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EXTENSIONS OF PARTIALLY ORDERED PARTIAL ABELIAN MONOIDS

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Abstract. The notion of a partially ordered partial abelian monoid is introduced and extensions of partially ordered abelian monoids by partially ordered abelian groups are studied. Conditions for the extensions to exist are found. The cases when both the above mentioned structures have the Riesz decomposition property, or are lattice ordered, are treated. Some applications to effect algebras and MV-algebras are shown.

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1. Introduction

In algebraic categories, the diagram

$$0 \longrightarrow A \xrightarrow{\tau} C \xrightarrow{\pi} B \longrightarrow 0$$

where A, B, C are objects in a category $\mathscr{C}$ and τ and π are morphisms such that τ is injective, π is surjective with kernel A, is called a *short exact sequence.* This short exact sequence is called an *extension of B by A* (or of A by B) in the category $\mathscr{C}$. Usually, it is convenient to identify the extension with the middle object C.

The extension theory of algebraic systems was started by O. Schreier [33], who proposed and solved the problem of group extensions: given two groups N and F, find all possible groups G such that N is an invariant subgroup of G and G/N is isomorphic to F (see [18] for a general theory of group extensions). Analogous problems have been discussed on some other algebraic systems. For example, the extension theory for rings was given in [9]. Extensions of C*-algebras have been studied in [19], [16], [26].

A general discussion of the extensions of partially ordered groups is in [12], for extensions of Riesz groups see [13], [15], for extensions of lattice ordered groups see [34].

The theory of partial abelian monoids is a relatively new part of algebra. In recent literature, it is followed with increasing interest, and finds applications in different fields, as a common generalization of abelian groups, boolean algebras, clans, certain BCK-algebras, orthomodular lattices and posets, MV-algebras, effect algebras. See, e.g., [10], [23], [32], [35], [8], [30], [17], [7], [21].

Extensions of a partial abelian monoid by an abelian group were introduced and studied in [14]. It was shown, in particular, that every cancellative, unital abelian monoid is an extension of an effect algebra by an abelian group. (See [11] for effect algebras and [35] for partial abelian monoids.)

An alternative theory for an abelian group and a D-set replacing the partial abelian monoid was given in [29]. It has been shown there that every unital D-set ([27]) is an extension of a D-poset by an abelian group (see [22] for the definition of a D-poset).

The special case of cohomology of power sets with applications in quantum probability was studied in [24].

In the present paper we introduce the notion of a partially ordered partial abelian monoid, as a common generalization of partially ordered abelian groups and effect algebras. A general theory of extensions in the category of (partially ordered) abelian monoids and the corresponding morphisms has not been developed yet. Here we start with some special cases. In particular, we study extensions of partially ordered abelian monoids by partially ordered abelian groups. Necessary and sufficient conditions are found under which the extension of an abelian monoid by an abelian group, both with the Riesz decomposition properties, also enjoys the Riesz decomposition property.

Also, necessary conditions are found under which an extension of a lattice ordered partial abelian monoid by a lattice ordered abelian group is lattice ordered.

Some applications to extensions of effect algebras and MV-algebras by partially ordered abelian groups are given.

2. Ordered partial abelian monoids

Definition 1. A partial abelian monoid (PAM, for short) is a nonempty set P endowed with a partial binary operation $\oplus$ with a neutral element 0 such that

(P1) $a \oplus b = b \oplus a$,

(P2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$,
(P3) $a \oplus 0 = a$, $\forall a \in P$.

In (P1) and (P2), we assume that if one side of the equality is defined, so is the other, and the equality holds.

For convenience, we will write $a \perp b$ if $a \oplus b$ is defined.

A PAM P is *cancellative* if

$$a \oplus b = a \oplus c \Rightarrow b = c.$$

Definition 2. A subset P^+ of a PAM P is called a positive cone of P if the following conditions are satisfied.
(C1) $0 \in P^+$.
(C2) $a, b \in P^+$, $a \perp b$ implies $a \oplus b \in P^+$.
(C3) $a, b \in P^+$, $a \oplus b = 0$ implies $a = 0 = b$.

Lemma 1. *Let P be a cancellative PAM with a positive cone P^+. Define a binary relation $\leqslant$ on P by $a \leqslant b$ if there is a $c \in P^+$ such that $a \oplus c = b$. Then $\leqslant$ is a partial order on P.*

Proof. Reflexivity follows from $a \oplus 0 = a$ (P3) and $0 \in P^+$ (C1). To prove antisymmetry, let $a \leqslant b$ and $b \leqslant a$. Then there are $c, d \in P^+$ such that $a \oplus c = b$ and $b \oplus d = a$. It follows that $a = (a \oplus c) \oplus d = a \oplus (c \oplus d)$ by associativity (P2), and by cancellativity, $c \oplus d = 0$. By (C3), $c = d = 0$, which entails $a = b$. The proof of transitivity is similar. $\square$

In what follows, we assume that P is a cancellative PAM and P^+ is its positive cone. We will call (P, P^+) an *ordered PAM* (a p.o. PAM for short).

