Yi Jia Tan

# Indecomposable matrices over a distributive lattice

*Czechoslovak Mathematical Journal*, Vol. 56 (2006), No. 2, 299–316

Persistent URL: `http://dml.cz/dmlcz/128068`

**Czechoslovak Mathematical Journal, 56 (131) (2006), 299–316**

# INDECOMPOSABLE MATRICES OVER A DISTRIBUTIVE LATTICE

Yi-jia Tan, Fuzhou

(Received June 26, 2003)

*Abstract.* In this paper, the concepts of indecomposable matrices and fully indecomposable matrices over a distributive lattice $L$ are introduced, and some algebraic properties of them are obtained. Also, some characterizations of the set $F_n(L)$ of all $n \times n$ fully indecomposable matrices as a subsemigroup of the semigroup $H_n(L)$ of all $n \times n$ Hall matrices over the lattice $L$ are given.

*Keywords*: distributive lattice, indecomposable matrix, fully indecomposable matrix, semigroup, characterization

*MSC 2000*: 15A33, 15A18

## 1. Introduction

The concept of indecomposable nonnegative matrices first appeared in 1912 in a paper by Frobenius [1] dealing with the spectral properties of nonnegative matrices, and the concept of fully indecomposable nonnegative matrices was introduced by Marcus and Minc [2]. Their properties and characterizations have been studied by many authors.

In 1973, Š. Schwarz [3] was the first to introduce the concepts of indecomposable Boolean matrices (or indecomposable relations) and fully indecomposable Boolean matrices (or fully indecomposable relations), and obtained some algebraic properties of them. Since then, a number of works in this area were published (see e.g. [4]–[10]).

In this paper we shall develop these concepts, introduce the concepts of indecomposable matrices and fully indecomposable matrices over a distributive lattice $L$ and

This work was supported by Foundation to the Educational Committee of Fujian (JB03056, JB02081), China, and the Natural Science Foundation of Fujian Province, China.

give some algebraic properties and characterizations of them. In Section 3, we obtain some algebraic properties of indecomposable matrices and fully indecomposable matrices over the lattice $L$. In Section 4, we shall show that the set $F_n(L)$ of all $n \times n$ fully indecomposable matrices over the lattice $L$ forms a nilpotent semigroup having the universal matrix as the zero element and that the index of nilpotency of $F_n(L)$ is equal to the number $n-1$. Also, we show that $F_n(L)$ is the maximal nilpotent ideal of the semigroup $H_n(L)$ of all $n \times n$ Hall matrices over the lattice $L$. Some results obtained in this paper generalize former results on Boolean matrices in [3].

## 2. Definitions and preliminary lemmas

Let $(L, \leqslant, \vee, \wedge)$ be a distributive lattice with the least and the greatest elements 0 and 1, respectively. The join $a \vee b$ and the meet $a \wedge b$ of $a$, $b$ in $L$ will be denoted by $a+b$ and $a \cdot b$ (or $ab$), respectively. It is clear that if $L$ is a linear lattice, especially the Boolean algebra $B_0 = \{0,1\}$ or the fuzzy algebra $F = [0,1]$, then $a + b = \max\{a,b\}$ and $ab = \min\{a,b\}$ for all $a$ and $b$ in $L$.

Let $V_n(L)$ $(n \geqslant 1)$ denote the set of all $n$-tuples ($n$-vectors) over the lattice $L$. For $\alpha = (a_1, a_2, \ldots, a_n)$, $\beta = (b_1, b_2, \ldots, b_n)$ in $V_n(L)$ we define $\alpha + \beta = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$ and $\alpha \leqslant \beta \Longleftrightarrow a_i \leqslant b_i$ for $i = 1, 2, \ldots, n$; we also define $\alpha < \beta \Longleftrightarrow \alpha \leqslant \beta$ and there exists $i \in \{1, 2, \ldots, n\}$ such that $a_i < b_i$. The *norm* of a vector $\alpha$ is defined by $\|\alpha\| = \sum\limits_{i=1}^{n} a_i$. Let $0 = (0, 0, \ldots, 0)$ and $e = (1, 1, \ldots, 1)$. The vector 0 is called the *zero vector* of $V_n(L)$. Let $e_i$ denote the $n$-tuple with 1 as its $i$th coordinate, 0 otherwise.

The multiplication of a vector $\alpha$ by a scalar $\lambda$ in $L$ is defined by $\lambda\alpha = (\lambda a_1, \ldots, \lambda a_n)$. The vector $\alpha$ is called a *constant vector* if $\alpha = \lambda e = (\lambda, \lambda, \ldots, \lambda)$ for some $\lambda$ in $L$, otherwise, $\alpha$ is called *nonconstant*.

Let $M_n(L)$ $(n \geqslant 1)$ be the set of $n \times n$ matrices over $L$ (*lattice matrices*). We shall denote by $A_{ij}$ or $a_{ij}$ the element of $L$ which is the $(i,j)$-entry of $A$ in $M_n(L)$. We define:

$A + B = C$ iff $c_{ij} = a_{ij} + b_{ij}$ for $i, j = 1, 2, \ldots, n$, $A \leqslant B$ iff $a_{ij} \leqslant b_{ij}$ for $i, j = 1, 2, \ldots, n$, $A < B$ iff $A \leqslant B$ and $a_{ij} < b_{ij}$ for some couple $i, j \in \{1, 2, \ldots, n\}$, $AB = C$ iff $c_{ij} = \sum\limits_{k=1}^{n} a_{ik} b_{kj}$ for $i, j = 1, 2, \ldots, n$, $A^T = C$ iff $c_{ij} = a_{ji}$ for $i, j = 1, 2, \ldots, n$,

$$I_n = (\delta_{ij}), \quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \quad i, j = 1, 2, \ldots, n,$$

$J_n = (a_{ij})$, where $a_{ij} = 1$ for $i, j = 1, 2, \ldots, n$. $J_n$ is called the *universal matrix*.

Further, $A^0 = I_n$, $A^{k+1} = A^k A$, $k = 0, 1, 2, \ldots$. We shall denote by $a_{ij}^k$ the element at the $(i, j)$-entry of $A^k$.

The following properties are derived immediately from these definitions.

a) $M_n(L)$ is a monoid with respect to multiplication.

b) $(M_n(L), +, \cdot)$ is a semiring and for any $A$, $B$, $C$ and $D$ in $M_n(L)$, $A + A = A$, and if $A \leqslant B$ and $C \leqslant D$ then $AC \leqslant BD$.

A matrix in $M_n(L)$ is called a *permutation matrix* if one of the elements of its every row and every column is 1 and the others are 0. A matrix $A$ in $M_n(L)$ is called *invertible* if there exists a matrix $B$ in $M_n(L)$ such that $AB = BA = I_n$. The matrix $B$ is called the *inverse* of $A$ and is denoted by $A^{-1}$.

It is clear that the set $S_n(L)$ of all invertible matrices in $M_n(L)$ is the group of the units of the monoid $M_n(L)$.

