\mathbf{A}_\kappa)$ coincide. $\square$

Example 6. Our lemmas make it possible to construct a variety of algebras that are not R-algebras. If a class of such algebras is given, we complete the family of nonnulary operations in any algebra in the above described way. Then, we test the resulting class: If it is a variety, also the original class is a variety.

Remark 2. Lemma 2 provides a possibility of constructing homomorphisms between algebras with a finite number of nonnullary operations, but it operates with infinite sets. Hence it need not be effective in contrast to constructions presented in [6]. Thus, the constructions of [6], though formally more complicated, are justified.

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