Let P and Q be PAMs. A mapping $\nu\colon P \to Q$ is a *morphism* (of PAMs) if $a \perp b$, $a, b \in P$ implies $\nu(a) \perp \nu(b)$ and $\nu(a \oplus b) = \nu(a) \oplus \nu(b)$. Clearly $\nu(0) = 0$ owing to $0 \perp 0$ and cancellativity. A morphism ν is *faithful* if $\nu(a) \perp \nu(b)$ implies $a \perp b$. A bijective faithful morphism is an isomorphism, i.e., its inverse mapping is also a morphism.

If P and Q are ordered PAMs, a morphism $\nu\colon P \to Q$ is a *positive morphism* if $a \in P^+$ implies $\nu(A) \in Q^+$. Clearly, a positive morphism preserves order ($a \leqslant b$ implies $\nu(a) \leqslant \nu(b)$). A morphism will be called *strongly positive* if $a \leqslant b$ iff $\nu(a) \leqslant \nu(b)$. A surjective strongly positive morphism is an order isomorphism.

Definition 3. Let G be a p.o. PAM. A subset A of G will be called a group order ideal (g.o. ideal) if it satisfies the following conditions.
(I1) For every a in A there is $\bar{a}$ in A such that $a \oplus \bar{a} = 0$.

(I2) For every $a \in A$ and every $g \in G$, $a \oplus g$ is defined.
(I3) If a, b belong to A, then $a \oplus b$ belongs to A.
(I4) If $a \oplus b$ belongs to A and a and b are positive, then a and b belong to A.

It is easy to check that A, with the operation $\oplus$ and the partial order inherited from G, is an abelian p.o. subgroup of G. Moreover, A is a convex subgroup of G in the sense that $a \leqslant x \leqslant b$ and $a, b \in A$ imply $x \in A$. Indeed, $b \ominus a = (b \ominus x) \oplus (x \ominus a)$, and $b \ominus a \in A$, $b \ominus x \geqslant 0$, $x \ominus a \geqslant 0$ implies by (I4) that $x \ominus a \in A$, which together with $a \in A$ and $x = a \oplus (x \ominus a)$ implies by (I3) that $x \in A$.

Define a binary relation $\sim_A$ on G by $a \sim_A b$ if there are $x, y \in A$ such that $a \oplus x = b \oplus y$. Owing to the group structure of A, we have $a \sim_A b$ iff $a = b \oplus x$ for some $x \in A$. In the next theorem, we prove that for any g.o. ideal A of G, $\sim_A$ is an equivalence. The set of all equivalence classes will be denoted by G/A and called the quotient of G with respect to A. The next theorem also shows that G/A can be naturally equipped with a structure of a p.o. PAM.

Theorem 1. *Let A be a g.o. ideal of a p.o. PAM G. The binary relation $\sim_A$ is an equivalence, and the quotient G/A can be equipped with a structure of a p.o. PAM. Moreover, the canonical mapping $\pi\colon G \to G/A$ is a faithful morphism such that $\pi(G^+) = (G/A)^+$.*

Proof. We will write $\sim$ instead of $\sim_A$. We also will replace $a \oplus b$ by $a + b$ iff at least one of a, b is in A. Reflexivity and symmetry of $\sim$ follow from the definition. To check transitivity, assume $u \sim v$ and $v \sim w$. Then there are $a, b \in A$ such that $v = u + a$ and $w = v + b$, so that $w = (u + a) + b$. Using associativity and (I3), we obtain $u \sim w$.

Let $[u]$ denote the equivalence class containing u. In G/A, define the binary relation $\perp$ by $[u] \perp [v]$ iff there are $u_1 \in [u]$ and $v_1 \in [v]$ with $u_1 \perp v_1$ in G. In this case we define $[u] \oplus [v] = [u_1 \oplus v_1]$. This definition does not depend on the choice of the representatives u_1, v_1. Indeed, let $u_2 \in [u]$, $v_2 \in [v]$, and $u_2 \perp v_2$. Then $u_1 \sim u_2$, $v_1 \sim v_2$ imply that there are $a, b \in A$ such that $u_2 = u_1 + a$, $v_2 = v_1 + b$, hence $u_2 \oplus v_2 = u_1 \oplus v_1 + (a + b)$, the latter equality entails that $u_2 \oplus v_2 \sim u_1 \oplus v_1$.

The operation $\oplus$ on G/A defined above is clearly commutative, and $[0] \oplus [u] = [u]$ for all $u \in G$. Obviously, the class $[u]$ can be identified with the set $u + A = \{u + a\colon\ a \in A\}$. If $[u] \perp [v]$, there are $a, b \in A$ such that $u + a \perp v + b$, so that $(u + a) \oplus (v + b)$ is defined. For any other representatives $u + x$, $v + y$ of $[u]$, $[v]$, respectively, we can write $u + x = u + a + (x - a)$, $v + y = v + b + (y - b)$, owing to the group structure of A. As $x - a$, $y - b$ belong to A, the element $(u + a) \oplus (v + b) + (x - a) + (y - b)$ is defined and equals $(u + x) \oplus (v + y)$. It follows that $[u] \perp [v]$ iff for any two representantives u_1, v_1 of $[u], [v]$, respectively, $u_1 \perp v_1$ holds.