**Remark 2.1.** A square matrix $A$ over the Boolean algebra $B_0$ is invertible iff $A$ is a permutation matrix.

A matrix $A$ in $M_n(L)$ is called a *Hall matrix* (see [11]) if there exists a matrix $P$ in $S_n(L)$ such that $P \leqslant A$. The matrix $A$ is called *reflexive* if $I_n \leqslant A$. It is clear that the set $H_n(L)$ of all $n \times n$ Hall matrices over $L$ forms a subsemigroup of the semigroup $M_n(L)$ and contains the group $S_n(L)$.

A set $S = \{a_1, a_2, \ldots, a_m\}$ of elements in $L$ is called a *decomposition* of 1 in $L$ if $\sum_{i=1}^{m} a_i = 1$; $S$ is called *orthogonal* if $a_i a_j = 0$ holds for all $i$ and $j$ provided that $i \neq j$; $S$ is called an *orthogonal decomposition* of 1 in $L$ if it is orthogonal and a decomposition of 1 in $L$.

A semigroup $S$ with zero element $z_0$ is called *nilpotent with the index of nilpotency $l$* if $S^l = \{z_0\}$ while $S^{l-1} \neq \{z_0\}$. A two-sided ideal (or ideal) $Q$ of a semigroup $S$ is called a *prime ideal* if $V \cdot W \subseteq Q$ implies either $V \subseteq Q$ or $W \subseteq Q$ for all two-sided ideals $V$, $W$ of $S$, where $V \cdot W = \{vw \colon v \in V, w \in W\}$ and $S^l = \{s_1 s_2 \ldots s_l \colon s_i \in S,\ i = 1, 2, \ldots, l\}$.

The following lemmas will be used:

**Lemma 2.1.** *Let $L$ be a distributive lattice. Then*

(1) *for $a, a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ in $L$, we have*

$$\sum_{i=1}^{n} a_i b_i = \prod_{U \subseteq N} \Bigl( \sum_{i \in U} a_i + \sum_{j \in N - U} b_j \Bigr) \quad \text{and} \quad a + \prod_{i=1}^{n} a_i = \prod_{i=1}^{n} (a + a_i)$$

*where $N = \{1, 2, \ldots, n\}$;*

(2) *for* $a_{ij} \in L$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, *we have*

$$\prod_{i\in N}\Big(\sum_{j\in M} a_{ij}\Big) = \sum_{\sigma\in m(N,M)} \prod_{i\in N} a_{i\sigma(i)}$$

*where* $M = \{1, 2, \ldots, m\}$ *and* $m(N, M)$ *is the set of all maps from* $N$ *to* $M$.

P r o o f. (1) can be obtained from Lemma 2.1 in [12]; (2) is the dual of Lemma 2.1 (2) in [13]. □

**Lemma 2.2.** *Let* $A \in M_n(L)$. *Then the following statements are all equivalent.*

(1) $A$ *is invertible;*
(2) $AA^T = A^TA = I_n$*;*
(3) *there exists a positive integer* $l$ *such that* $A^l = I_n$*;*
(4) *each row and each column of* $A$ *is an orthogonal decomposition of* 1 *in* $L$.

P r o o f. The proof of Lemma 2.2 can be found in [14]. □

**Remark 2.2.** Note that if $A$ is invertible in $M_n(L)$ then $A^{-1} = A^T$.

**Lemma 2.3.** *Let* $P = (p_{ij}) \in S_n(L)$. *Then*

(1) *for any* $\alpha, \beta \in V_n(L)$, $\alpha < \beta \Rightarrow \alpha P < \beta P$*;*
(2) *for any* $\alpha$ *in* $V_n(L)$, $\alpha$ *is a constant vector iff* $\alpha P$ *is a constant vector;*
(3) $\sum\limits_{i\in U,\, j\in V} p_{ij} = 1$ *for any* $U, V \subseteq N$ *with* $|U| + |V| \geqslant n + 1$.

P r o o f. (1) First, it is clear that $\alpha < \beta$ implies $\alpha P \leqslant \beta P$. Suppose that $\alpha P = \beta P$. Then $\alpha PP^{-1} = \beta PP^{-1}$, and so $\alpha = \beta$, which contradicts our hypothesis. This proves (1).

(2) Let $\alpha = \lambda e$ for some $\lambda$ in $L$. Then $\alpha P = (\lambda e)P = \lambda e$ (by Lemma 2.2 (4)). Conversely, if $\alpha P = \lambda e$ for some $\lambda \in L$ then $\alpha = (\lambda e)P^{-1} = \lambda e$. This proves (2).

(3) Since $p_{i1} + p_{i2} + \ldots + p_{in} = 1$ for any $i \in U$, we have

$$\begin{aligned} 1 &= \prod_{i\in U}\Big(\sum_{j\in N-V} p_{ij} + \sum_{j\in V} p_{ij}\Big) \\ &\leqslant \prod_{i\in U}\Big(\sum_{k\in N-V} p_{ik} + \sum_{i\in U,\, j\in V} p_{ij}\Big) \\ &= \prod_{i\in U}\Big(\sum_{k\in N-V} p_{ik}\Big) + \sum_{i\in U,\, j\in V} p_{ij} \quad \text{(by Lemma 2.1 (1))} \\ &= \sum_{\sigma\in m(U,N-V)} \prod_{i\in U} p_{i\sigma(i)} + \sum_{i\in U,\, j\in V} p_{ij} \quad \text{(by Lemma 2.1 (2))}, \end{aligned}$$

where $m(U, N - V)$ is the set of all maps of $U$ to $N - V$.

Since $|U| \geqslant (n - |V|) + 1 = |N - V| + 1$, for any $\sigma \in m(U, N - V)$ there must be a couple $s, t \in U$ such that $\sigma(s) = \sigma(t)$. Therefore $\prod_{i \in U} p_{i\sigma(i)} = 0$ for all $\sigma$ in $m(U, N - V)$ (by Lemma 2.2 (4)). Hence $\sum_{i \in U,\, j \in V} p_{ij} = 1$. This proves (3). □

**Lemma 2.4.** *Let $A \in H_n(L)$. Then*

(1) *there exists a matrix $P$ in $S_n(L)$ such that $\alpha \leqslant \alpha(PA)$ for any $\alpha$ in $V_n(L)$;*

(2) *if $I_n \leqslant A$, then $A^k = A^{n-1}$ holds for $k \geqslant n$.*

P r o o f. (1) Let $A \in H_n(L)$. Then there exists a matrix $Q$ in $S_n(L)$ such that $Q \leqslant A$. Let $Q^{-1} = P$. Then $P \in S_n(L)$ and $I_n \leqslant PA$. Clearly, $\alpha \leqslant \alpha(PA)$ for any $\alpha$ in $V_n(L)$.

(2) can be obtained from Theorem 4 in [14]. □

## 3. Indecomposable lattice matrices and fully indecomposable lattice matrices

In this section, we shall introduce the concepts of indecomposable matrices and fully indecomposable matrices over a lattice $L$, and discuss some of their properties.

To do this, we first recall the notions of indecomposable Boolean matrices and fully indecomposable Boolean matrices and give some of their characterizations.