To check the associativity of $\oplus$, assume that $([u] \oplus [v]) \oplus [w]$ is defined. This is equivalent to the existence of $((u+a) \oplus (v+b)) \oplus (w+c)$ for any $a, b, c \in A$. Owing to the associativity of $\oplus$ in G, the associativity in G/A follows.

To prove cancellativity, assume $[u] \oplus [v] = [u] \oplus [z]$. This entails $(u+a) \oplus (v+b) \sim (u+c) \oplus (z+d)$ for any $a, b, c, d \in A$, that is, $u \oplus v = u \oplus z + e$, $e = c+d-a-b \in A$, where we write $-x$ for the element $\bar{x}$, $x \in A$, and by cancellativity in G we obtain $v \sim z$.

We have proved that G/A is a cancellative PAM. Moreover, the mapping $\pi\colon G \to G/A$, $g \mapsto [g]$ is a faithful morphism of PAMs.

To introduce ordering in G/A, define $[g] \geqslant [0]$ if $g + a \in G^+$ for some $a \in A$, i.e., if $[g]$ has a positive representative. Let $(G/A)^+ = \{[g]\colon [g] \geqslant [0]\}$. Clearly, $[0] \in (G/A)^+$. If $[g], [h] \in (G/A)^+$ and $[g] \perp [h]$, then $g + a, h + b \in G^+$ for some $a, b \in A$, which implies that $g \oplus h + a + b \in G^+$, and hence $[g] \oplus [h] \in (G/A)^+$. Finally, if $[g] \oplus [h] = [0]$ and $[g], [h] \in (G/A)^+$, then $(g+a) \oplus (h+b) \in A$ for some $a, b \in A$ where $g + a, h + b \in G^+$, which entails, by (I4), that $g + a, h + b \in A$, and hence $g, h \in A$. This proves that $(G/A)^+$ satisfies conditions (C1), (C2), (C3) of a positive cone, and so it induces a partial order in (G/A). Evidently, $g \in G^+$ implies $[g] \in (G/A)^+$, hence π is a positive morphism. Conversely, if $[g] \in (G/A)^+$, then there is a representative $g + a \in G^+$, consequently, $\pi(G^+) = (G/A)^+$ $\square$

The partial order in G/A induced by $(G/A)^+$ will be called the *quotient ordering.*

Following [14] we define, in analogy with the extensions of abelian groups, an extension of a partially ordered PAM by a partially ordered abelian group.

Definition 4. Let A be a partially ordered abelian group. Let G and Δ be p.o. PAMs. We say that G is an ordered extension of Δ by A if there is an injective positive morphism τ of A into G such that $A^+ = \tau^{-1}(G^+)$, and there is a morphism π of G onto Δ with kernel A such that $\pi(G^+) = \Delta^+$.

So we have the sequence

$$0 \longrightarrow A \xrightarrow{\tau} G \xrightarrow{\pi} \Delta \longrightarrow 0. \tag{1}$$

Since τ is injective and π is surjective with kernel A, (1) is a short exact sequence in the category of ordered PAMs and positive morphisms.

Like in the category of p.o. abelian groups, the extensions of A by Δ can be described by cocycles and special subsets of A.

Definition 5. Let (Δ, Δ^+) be an ordered PAM (with the neutral element denoted by θ), and let (A, A^+) be an abelian partially ordered group (with the group operation denoted by $+$ and the neutral element 0). Let

$$f\colon \{(\alpha, \beta) \in (\Delta \times \Delta)\colon \alpha \perp \beta\} \to A$$

be a mapping satisfying

(c1) $f(\alpha,\theta)=0$ for all $\alpha\in\Delta$,

(c2) $f(\alpha,\beta)=f(\beta,\alpha)$ for all $\alpha,\beta\in\Delta$ such that $\alpha\oplus\beta$ is defined,

(c3) $f(\alpha,\beta)+f(\alpha\oplus\beta,\gamma)=f(\alpha,\beta\oplus\gamma)+f(\beta,\gamma)$ for all $\alpha,\beta,\gamma\in\Delta$ such that $\alpha\oplus\beta\oplus\gamma$ is defined.

Then f is called an A-cocycle on Δ.

Definition 6. Let (Δ,Δ^+) be an ordered PAM and (A,A^+) an abelian partially ordered group. We say that

$$G=\{(\alpha,a)\colon\ \alpha\in\Delta, a\in A\}$$

with a partially defined binary operation $\oplus$ is an f-product of Δ by A if the following conditions are satisfied:

(E1) $(\alpha,a)=(\beta,b)$ iff $a=b$ and $\alpha=\beta$,

(E2) $(\alpha,a)\oplus(\beta,b)$ is defined iff $\alpha\oplus\beta$ is defined, and

$$(\alpha,a)\oplus(\beta,b)=(\alpha\oplus\beta,a+b+f(\alpha,\beta)),$$

where f is any A-cocycle on Δ.