**Definition 3.1.** Let $A \in M_n(B_0)$. $A$ is said to be *decomposable* if there exists a permutation matrix $P$ such that

$$PAP^T = \begin{bmatrix} B & O \\ C & D \end{bmatrix}$$

where $B$ and $D$ are square. Otherwise, $A$ is called *indecomposable*; $A$ is said to be *partly decomposable* if there exist permutation matrices $P$ and $Q$ such that

$$PAQ = \begin{bmatrix} B & O \\ C & D \end{bmatrix}$$

where $B$ and $D$ are square. Otherwise, $A$ is called *fully indecomposable.*

**Remark 3.1.** Note that a matrix $A \in M_n(B_0)$ is indecomposable if and only if there is no proper nonempty subset $U$ of the set $N = \{1, 2, \ldots, n\}$ such that $a_{ij} = 0$ for all $i \in U$ and $j \in N - U$.

**Remark 3.2.** Note that any fully indecomposable Boolean matrix is indecomposable.

**Proposition 3.1.** *Let $A \in M_n(B_0)$. Then*

(1) *$A$ is indecomposable if and only if*

$$(I_n + A)^{n-1} = J_n,$$

(2) *$A$ is fully indecomposable if and only if for any $k$ in $\{1, 2, \ldots, n-1\}$, every $k \times n$ $(n \times k)$ submatrix of $A$ has at least $k+1$ columns $(k+1$ rows$)$ which are not zero vectors,*

(3) *$A$ is fully indecomposable if and only if there exists a permutation matrix $P$ such that $I_n \leqslant PA$ and $PA$ is indecomposable.*

Proof. (1) *Sufficiency*: Suppose that $A$ is decomposable. Then, by Remark 3.1, there exists a proper nonempty subset $U$ of $N$ such that $a_{ij} = 0$ for all $i \in U$ and $j \in N - U$. Now let $u \in U$ and $v \in N - U$. Since $J_n = (I_n + A)^{n-1} = I_n + A + \ldots + A^{n-1}$, we have $(I_n + A + \ldots + A^{n-1})_{uv} = 1$, and so there exists a $k$ in $\{1, 2, \ldots, n-1\}$ such that $(A^k)_{uv} = 1$. But

$$(A^k)_{uv} = \sum_{1 \leqslant i_1, \ldots, i_{k-1} \leqslant n} a_{ui_1} a_{i_1 i_2} \ldots a_{i_{k-1} v},$$

hence there exists a sequence $i_1, \ldots, i_{k-1}$ such that $a_{ui_1} = a_{i_1 i_2} = \ldots = a_{i_{k-1} v} = 1$. Let $i_t$ be the last member in the sequence $i_0, i_1, \ldots, i_{k-1}, i_k$ which is in $U$ (taking $i_0 = u$ and $i_k = v$). Then $i_t \in U$ and $i_{t+1} \in N - U$. But $a_{i_t i_{t+1}} = 1$, a contradiction.

*Necessity*: Suppose that $A$ is indecomposable. Then by Proposition 5.2.3 in [15] we have that for any $i, j \in N$, there exists a sequence $\gamma_1, \ldots, r_{k(i,j)-1}$ such that $a_{i\gamma_1} = a_{\gamma_1 \gamma_2} = \ldots = a_{\gamma_{k(i,j)-1} j} = 1$ (including the empty sequence with $a_{ij} = 1$). Therefore $(A^{k(i,j)})_{ij} = 1$. Let $k = \max_{i,j \in N} \{k(i,j)\}$. Then $((I_n + A)^k)_{ij} = (I_n + A + \ldots + A^k)_{ij} = 1$ for all $i$, $j$ in $N$, and so $(I_n + A)^k = J_n$. Since $I_n \leqslant I_n + A$, we have $(I_n + A)^m = (I_n + A)^{n-1}$ for all $m \geqslant n$ (by Lemma 2.4 (2)). If $k \geqslant n$, then $(I_n + A)^{n-1} = (I_n + A)^k = J_n$; if $k \leqslant n-1$, then $J_n = (I_n + A)^k \leqslant (I_n + A)^{n-1}$, and so $(I_n + A)^{n-1} = J_n$. This proves (1).

(2) By Definition 3.1, $A$ is partly decomposable if and only if $A$ contains an $s \times (n-s)$ zero submatrix with $1 \leqslant s \leqslant n-1$. That is to say, $A$ is fully indecomposable if and only if for any $s \times t$ zero submatrix of $A$ we have $s + t \leqslant n-1$. Therefore, $A$ is fully indecomposable if and only if for any $k$ in $\{1, 2, \ldots, n-1\}$, every $k \times n$ $(n \times k)$ submatrix of $A$ has at least $k+1$ columns $(k+1$ rows$)$ which are not zero vectors. This proves (2).

(3)*Sufficiency*: Let $B = PA$. Then $I_n \leqslant B$ and $B$ is indecomposable. Let $B[U|V]$ denote the $|U| \times |V|$ submatrix of $B$ consisting precisely of those elements $b_{ij}$ of $B$ for which $i \in U$ and $j \in V$, where $U$ and $V$ are nonempty subsets of the set $N$. Then for any proper nonempty subset $U$ of $N$, the matrix $B[U|N-U]$ is not the zero matrix (by Remark 3.1) and $I_k \leqslant B[U|U]$, where $k = |U|$, and so the matrix $B[U|N]$ has at least $k+1$ columns which are not zero vectors. By (2), $B$ is fully indecomposable and so is $A$.

*Necessity*: Suppose that $A$ is fully indecomposable. Then the first row of $A$ has at least two elements which are 1, say $a_{1j_1} = a_{1j_1'} = 1$, where $j_1 \neq j_1'$. By (2), the $j_1$th column of $A$ has at least two elements which are 1. Assume that $a_{1j_1} = a_{2j_1} = 1$ without loss of generality. By (2), the second row of $A$ has at least two elements $a_{2j_2}$ and $a_{2j_2'}$ such that $a_{2j_2} = a_{2j_2'} = 1$ and $j_2 \neq j_1$. Similarly, the $k$th row $(3 \leqslant k \leqslant n)$ of $A$ has at least two elements $a_{kj_k}$ and $a_{kj_k'}$ such that $a_{kj_k} = a_{kj_k'} = 1$ and $j_k \notin \{j_1, j_2, \ldots, j_{k-1}\}$. Therefore, we have that $a_{1j_1} = a_{2j_2} = \ldots = a_{nj_n} = 1$ and that $j_1, j_2, \ldots, j_n$ are distinct. Now put $\bar{A} = (\bar{a}_{il})_{n\times n}$ such that

$$\bar{a}_{il} = \begin{cases} a_{il} & \text{if } l = j_i, \\ 0 & \text{if } l \neq j_i. \end{cases}$$

It is clear that $\bar{A}$ is a permutation matrix and $\bar{A} \leqslant A$. Let $P = (\bar{A})^{-1}$. Then $P$ is a permutation matrix and $I_n \leqslant PA$. Clearly, $PA$ is indecomposable. This proves (3). □

By Proposition 3.1, the indecomposable Boolean matrices and the fully indecomposable Boolean matrices can be described as follows:

**Definition 3.1′.** Let $A \in M_n(B_0)$. $A$ is called *indecomposable* if $(I_n + A)^{n-1} = J_n$; $A$ is called *fully indecomposable* if there exists a permutation matrix $P$ such that $I_n \leqslant PA$ and $PA$ is indecomposable.