Lemma 2. *Let G be an f-product of a PAM Δ by a group A. Then G is a PAM.*

Proof. We have to check conditions (P1), (P2) and (P3).

(P1): Assume $\alpha\perp\beta$. Then, for any $a,b\in A$, $(\alpha,a)\oplus(\beta,b)$ is defined, and

$$(\alpha,a)\oplus(\beta,b)=(\alpha\oplus\beta,a+b+f(\alpha,\beta))=(\beta\oplus\alpha,b+a+f(\beta,\alpha))=(\beta,b)\oplus(\alpha,a)$$

by (c2).

(P2): Assume that $\alpha\oplus\beta\oplus\gamma$ is defined. Then

$$\begin{aligned}[(\alpha,a)\oplus(\beta,b)]\oplus(\gamma,c)&=(\alpha\oplus\beta,a+b+f(\alpha,\beta))\oplus(\gamma,c)\\&=((\alpha\oplus\beta)\oplus\gamma,a+b+c+f(\alpha,\beta)+f(\alpha\oplus\beta,\gamma))\\&=((\alpha\oplus(\beta\oplus\gamma),a+b+c+f(\alpha,\beta\oplus\gamma)+f(\beta,\gamma))\\&=((\alpha,a)\oplus(\beta\oplus\gamma,b+c+f(\beta,\gamma))\\&=(\alpha,a)\oplus[(\beta,b)\oplus(\gamma,c)].\end{aligned}$$

(P3):

$$(\alpha,a)\oplus(0,\theta)=(\alpha\oplus\theta,a+0+f(\alpha,\theta))=(\alpha,a).$$

□

Like in partially ordered groups, the order relation in G can be given in terms of the sets $P_\alpha = \{\text{the set of all } a \in A \text{ such that } (\alpha, a) \geqslant (\theta, 0)\}$, where the system of sets $P_\alpha, \alpha \in \Delta$ satisfies

(p1) $P_\alpha \neq \emptyset$ iff $\alpha \in \Delta^+$,

(p2) $P_\theta = A^+$,

(p3) $P_\alpha + P_\beta + f(\alpha, \beta) \subseteq P_{\alpha \oplus \beta}$ whenever $\alpha \perp \beta$.

Lemma 3. *Put $G^+ = \cup_{\alpha \in \Delta^+}(\alpha, P_\alpha)$. Then G^+ is a positive cone in G.*

P r o o f. We have to check the properties (C2) and (C3). To check (C2), assume that $(\alpha, a), (\beta, b) \in G^+$ and $(\alpha, a) \perp (\beta, b)$. Then $\alpha \perp \beta$ and $a \in P_\alpha$, $b \in P_\beta$. By (p3), $P_\alpha + P_\beta + f(\alpha, \beta) \subset P_{\alpha \oplus \beta}$, and since

$$(\alpha, a) \oplus (\beta, b) = (\alpha \oplus \beta, a + b + f(\alpha, \beta)),$$

we obtain $a + b + f(\alpha, \beta) \in P_{\alpha \oplus \beta}$. Hence $(\alpha, a) \oplus (\beta, b) \in G^+$.

As for (C3), assume that $(\alpha, a) \oplus (\beta, b) = 0$ for some $(\alpha, a), (\beta, b) \in G^+$. Then $(\alpha \oplus \beta, a + b + f(\alpha, \beta) = (\theta, 0)$, hence $\alpha \oplus \beta = \theta$ and $a + b + f(\alpha, \beta) = 0$. Since $a \in P_\alpha$, $b \in P_\beta$, we get by (p1) that $\alpha, \beta \in \Delta^+$, and from $\alpha \oplus \beta = \theta$ we get $\alpha = \beta = \theta$, hence $f(\alpha, \beta) = 0$. Now $a, b \in A^+$, $a + b = 0$ implies $a = b = 0$. Therefore $(\alpha, a) = (\beta, b) = (\theta, 0)$. $\square$

Observe that $(\alpha, a) \geqslant (\beta, b)$ iff there is $(\delta, d) \in G^+$ such that $(\alpha, a) = (\beta, b) \oplus (\delta, d) = (\beta \oplus \delta, b + d + f(\beta, \delta))$, and this yields

$$\delta = \alpha \ominus \beta \in \Delta^+, \quad d = a - b - f(\beta, \alpha \ominus \beta) \in P_{\alpha \ominus \beta}.$$

We note that P_α is an upper class, i.e., $a \in P_\alpha$, $a \leqslant b$ implies $b \in P_\alpha$, $\alpha \in \Delta$, $a, b \in A$. Indeed,

$$(\alpha, b) \ominus (\alpha, a) = (\alpha \ominus \alpha, b - a - f(\alpha, \alpha)) = (\theta, b - a) \geqslant (\theta, 0),$$

so that $(\alpha, b) \geqslant (\alpha, a) \geqslant (\theta, 0)$.

Let $\Delta = G/A$. In agreement with [4], we will call a subset $S \subset G$ a *transversal* in G for G/A if every class in Δ intersects S in exactly one element.

Theorem 2. *Let A be a p.o. abelian group, Δ a p.o. PAM. A p.o. PAM G is a p.o. extension of Δ by A if and only if there are a cocycle f and a family of sets P such that G is an f-product of A by Δ ordered by P.*

Proof. First assume that G is a p.o. extension of Δ by A. Then there is an order preserving embedding of A into G. Without loss of generality we may assume that $A \subset G$. We will prove that A is a g.o. ideal of G. Properties (I1), (I2) and (I3) of Definition 5 follow directly from the group structure of A. Let $x, y \in G^+$ and $x \oplus y \in A$. Since $\pi(G^+) = \Delta^+$, we have $\pi(x), \pi(y) \in \Delta^+$ and $\pi(x) \oplus \pi(y) = \theta$, so that $\pi(x) = \pi(y) = \theta$, and hence $x, y \in A$, so (I4) holds. This proves that A is a g.o. ideal of G. Assume that $g \in G$ is such that $\pi(g) = \alpha \in \Delta$. Then $\pi(g+a) = \alpha$ for every $a \in A$. If $\pi(g) = \pi(h)$, $g, h \in G$, then $\pi(g) \ominus \pi(h) = \theta$, and since π is faithful, $g \ominus h$ is defined and belongs to A. This implies that Δ can be identified with the quotient G/A, and every $g \in G$ can be written in the form (α, a), $\alpha \in \Delta$, $a \in A$. Let ϱ be any transversal which chooses a (unique) representative $\varrho([g])$ in every class $[g] \in G/A$ such that $\varrho([0]) = (\theta, 0)$. Then there is a mapping $t\colon \Delta \to A$ such that $\varrho([g]) = (\alpha, t(\alpha))$ if $\pi(g) = \alpha$, and $t(\theta) = 0$. If $\alpha \perp \beta$, then there is a unique element $c \in A$ such that

$$(\alpha, t(\alpha)) \oplus (\beta, t(\beta)) = (\alpha \oplus \beta, t(\alpha) + t(\beta) + c)$$

where c depends on α and β. Put $c = f(\alpha, \beta)$. The function $f(\alpha, \beta)$ with the domain $\{(\alpha, \beta)\colon\ \alpha \perp \beta\} \subseteq \Delta \times \Delta$ is a cocycle. The proof is left to the reader as an easy exercise.

Let $P_\alpha = \{a \in A\colon\ (\alpha, a) \geqslant (\theta, 0)\}$. Since $\pi(G^+) = \Delta^+$, we have $P_\alpha \neq \emptyset$ iff $\alpha \in \Delta^+$, and $P_\theta = \{a \in A\colon\ (\theta, a) \geqslant (\theta, 0)\} = A^+$. If $\alpha \perp \beta$, then $(\alpha, a) \geqslant (\theta, 0)$, $(\beta, b) \geqslant (\theta, 0)$ implies $(\alpha, a) \oplus (\beta, b) \geqslant (\theta, 0)$, hence $(\alpha \oplus \beta, a + b + f(\alpha, \beta)) \geqslant (\theta, 0)$. It follows that $P_\alpha + P_\beta + f(\alpha, \beta) \subseteq P_{\alpha \oplus \beta}$.

This concludes the proof that any ordered extension G of A by Δ is an f-product of A by Δ ordered by a system P of subsets of A.

Conversely, let G be an f-product of A by Δ. By Lemma 2 and Lemma 3, G is a PAM with an ordering cone $G^+ = \bigcup\limits_{\alpha \in \Delta^+} (\alpha, P_\alpha)$. Consider the set $\{(\theta, a)\colon\ a \in A\} \subset G$. For $(\theta, a), (\theta, b)$ we have $(\theta, a) \oplus (\theta, b) = (\theta, a + b + f(\theta, \theta)) = (\theta, a + b)$, hence there is a bijective group homomorphism $\tau\colon\ a \mapsto (\theta, a)$. It is easy to check that $I = \{(\theta, a)\colon\ a \in A\}$ is a g.o. ideal in G. Indeed, (I1)–(I3) are immediate. To check (I4), let $g, h \in G^+$ and $g \oplus h \in I$. Let $g = (\alpha, a)$, $h = (\beta, b)$, $\alpha, \beta \in \Delta^+$, $a \in P_\alpha$, $b \in P_\beta$, then $(\alpha, a) \oplus (\beta, b) = (\alpha \oplus \beta, a + b + f(\alpha, \beta))$ implies $\alpha \oplus \beta = \theta$, hence $\alpha = \beta = \theta$, and therefore $g, h \in I$. Let $x = (\alpha, a)$, $y = (\beta, b)$ and $x \sim_I y$. Then there is $z \in I$, $z = (\theta, c)$ with $y = x \oplus z$, that is, $(\beta, b) = (\alpha, a) \oplus (\theta, c) = (\alpha, a + c)$, which yields $\alpha = \beta$,