Now we introduce the concepts of indecomposable matrices and fully indecomposable matrices over a lattice $L$.

**Definition 3.2.** Let $A \in M_n(L)$. $A$ is said to be *indecomposable* if $(I_n + A)^{n-1} = J_n$; $A$ is said to be *fully indecomposable* if there exists a $P$ in $S_n(L)$ such that $I_n \leqslant PA$ and $PA$ is indecomposable.

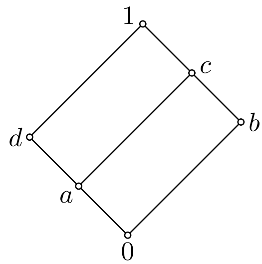
The sets of indecomposable matrices and fully indecomposable matrices in $M_n(L)$ are denoted by $I_n(L)$ and $F_n(L)$, respectively.

**Example 3.1.** Consider the lattice $L = \{0, a, b, c, d, 1\}$ whose diagram is shown below:

It is easy to see that $L$ is a distributive lattice.

Now let

$$A = \begin{bmatrix} 0 & d & b \\ c & 0 & d \\ d & 1 & 0 \end{bmatrix}$$




and

$$B = \begin{bmatrix} b & 1 & d \\ d & b & 1 \\ 1 & d & b \end{bmatrix}.$$

Then

$$I_3 + A = \begin{bmatrix} 1 & d & b \\ c & 1 & d \\ d & 1 & 1 \end{bmatrix} \quad \text{and} \quad (I_3 + A)^2 = J_3,$$

and so $A$ is indecomposable.

Let $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. It is clear that $P \in S_n(L)$ and $PB = \begin{bmatrix} 1 & d & b \\ b & 1 & d \\ d & b & 1 \end{bmatrix}$.

Therefore $I_n \leqslant PB$. Since $(PB)^2 = J_3$, we have that $A$ is fully indecomposable.

**Proposition 3.2.** *Let $A \in I_n(L)$. Then*

(1) *for any $P$ in $S_n(L)$, we have $PAP^T \in I_n(L)$;*

(2) *if $\sum_{i=1}^{n} a_{ii} = 1$, then $A^{2n-1} = J_n$.*

Proof. (1) Let $A \in I_n(L)$. Then $(I_n + A)^{n-1} = J_n$. Therefore

$$(I_n + PAP^T)^{n-1} = (P(I_n + A)P^T)^{n-1} = P(I_n + A)^{n-1}P^T = PJ_nP^T = J_n,$$

and so

$$PAP^T \in I_n(L).$$

This proves (1).

(2) Since $A = (a_{ij}) \in I_n(L)$ we have

$$I_n + A + A^2 + \ldots + A^{n-1} = J_n,$$

and so

$$a_{ij} + a_{ij}^2 + \ldots + a_{ij}^{n-1} = 1 \quad \text{for all } i \neq j. \tag{3.1}$$

For any $i$ and $j$ in $N = \{1, 2, \dots, n\}$, we have

$$\begin{aligned} a_{ij}^{2n-1} &= \sum_{1 \leqslant i_1, \dots, i_{2n-2} \leqslant n} a_{ii_1} a_{i_1 i_2} \dots a_{i_{2n-2}j} \geqslant \sum_{k=1}^{n} \sum_{p+d+r=2n-1} a_{ik}^p a_{kk}^d a_{kj}^r \\ &\geqslant \sum_{k=1}^{n} a_{kk} \sum_{p+r \leqslant 2n-2} a_{ik}^p a_{kj}^r \quad (\text{because } \ a_{kk}^d \geqslant a_{kk}) \\ &\geqslant \sum_{k=1}^{n} a_{kk} \left( \sum_{p=1}^{n-1} a_{ik}^p \right) \left( \sum_{r=1}^{n-1} a_{kj}^r \right). \end{aligned}$$

*Case I*: $i \neq j$. In this case

$$\begin{aligned} a_{ij}^{2n-1} &\geqslant \sum_{k \neq i,j} a_{kk} + a_{ii} \left( \sum_{p=1}^{n-1} a_{ii}^p \right) + a_{jj} \left( \sum_{r=1}^{n-1} a_{jj}^r \right) \quad (\text{by } \ (3.1)) \\ &= \sum_{k=1}^{n} a_{kk} = 1. \end{aligned}$$

*Case II*: $i = j$. In this case

$$a_{ii}^{2n-1} \geqslant \sum_{k \neq i} a_{kk} + a_{ii} \left( \sum_{p=1}^{n-1} a_{ii}^p \right) = \sum_{k=1}^{n} a_{kk} = 1.$$

Therefore $A^{2n-1} = J_n$. This proves (2). □

**Proposition 3.3.** *Let $A = (a_{ij}) \in F_n(L)$. Then*

(1) $A \in H_n(L) \cap I_n(F)$;

(2) *for any $P_1$, $P_2$ in $S_n(L)$, $P_1 A P_2 \in F_n(L)$;*

(3) *for any nonempty subsets $U$, $V$ of $N = \{1, 2, \dots, n\}$ with $|U| + |V| \geqslant n$, we have*

$$\sum_{i \in U,\, j \in V} a_{ij} = 1.$$

Proof. (1) Clearly, $A \in H_n(L)$. Now we shall show that $A \in I_n(L)$. Since $A \in F_n(L)$, there exists a matrix $P$ in $S_n(L)$ such that $I_n \leqslant PA$ and $PA$ is indecomposable. Therefore

$$\begin{aligned} I_n + A &= I_n + P^{-1}(PA) = I_n + P^{-1}(I_n + PA) \quad (\text{because } \ I_n \leqslant PA) \\ &= (I_n + A) + (I_n + P^{-1}). \end{aligned}$$

By Lemma 2.2 (3), there exists a positive integer $l$ such that $P^l = I_n$. Thus $P^{l-1} = P^{-1}$. Since the integers $l$ and $l-1$ are relatively prime, there exists a positive integer $u$ such that $u(l-1) \equiv 1 \pmod{l}$, and so $P^{u(l-1)} = P$. Now

$$\begin{aligned}(I_n + A)^u &= ((I_n + A) + (I_n + P^{l-1}))^u \geqslant (I_n + A)^u + (I_n + P^{l-1})^u \\ &\geqslant A + P^{u(l-1)} = A + P.\end{aligned}$$

Thus

$$(I_n + A)^{2u} \geqslant (A + P)^2 \geqslant PA = I_n + PA,$$

and so

$$\begin{aligned}(I_n + A)^{n-1} &= (I_n + A)^{2u(n-1)} \quad \text{(by Lemma 2.4 (2))} \\ &\geqslant (I_n + PA)^{n-1} = J_n \quad \text{(because } PA \text{ is indecomposable).}\end{aligned}$$

Then $(I_n + A)^{n-1} = J_n$, i.e., $A$ is indecomposable. This proves (1).