$b = a \oplus c$. So there is a bijection between G/I and Δ. Moreover, $[(\alpha, a)] \perp [(\beta, b)]$ iff $(\alpha, a) \perp (\beta, b)$ iff $\alpha \perp \beta$, which implies that there is a faithful bijective morphism between G/I and Δ. Moreover, $(\beta, b) \leqslant (\alpha, a)$ iff there is $(\delta, d) \geqslant (\theta, 0)$, where $\delta = \alpha \ominus \beta \in \Delta^+$, $d = a - b - f(\beta, \alpha \ominus \beta) \in P_{\alpha \ominus \beta}$, hence $\pi(G^+) \subseteq \Delta^+$, where $\pi \colon G \to G/I$ is the canonical mapping. Conversely, if $\delta \in \Delta^+$, then there is a $d \in P_\delta$, and hence $(\delta, d) \in G^+$. This proves that $\pi(G^+) = \Delta^+$. $\square$

Two extensions G and G' of A by Δ corresponding to f, P and f', P', respectively, are called *o-equivalent* if there is a function $t \colon \Delta \to A$ such that $t(\theta) = 0$,

$$f'(\alpha, \beta) = f(\alpha, \beta) + t(\alpha \oplus \beta) - t(\alpha) - t(\beta) \tag{2}$$

whenever $\alpha \perp \beta$, and

$$P'_\alpha = P_\alpha + t(\alpha) \tag{3}$$

for every $\alpha \in \Delta$. This is equivalent to the existence of a positive isomorphism Φ of G onto G' such that $\Phi(\alpha, a) = (\alpha, a + t(\alpha))$, where $t \colon \Delta \to A$ is any mapping such that $t(\theta) = 0$. This isomorphism leaves the elements of A and the equivalence classes in Δ fixed.

Indeed, assume that (G, f, P) and (G', f', P') are equivalent. Define $\Phi \colon G \to G'$ by $\Phi(\alpha, a) = (\alpha, a + t(\alpha))$. Clearly, Φ leaves the class $[\alpha]$ invariant, and $\Phi((\theta, a)) = (\theta, a + t(\theta)) = (\theta, a)$ owing to $t(\theta) = 0$. If $\alpha \perp \beta$, then

$$\begin{aligned}
\Phi((\alpha, a) \oplus (\beta, b)) &= \Phi(\alpha \oplus \beta, a + b + f(\alpha, \beta)) \\
&= (\alpha \oplus \beta, a + b + f(\alpha, \beta) + t(\alpha \oplus \beta)) \\
\Phi(\alpha, a) \oplus \Phi(\beta, b) &= (\alpha, a + t(\alpha)) \oplus' (\beta, b + t(\beta)) \\
&= (\alpha \oplus \beta, a + t(\alpha) + b + t(\beta) + f'(\alpha, \beta)),
\end{aligned}$$

and owing to equation (2), Φ is a morphism. Considering the inverse mapping $\Phi^{-1} \colon G' \to G$, $\Phi^{-1}(\alpha, a) = (\alpha, a - t(\alpha))$, we prove that Φ is a faithful morphism. Now $(\alpha, a) \in G^+$ iff $a \in P_\alpha$, which yields $a + t(\alpha) \in P_\alpha + t(\alpha)$, i.e., $\Phi(\alpha, a) \in G^+$. Hence Φ is a positive isomorphism.

The proof of the converse implication is similar. It is easy to see that if we choose another transversal ϱ' instead of ϱ in the proof of Theorem 2, we obtain an equivalent extension. $\square$

3. Examples

Example 1. Let (A, A^+) be an abelian p.o. group, (Δ, Δ^+) a p.o. PAM. Consider the cartesian product $G := \Delta \times A$, endowed with the partial order $(\alpha, a) \leqslant (\beta, b)$ iff $\alpha \leqslant \beta$ and $a \leqslant b$, and with $(\alpha, a) \oplus (\beta, b)$ defined iff $\alpha \perp \beta$, and then $(\alpha, a) \oplus (\beta, b) = (\alpha \oplus \beta, a + b)$. Then G is a p.o. extension of Δ by A.

To prove it, put $f(\alpha, \beta) = 0$ for all $\alpha, \beta \in \Delta$ such that $\alpha \perp \beta$. Then f satisfies conditions (c1), (c2), (c3) of Definition 5. For every $\alpha \in \Delta^+$ define $P_\alpha = A^+$. Then $G^+ := \bigcup\{(\alpha, a)\colon\ \alpha \in \Delta^+, a \in A^+\} = \Delta^+ \times A^+$, and the corresponding partial order is given by $(\alpha, a) \leqslant (\beta, b)$ iff $\beta \ominus \alpha \in \Delta^+$ and $b - a \in A^+$.

This extension is called the *direct sum* of the abelian group A and the p.o. PAM Δ. Let G be an extension corresponding to f and P which is equivalent to the direct sum extension. Then we have

$$f(\alpha, \beta) = t(\alpha \oplus \beta) - t(\alpha) - t(\beta) \tag{4}$$

whenever $\alpha \perp \beta$, and

$$P_\alpha = A^+ + t(\alpha) \tag{5}$$

for every α, where $t\colon\ \Delta \to A$ with $t(\theta) = 0$ is any function. A cocycle defined by (4) is called a trivial cocycle.

Example 2. Let A be an abelian group, Δ an ordered PAM, and let f be any cocyle. Put

$$P_\alpha = \begin{cases} A\colon\ \alpha > \theta \\ A^+\colon\ \alpha = \theta \\ \emptyset\colon\ \text{otherwise.} \end{cases}$$

As is readily seen, the conditions (p1), (p2) and (p3) are fulfilled. Accordingly, in the extension PAM G we have:

$$(\alpha, a) \leqslant (\beta, b) \text{ if and only if } \alpha \leqslant \beta, \text{or } \alpha = \beta \text{ and } a \leqslant b.$$

Therefore it is natural to say that G is a *lexicographic extension*. Observe that an extension is lexicographic iff $G^+ = \tau(A^+) \cup \pi^{-1}(\Delta^+ \setminus \{0\})$.

Example 3. Let us first formulate a proposition.

Proposition 1. *Let G be a cancellative p.o. PAM with a positive cone G^+. Let $A \subset G$ be the set of all elements $a \in G$ such that $a \oplus b = 0$ for some $b \in G$. Then*

(i) *$(A, \oplus)$ is a p.o. group.*

(ii) *For $u, v \in G$ define $u \sim_A v$ iff there is $a \in A$ with $u \oplus a = v$. Then $\sim_A$ is an equivalence relation, and the quotient G/A can be endowed with a structure of a p.o. PAM with a positive cone $(G^+ \oplus A)/A$.*

Proof. (i) Let $a \in A$. Then there is $b \in G$ with $a \oplus b = 0$. Owing to cancellativity, this element b is uniquely defined. For any $u \in G$, $u = u \oplus 0 = u \oplus (a \oplus b) = (u \oplus a) \oplus b$. It follows that $u \oplus a$ is defined for every $u \in G$ and $a \in A$. In particular, $a \oplus b$ is defined for every $a, b \in A$. Moreover, there are $c, d \in A$ with $a \oplus c = b \oplus d = 0$, hence $(a \oplus b) \oplus (c \oplus d) = 0$, which entails $a \oplus b \in A$. Clearly, $0 \in A$, and A with the operation $\oplus$ inherited from G becomes an abelian group. Define $A^+ := A \cap G^+$. Then A^+ is closed under $\oplus$, and if $x, 0 \ominus x \in A \cap G^+$, then $x \oplus (0 \ominus x) \in G^+$ implies $x = 0 \ominus x = 0$. Therefore A^+ is a positive cone in A. In addition, the p.o. group (A, A^+) is an ordered subgroup of (G, G^+).

(ii) The relation $u \sim_A v$ iff $u = v \oplus a$, $u, v \in G$, $a \in A$, is clearly symmetric and reflexive, and transitivity easily follows from the fact that A is a group. In addition, the equivalence classes can be written as $[g] = \{g \oplus a \colon a \in A\}$, i.e., "cosets" of G with respect to A. Then the set $C := (G^+ \oplus A)/A$ is a cone in $G/A = \{g \oplus A \colon g \in G\}$. Indeed, since G^+ contains 0 and is closed under existing $\oplus$-sums, the same is true for C. Assume that $x, y \in G/A$ lie in C and $x \oplus y = 0/A$. Then there exist $a, b \in G^+ \oplus A$ such that $a \perp b$, $x = a \oplus A$, $y = b \oplus A$, and $a \oplus b \oplus A = x \oplus y = 0/A$. It follows that $a \oplus b \in A$. Then there is a $c \in A$ with $(a \oplus b) \oplus c = 0 = a \oplus (b \oplus c)$, hence $a \in A$. This proves that $(G^+ \oplus A)/A$ is a positive cone in G/A. $\square$

Consider the sequence

$$0 \longrightarrow A \xrightarrow{\tau} G \xrightarrow{\pi} G/A \longrightarrow 0 \tag{6}$$

where $\tau \colon A \to G$ is the identity mapping, $\pi \colon G \to G/A$ is the quotient mapping $g \to g \oplus A$. Hence (6) is a short exact sequence, and G is the extension of the PAM G/A by the abelian group A.

4. Extensions of PAMs with the Riesz decomposition properties

Definition 7. We say that a p.o. PAM P possesses the Riesz decomposition property if for $a, b, c \in P^+$, $c \leqslant a \oplus b$, there are $0 \leqslant a_1 \leqslant a$, $0 \leqslant b_1 \leqslant b$ such that $c = a_1 \oplus b_1$.