(2) Let $A \in F_n(L)$. Then there exists a $P$ in $S_n(L)$ such that $I_n \leqslant PA$ and $PA \in I_n(L)$. Let $Q = P_2^{-1} P P_1^{-1}$. Then $Q \in S_n(L)$ and

$$Q(P_1 A P_2) = (P_2^{-1} P P_1^{-1})(P_1 A P_2) = P_2^{-1}(PA)P_2 \geqslant P_2^{-1} I_n P_2 = I_n.$$

Furthermore, $Q(P_1 A P_2) = P_2^{-1}(PA)P_2$ is indecomposable since $PA$ is indecomposable. Therefore, $P_1 A P_2$ is fully indecomposable. This proves (2).

(3) By the definition of $A$, there exists $P$ in $S_n(L)$ such that $I_n \leqslant PA$ and $PA$ is indecomposable. Thus, we have $I_n < PA$. Let $PA = B = (b_{ij})$. Then

$$b_{ii} = 1 \quad \text{for } i = 1, 2, \ldots, n,$$

and so

$$\sum_{i \in U,\, j \in V} b_{ij} = 1 \quad \text{for any } U \text{ and } V \quad \text{with } U \cap V \neq \emptyset.$$

If $U \cap V = \emptyset$, then $U \cup V = N$. Let now $\alpha = \sum_{i \in U} e_i + \lambda \sum_{i \in V} e_i$, where $\lambda \in L$ and $\lambda \neq 1$. Then

$$\alpha B = \left(\sum_{i \in U} b_{i1} + \lambda \sum_{i \in V} b_{i1}, \ldots, \sum_{i \in U} b_{in} + \lambda \sum_{i \in V} b_{in}\right).$$

For any $j \in N$, if $j \in U$, then $\sum_{i \in U} b_{ij} + \lambda \sum_{i \in V} b_{ij} = 1$; if $j \in V$, then $\sum_{i \in U} b_{ij} + \lambda \sum_{i \in V} b_{ij} = \sum_{i \in U} b_{ij} + \lambda$. Since $\alpha < \alpha B$, there is $j \in V$ such that $\sum_{i \in U} b_{ij} + \lambda > \lambda$, and so $\sum_{i \in U,\, j \in V} b_{ij} + \lambda > \lambda$ for all $\lambda \in L$ with $\lambda \neq 1$. Thus $\sum_{i \in U,\, j \in V} b_{ij} = 1$.

Now $A = P^{-1}B$. Let $P^{-1} = (d_{ij})$. Then

$$\begin{aligned}\sum_{i\in U,\, j\in V} a_{ij} &= \sum_{i\in U}\sum_{j\in V}\sum_{t=1}^{n} d_{it}b_{tj} = \sum_{t=1}^{n}\Big(\sum_{i\in U} d_{it}\Big)\Big(\sum_{j\in V} b_{tj}\Big)\\ &= \prod_{W\subseteq N}\Big(\sum_{i\in U,\, s\in W} d_{is} + \sum_{j\in V,\, t\in N-W} b_{tj}\Big) \quad \text{(by Lemma 2.1 (1))}.\end{aligned}$$

Let $\Delta(W) = \sum_{i\in U,\, s\in W} d_{is} + \sum_{j\in V,\, t\in N-W} b_{tj}.$

If $(N-W)\cap V \neq \emptyset$ or $|V| + |N-W| \geqslant n$, then $\sum_{j\in V,\, t\in N-W} b_{tj} = 1$, and so $\Delta(W) = 1$.

If $(N-W)\cap V = \emptyset$ and $|V|+|N-W| \leqslant n-1$, then $V \subsetneq W$, and so $|W| \geqslant |V|+1$. Thus $|U|+|W| \geqslant |U|+|V|+1 \geqslant n+1$, and so $\sum_{i\in U,\, s\in W} d_{is} = 1$ (by Lemma 2.3 (3)). Therefore $\Delta(W) = 1$ for any $W \subseteq N$. Hence $\sum_{i\in U,\, j\in V} a_{ij} = 1$. This proves (3). □

**Proposition 3.4.** *Let $A \in M_n(L)$. Then $A \in F_n(L)$ iff there exists a $P$ in $S_n(L)$ such that*

$$\alpha < \alpha PA$$

*for any nonconstant vector $\alpha$ in $V_n(L)$.*

P r o o f. Suppose that there exists a matrix $P$ in $S_n(L)$ such that $\alpha < \alpha PA$ for any nonconstant vector $\alpha$ in $V_n(L)$. Take $\alpha = e_1, e_2, \dots, e_n$. Then

$$\begin{pmatrix} e_1\\ e_2\\ \vdots\\ e_n\end{pmatrix} < \begin{pmatrix} e_1\\ e_2\\ \vdots\\ e_n\end{pmatrix} PA.$$

Therefore, we have $I_n < PA$, and so $\alpha \leqslant \alpha PA \leqslant \dots \leqslant \alpha(PA)^{n-1} \leqslant \alpha(PA)^n$ for any $\alpha$ in $V_n(L)$. By Lemma 2.4 (2), we have $(PA)^{n-1} = (PA)^n$. Therefore $\alpha(PA)^{n-1} = \alpha(PA)^{n-1}(PA)$ for any $\alpha$ in $V_n(L)$, and so $\alpha(PA)^{n-1} = \lambda_\alpha e$ for some $\lambda_\alpha$ in $L$. If we take $\alpha = e_1, e_2, \dots, e_n$, then $e_i(PA)^{n-1} = e$, and so

$$\begin{pmatrix} e_1\\ e_2\\ \vdots\\ e_n\end{pmatrix} (PA)^{n-1} = J_n.$$

Thus $(I_n + PA)^{n-1} = (PA)^{n-1} = J_n$. Hence $A \in F_n(L)$.

Conversely, suppose that $A \in F_n(L)$. Then there exists a matrix $P$ in $S_n(L)$ such that $I_n \leqslant PA$ and $PA \in I_n(L)$, and so $\alpha \leqslant \alpha PA$ for all vectors $\alpha$ in $V_n(L)$. If $\alpha = \alpha PA$, then

$$\alpha = \alpha PA = \alpha(PA)^2 = \ldots = \alpha(PA)^{n-1} = \alpha J_n = \|\alpha\| e.$$

Therefore $\alpha < \alpha PA$ for any nonconstant vector $\alpha$ in $V_n(L)$. This proves the proposition. □

At the end of this section, we will introduce the concepts of weakly indecomposable matrices and weakly fully indecomposable matrices over the lattice $L$.

**Definition 3.3.** Let $A \in M_n(L)$. $A$ is called *weakly decomposable* if there exists a matrix $P$ in $S_n(L)$ such that

$$PAP^T = \begin{bmatrix} B & O \\ C & D \end{bmatrix}$$

where $B$ and $D$ are square. Otherwise, $A$ is called *weakly indecomposable*; $A$ is called *weakly partly decomposable* if there exist matrices $P$ and $Q$ in $S_n(L)$ such that

$$PAQ = \begin{bmatrix} B & O \\ C & D \end{bmatrix}$$

where $B$ and $D$ are square. Otherwise, $A$ is called *weakly fully indecomposable.*

**Remark 3.3.** Note that any indecomposable matrix is weakly indecomposable and any fully indecomposable matrix is weakly fully indecomposable over the lattice $L$. However, the converse is not true.

**Example 3.2.** Consider the lattice $L$ from Example 3.1. Let $A = \begin{bmatrix} a & c \\ d & b \end{bmatrix}$, $B = \begin{bmatrix} a & c \\ d & a \end{bmatrix} \in M_2(L)$. For any $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ and $D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$ in $S_n(L)$, using Lemma 2.2 (4), we have

$$PAP^T = \begin{pmatrix} p_{11}a + p_{12}b & p_{12}p_{21}d + p_{11}p_{22}c \\ p_{11}p_{22}d + p_{12}p_{21}c & p_{21}a + p_{22}b \end{pmatrix}.$$

Since

$$\begin{aligned} p_{12}p_{21}d + p_{11}p_{22}c &\geqslant (p_{11}p_{22} + p_{12}p_{21})a \\ &= (p_{12} + p_{11}p_{22})(p_{21} + p_{11}p_{22})a \quad \text{(by Lemma 2.1 (1))} \\ &= (p_{12} + p_{11})(p_{12} + p_{22})(p_{21} + p_{11})(p_{21} + p_{22})a \\ &\qquad \text{(by Lemma 2.1 (1))} \\ &= a \quad \text{(by Lemma 2.2 (4))} > 0, \end{aligned}$$

$A$ is weakly indecomposable. But $(I_2 + A)^{2-1} = \begin{bmatrix} 1 & c \\ d & 1 \end{bmatrix} \neq J_2$, hence $A$ is not indecomposable.

Also,

$$\begin{aligned} PBD &\geqslant \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \\ &= \begin{bmatrix} (p_{11}+p_{12})a & (p_{11}+p_{12})a \\ (p_{21}+p_{22})a & (p_{21}+p_{22})a \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \\ &= \begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \quad \text{(by Lemma 2.2 (4))} \\ &= \begin{bmatrix} (d_{11}+d_{21})a & (d_{12}+d_{22})a \\ (d_{11}+d_{21})a & (d_{12}+d_{22})a \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix} \quad \text{(by Lemma 2.2 (4))}. \end{aligned}$$

Therefore, $B$ is weakly fully indecomposable.

For any $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \in S_n(L)$, we have

$$PB = \begin{bmatrix} p_{11}a + p_{12}d & p_{11}c + p_{12}a \\ p_{21}a + p_{22}d & p_{21}c + p_{22}a \end{bmatrix}.$$

Since $p_{11}a + p_{12}d \leqslant a + d = d < 1$ and $p_{21}c + p_{22}a \leqslant c + a = c < 1$, we have that $I_2 \nleqslant PB$. Thus $B$ is not fully indecomposable.

**Remark 3.4.** If $L$ is the Boolean algebra $B_0$, then the concept of indecomposable matrices concides with that of weakly indecomposable matrices and the concept of fully indecomposable matrices concides with that of weakly fully indecomposable mtrices over $L$.

## 4. The semigroup of fully indecomposable lattice matrices

In this section, we shall give some charactorizations of $F_n(L)$ as a semigroup.

**Theorem 4.1.**

(1) *$F_n(L)$ is a nilpotent semigroup having $J_n$ as the zero element.*

(2) *The index of nilpotency of $F_n(L)$ is equal to the number $n-1$.*

Proof. (1) $J_n$ is clearly the zero element of $F_n(L)$ since $J_n \in F_n(L)$ and for any $A \in F_n(L)$ we have $AJ_n = J_nA = J_n$. Suppose that $A, B \in F_n(L)$. Then there exist $P_1$, $P_2$ in $S_n(L)$ such that $I_n \leqslant P_1A$, $I_n \leqslant P_2B$, $(P_1A)^{n-1} = J_n$ and $(P_2B)^{n-1} = J_n$. Therefore

$$\begin{aligned} I_n &\leqslant P_2B = P_2I_nB \leqslant P_2(P_1A)B = (P_2P_1)(AB), \\ J_n &= (P_2B)^{n-1} = (P_2I_nB)^{n-1} \leqslant (P_2(P_1A)B)^{n-1} = ((P_2P_1)(AB))^{n-1}, \end{aligned}$$

and so

$$J_n = ((P_2P_1)(AB))^{n-1}.$$

Let now $P = P_2P_1$. Then $P \in S_n(L)$, $I_n \leqslant P(AB)$ and $J_n = (P(AB))^{n-1}$, and so $AB \in F_n(L)$. Hence, $F_n(L)$ is a semigroup.

Suppose that $A_1, A_2, \ldots, A_{n-1} \in F_n(L)$. Let $T = A_1A_2 \ldots A_{n-1}$, $A_l = (a_{ij}^{(l)})$, $l = 1, 2, \ldots, n-1$. Then

$$t_{ij} = \sum_{1 \leqslant i_1, \ldots, i_{n-2} \leqslant n} a_{ii_1}^{(1)} a_{i_1i_2}^{(2)} \ldots a_{i_{n-2}j}^{(n-1)}.$$

Let $\Delta_{ij}^{(0)} = t_{ij}$ and $\Delta_{i_lj}^{(l)} = \sum_{1 \leqslant i_{l+1}, \ldots, i_{n-2} \leqslant n} a_{i_li_{l+1}}^{(l+1)} \ldots a_{i_{n-2}j}^{(n-1)}$, $l = 1, 2, \ldots, n-2$. It is clear that

$$\Delta_{i_lj}^{(l)} = \sum_{i_{l+1}=1}^{n} a_{i_li_{l+1}}^{(l+1)} \Delta_{i_{l+1}j}^{(l+1)}.$$

Hence

$$\begin{aligned} t_{ij} &= \sum_{i_1=1}^{n} a_{ii_1}^{(1)} \Delta_{i_1j}^{(1)} \\ &= \prod_{U_1 \subseteq N} \left( \sum_{i_1 \in U_1} a_{ii_1}^{(1)} + \sum_{j_1 \in N - U_1} \Delta_{j_1j}^{(1)} \right) \quad \text{(by Lemma 2.1 (1))} \\ &= \prod_{U_1 \subseteq N} \left( \sum_{i_1 \in U_1} a_{ii_1}^{(1)} + \sum_{j_1 \in N - U_1} \left( \sum_{i_2=1}^{n} a_{j_1i_2}^{(2)} \Delta_{i_2j}^{(2)} \right) \right) \\ &= \prod_{U_1 \subseteq N} \left( \sum_{i_1 \in U_1} a_{ii_1}^{(1)} + \sum_{i_2=1}^{n} \left( \sum_{j_1 \in N - U_1} a_{j_1i_2}^{(2)} \right) \Delta_{i_2j}^{(2)} \right) \\ &= \prod_{U_1 \subseteq N} \left( \sum_{i_1 \in U_1} a_{ii_1}^{(1)} + \prod_{U_2 \subseteq N} \left( \sum_{i_2 \in U_2} \left( \sum_{j_1 \in N - U_1} a_{j_1i_2}^{(2)} \right) + \sum_{j_2 \in N - U_2} \Delta_{j_2j}^{(2)} \right) \right) \\ &= \prod_{U_1, U_2 \subseteq N} \left( \sum_{i_1 \in U_1} a_{ii_1}^{(1)} + \sum_{i_2 \in U_2} \sum_{j_1 \in N - U_1} a_{j_1i_2}^{(2)} + \sum_{j_2 \in N - U_2} \Delta_{j_2j}^{(2)} \right) \\ &\qquad \text{(by Lemma 2.1 (1)).} \end{aligned}$$

Repeating this process we can obtain that

$$t_{ij} = \prod_{U_1,\dots,U_{n-2}\subseteq N} \Bigg( \sum_{i_1\in U_1} a^{(1)}_{ii_1} + \sum_{i_2\in U_2,\, j_1\in N-U_1} a^{(2)}_{j_1 i_2} + \dots + \sum_{i_{n-2}\in U_{n-2},\, j_{n-3}\in N-U_{n-3}} a^{(n-2)}_{j_{n-3} i_{n-2}} + \sum_{j_{n-2}\in N-U_{n-2}} a^{(n-1)}_{j_{n-2} j} \Bigg).$$

For any $U_1, U_2, \dots, U_{n-2} \subseteq N$, let

$$\Delta(U_1,\dots,U_{n-2}) = \sum_{i_1\in U_1} a^{(1)}_{ii_1} + \sum_{i_2\in U_2,\, j_1\in N-U_1} a^{(2)}_{j_1 i_2} + \dots + \sum_{i_{n-2}\in U_{n-2},\, j_{n-3}\in N-U_{n-3}} a^{(n-2)}_{j_{n-3} i_{n-2}} + \sum_{j_{n-2}\in N-U_{n-2}} a^{(n-1)}_{j_{n-2} j}.$$

If $|U_1| \geqslant n-1$, then $\sum_{i_1\in U_1} a^{(1)}_{ii_1} = 1$ (by Proposition 3.3 (3)), and so $\Delta(U_1,\dots,U_{n-2}) = 1$. Similarly, if $|N-U_{n-2}| \geqslant n-1$, we have that $\sum_{j_{n-2}\in N-U_{n-2}} a^{(n-1)}_{j_{n-2} j} = 1$, and so $\Delta(U_1,\dots,U_{n-2}) = 1$. This means that $\Delta(U_1,\dots,U_{n-2}) = 1$ if $|U_1| \geqslant n-1$ or $|N-U_{n-2}| \geqslant n-1$. Let now $|U_1| \leqslant n-2$ and $|N-U_{n-2}| \leqslant n-2$. Since

$$|U_1| + (|U_2| + |N-U_1|) + \dots + (|U_{n-2}| + |N-U_{n-3}|) + |N-U_{n-2}| = (n-2)n,$$

we have

$$(|U_2| + |N-U_1|) + \dots + (|U_{n-2}| + |N-U_{n-3}|) \geqslant (n-2)n - 2(n-2) = (n-2)^2.$$

Hence there must be an $l$ in $\{1,2,\dots,n-3\}$ such that $|U_{l+1}| + |N-U_l| \geqslant n$. Since $A_{l+1} \in F_n(L)$, we have

$$\sum_{i_{l+1}\in U_{l+1},\, j_l\in N-U_l} a^{(l+1)}_{j_l i_{l+1}} = 1 \quad \text{(by Proposition 3.3 (3))},$$

and so $\Delta(U_1,\dots,U_{n-2}) = 1$.

Therefore, we have $\Delta(U_1,\dots,U_{n-2}) = 1$ for all $U_1,\dots,U_{n-2} \subseteq N$, and so

$$t_{ij} = \prod_{U_1,U_2,\dots,U_{n-2}\subseteq N} \Delta(U_1,\dots,U_{n-2}) = 1, \quad \text{i.e., } T = J_n.$$

Hence $F_n(L)$ is a nilpotent semigroup having $J_n$ as the zero element and $(F_n(L))^{n-1} = \{J_n\}$. This proves (1).

(2) By (1), the index of nilpotency of $F_n(L) \leqslant n-1$. To show that the index of nilpotency is exactly $n-1$, it is sufficient to show that for any $n > 1$ there is an $A \in F_n(L)$ such that $A^{n-2} \neq J_n$. It is easy to prove that the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

has this property. This proves (2). □

**Remark 4.1.** Theorem 4.1 is a generalization of Theorem 1.2 in [3].

**Corollary 4.1.** *For any $A \in F_n(L)$ we have $A^{n-1} = J_n$.*

We now give some characterizations of $F_n(L)$ as a subsemigroup of $H_n(L)$.

**Theorem 4.2.** *The set $F_n(L)$ is a two-sided ideal of $H_n(L)$.*

Proof. Suppose that $A \in F_n(L)$ and $B \in H_n(L)$. Then there exist $P_1$ and $P_2$ in $S_n(L)$ such that $I_n \leqslant P_1A$, $I_n \leqslant P_2B$ and $(P_1A)^{n-1} = J_n$. Therefore $I_n \leqslant P_2B \leqslant P_2(P_1A)B = (P_2P_1)(AB)$, $I_n \leqslant P_1A \leqslant P_1(P_2B)A = (P_1P_2)(BA)$, $J_n = (P_1A)^{n-1} \leqslant (P_1(P_2B)A)^{n-1} = ((P_1P_2)(BA))^{n-1}$, and so $((P_1P_2)(BA))^{n-1} = J_n$. Also,

$$J_n = (P_1A)^{n-1} \leqslant ((P_1A)(BP_2))^{n-1} = (P_1(AB)P_2)^{n-1} = P_2^{-1}((P_2P_1)(AB))^{n-1}P_2.$$

This implies $J_n \leqslant ((P_2P_1)(AB))^{n-1}$. Thus $((P_2P_1)(AB))^{n-1} = J_n$. Since $P_1P_2, P_2P_1 \in S_n(L)$, we have $AB, BA \in F_n(L)$. This proves that $F_n(L)$ is a two-sided ideal of $H_n(L)$. □

**Remark 4.2.** Theorem 4.2 is a generalization of Theorem 2.3 in [3].

**Definition 4.1.** A matrix $A$ in $H_n(L)$ is called *strongly nilpotent* if $P_1AP_2$ is nilpotent for any $P_1$ and $P_2$ in $S_n(L)$, i.e., $(P_1AP_2)^k = J_n$ for some positive integer $k$.

**Theorem 4.3.** *The semigroup $F_n(L)$ is exactly the set of all strongly nilpotent elements in $H_n(L)$.*

Proof. Let $A \in F_n(L)$. Then by Proposition 3.3 (2), $P_1AP_2 \in F_n(L)$ for any $P_1$, $P_2$ in $S_n(L)$, and so $P_1AP_2$ is nilpotent for any $P_1$, $P_2$ in $S_n(L)$ by Corollary 4.1.

Conversely, let $A \in H_n(L)$ and let $P_1AP_2$ be nilpotent for any $P_1$, $P_2$ in $S_n(L)$. Since $A \in H_n(L)$, there exists a $P$ in $S_n(L)$ such that $I_n \leqslant PA$, and so $\alpha \leqslant \alpha PA$

for any $\alpha$ in $V_n(L)$. If $\alpha = \alpha PA$, then $\alpha = \alpha(PA) = \alpha(PA)^2 = \ldots = \alpha(PA)^k = \ldots$. But $PA$ is nilpotent, hence there exists an integer $k$ such that $(PA)^k = J_n$ and so $\alpha = \alpha(PA)^k = \alpha J_n = \|\alpha\| e$. Therefore $\alpha < \alpha(PA)$ if $\alpha$ is noncostant, and so $A \in F_n(L)$ by Proposition 3.4. □

**Theorem 4.4.**
(1) *$F_n(L)$ is the maximal nilpotent ideal of $H_n(L)$.*
(2) *The semigroup $F_n(L)$ is precisely the intersection of all prime ideals of $H_n(L)$.*

Proof. (1) Suppose that $U$ is a nilpotent ideal of $H_n(L)$ and $F_n(L) \subsetneq U$. Then there is a nilpotent element $A \in U - F_n(L)$. Since $A \in U$, we have also $P_1AP_2 \in U$ for any $P_1$, $P_2$ in $S_n(L) \subseteq H_n(L)$. On the other hand, since $A \notin F_n(L)$, $A$ is not strongly nilpotent, and so there is a couple $P_3, P_4$ in $S_n(L)$ such that $(P_3AP_4)^k < J_n$ for all $k$. That is, $P_3AP_4$ is not nilpotent, a contradiction with the supposition that $U$ is nilpotent.

(2) We first prove that $F_n(L)$ is contained in any prime ideal of $H_n(L)$. Let $Q$ be a prime ideal of $H_n(L)$. Since $F_n(L)^{n-1} = \{J_n\}$ and $J_n \in Q$, $F_n(L) \cdot F_n(L)^{n-2} \subseteq Q$ implies either $F_n(L) \subseteq Q$, in which case our statement is proved, or $F_n(L)^{n-2} \subseteq Q$. This implies $F_n(L) \cdot F_n(L)^{n-3} \subseteq Q$, hence again either $F_n(L) \subseteq Q$ or $F_n(L)^{n-3} \subseteq Q$. Repeating this argument we find $F_n(L) \subseteq Q$.

Our assertion will be proved if we are able to prove that for any $B \in H_n(L) - F_n(L)$ there is a prime ideal $Q_B$ such that $B \notin Q_B$.

Note first that if $B \in H_n(L) - F_n(L)$, then $P_1BP_2 \in H_n(L) - F_n(L)$ for any $P_1$, $P_2$ in $S_n(L)$. For, if there were $P_3BP_4 \in F_n(L)$ for some $P_3$, $P_4$ in $S_n(L)$, this would imply $P_3^{-1}(P_3BP_4)P_4^{-1} = B \in F_n(L)$, contrary to the choice of $B$.

Now since $B \notin F_n(L)$, there are $P_5$, $P_6$ in $S_n(L)$ such that the matrix $C = P_5BP_6$ is not nilpotent. Hence no member of the sequence

$$C, C^2, \ldots, C^k, \ldots \tag{4.1}$$

is contained in $F_n(L)$.

Let $Q_B$ be the largest ideal of $H_n(L)$ which does not meet any element of the sequence (4.1). Then $Q_B$ is not empty since it contains $F_n(L)$. We state that $Q_B$ is a prime ideal of $H_n(L)$. Suppose for an indirect proof that there are two ideals $V$ and $W$ of $H_n(L)$ such that $V \not\subseteq Q_B$, $W \not\subseteq Q_B$ and $V \cdot W \subseteq Q_B$. Since $Q_B \subsetneq Q_B \cup V$ and $Q_B \subsetneq Q_B \cup W$, there are some powers $C^u$ and $C^v$ such that $C^u \in Q_B \cup V$, $C^v \in Q_B \cup W$, and so $C^u \in V$, $C^v \in W$. Therefore $C^{u+v} \in V \cdot W \subseteq Q_B$, contrary to the construction of $Q_B$. Now $B$ is not contained in the ideal $Q_B$, since otherwise $P_5BP_6 = C$ would be contained in $Q_B$, contrary to the choice of $C$. This completes the proof of our statement. □

**Remark 4.3.** Theorem 4.4 generilizes Theorems 2.7 and 2.8 in [3].

**Acknowledgements.** The author thanks the referees for their suggestions which have made the paper more readable.

*References*

[1] *G. Frobenius*: Über Matrizen aus nichtnegativen Elementen. Sitzb. d. Preuss, Akad. d. Wiss. (1912), 456–477. Jfm 43.0204.09

[2] *M. Marcus, H. Minc*: Disjoint pairs of sets and incidence matrices. Illinois J. Math. *7* (1963), 137–147. Zbl 0122.24901

[3] *Š. Schwarz*: The semigroup of fully indecomposable relations and Hall relations. Czechoslovak Math. J. *23* (1973), 151–163. Zbl 0261.20057

[4] *C. Y. Chao*: On a conjecture of the semigroup of fully indecomposable relations. Czechoslovak Math. J. *27* (1977), 591–597. Zbl 0384.20051

[5] *C. Y. Chao, M. C. Zhang*: On the semigroup of fully indecomposable relations. Czechoslovak Math. J. *33* (1983), 314–319. Zbl 0524.20045

[6] *J. Y. Shao, Q. Li*: On the indices of convergence of an irreducible Boolean matrix. Linear Algebra Appl. *97* (1987), 185–210. Zbl 0634.15008

[7] *J. Y. Shao, Q. Li*: The index set for the class of irreducible Boolean matrices with given period. Linear and Multilinear Algebra *22* (1988), 285–303. Zbl 0637.15013

[8] *R. A. Brualdi, B. L. Liu*: Fully indecomposable exponents of primitive matrices. Proc. Amer. Math. Soc. *112* (1991), 1193–1202. Zbl 0736.05038

[9] *X. J. Wu, J. Y. Shao*: The index set of convergence of irreducible Boolean matrices. J. Math. Res. Exposition *12* (1992), 441–447. (In Chinese.) Zbl 0777.15011

[10] *J. Y. Shao*: On the indices of convergence of irreducible and nearly reducible Boolean matrices. Acta Math. Appl. Sinica *15* (1992), 333–344. (In Chinese.) Zbl 0762.05028

[11] *Y. J. Tan*: The semigroup of Hall matrices over a distributive lattice. Semigroup Forum *61* (2000), 303–314. Zbl 0970.20047

[12] *Y. J. Tan*: Primitive lattice matrices. Southeast Asian Bull. Math.. To appear.

[13] *Y. J. Tan*: On compositions of lattice matrices. Fuzzy Sets and Systems *129* (2002), 19–28. Zbl 1014.15011

[14] *Y. Giveón*: Lattice matrices. Information and Control *7* (1964), 477–484. Zbl 0154.01103

[15] *K. H. Kim*: Boolean Matrix Theory and Applications. Marcel Dekker, New York, 1982. Zbl 0495.15003

*Author's address*: Department of Mathematics, Fuzhou University, Fuzhou, 350002, P.R. China, e-mail: `yjtan@fzu.edu.cn`.