In what follows, we will consider an *upward directed* PAM P, i.e., we will assume that for any $x, y \in P$ there is $z \in P$ such that $x, y \leqslant z$. If P is upward directed, then every element $x \in P$ can be expressed in the form $x = x_1 \ominus x_2$, where $x_1, x_2 \in P^+$. Indeed, for any $x \in P$ there is $z \in P$ such that $0 \leqslant z$, $x \leqslant z$, and so $z = x \oplus (z \ominus x)$, which yields $x = z \ominus (z \ominus x)$ with $z, z \ominus x \in P^+$. For the extensions we have the following lemma.

Lemma 4. *Let A be a directed abelian group, Δ an upward directed p.o. PAM, and let G be a p.o. extension of Δ by A. Then G is upward directed.*

P r o o f. Let $(\alpha, a), (\beta, b)$ be arbitrary elements from G. Since Δ is upward directed, there is $\gamma \in \Delta$ such that $\alpha, \beta \leqslant \gamma$. Then there are $c \in P_{\gamma \ominus \alpha}$ and $d \in P_{\gamma \ominus \beta}$. We have

$$(\alpha, a) \oplus (\gamma \ominus \alpha, c) = (\gamma, a + c + f(\alpha, \gamma \ominus \alpha)) =: (\gamma, u),$$
$$(\beta, b) \oplus (\gamma \ominus \beta, d) = (\gamma, b + d + f(\beta, \gamma \ominus \beta)) =: (\gamma, v).$$

As A is directed, there is $z \in A$ such that $z \geqslant u, v$. Then $(\gamma, z) \geqslant (\alpha, a), (\beta, b)$. $\square$

In accordance with [11] we introduce the following definition.

Definition 8. Let (P, P^+) be an upward directed p.o. PAM. A p.o. abelian group G is called a universal group for P if there is an injective faithful positive morphism $\varphi\colon P \to G$ satisfying the following conditions:

(i) $\varphi(P^+)$ is a convex subset of G^+ and for every $g \in G^+$ there are elements $a_1, \ldots, a_n$ in P^+ such that $g = \varphi(a_1) \oplus \ldots \oplus \varphi(a_n)$.

(ii) Let K be any abelian group and $\kappa\colon P \to K$ a mapping such that $a \perp b$ implies $\kappa(a \oplus b) = \kappa(a) + \kappa(b)$. Then κ uniquely extends to a group homomorphism $\kappa^*\colon G \to K$ such that $\kappa^* \circ \varphi = \kappa$.

We note that if K is partially ordered and κ is positive, then κ^* is a positive group homomorphism, owing to property (i).

Let P be a p.o. PAM with an ordering cone P^+. It is easy to check that P^+ is a generalized effect algebra in the sense of [8, Def. 1.2.1]. Therefore, the following statement can be derived (cf. [8, Theorem 1.7.13] and [8, Prop. 1.7.15]).

Theorem 3. *Let (P, P^+) be an upward directed PAM with the Riesz decomposition property. Then there is an abelian group G with the Riesz decomposition property and an injective faithful positive morphism $\varphi\colon P \to G$ such that (G, φ) is a universal group for P.*

Notice that the group G in Theorem 3 is an interpolation group, that is, if a_1, a_2, b_1, b_2 are elements of G such that any of a_i is under any of b_j, $i, j \in \{1, 2\}$, then there is an element $c \in G$, $a_1, a_2 \leqslant c \leqslant b_1, b_2$. This interpolation property is inherited by P. Unlike the partially ordered groups, the interpolation property need not imply the Riesz decomposition property (see [8], [31]).

Proof of the following theorem is an appropriate adaptation of [12, Theorem 6.1].

Theorem 4. *Let A be a directed abelian group with the Riesz decomposition property, Δ a p.o. PAM with the Riesz decomposition property, and G a p.o. extension of Δ by A corresponding to $f(\alpha, \beta), P_\alpha$. Then G is a p.o. PAM with the Riesz decomposition property if and only if the following two conditions are satisfied:*

(a) *For every $\alpha \in \Delta^+$, P_α is lower directed.*

(b) *$P_\alpha + P_\beta + f(\alpha, \beta) = P_{\alpha \oplus \beta}$ for all $\alpha, \beta \in \Delta^+$, $\alpha \perp \beta$.*

Proof. First assume that G has the Riesz decomposition property. Let $a, b \in P_\alpha$ and choose $c \in A$ with $c \leqslant a, b$. Then each of $(\theta, 0), (\alpha, c)$ is under each of $(\alpha, a), (\alpha, b)$. Since the Riesz decomposition property implies the interpolation property, some (δ, d) can be inserted in between them. Evidently, $\delta = \alpha$ and $d \in P_\alpha$ satisfies $d \leqslant a, b$. Hence (a) is a necessary condition.

We have, by (p3), $P_\alpha + P_\beta + f(\alpha, \beta) \subset P_{\alpha \oplus \beta}$. To every $c \in P_{\alpha \oplus \beta}$ we can find elements a, b such that $a \in P_\alpha$, $b \in P_\beta$ and $c \leqslant a + b + f(\alpha, \beta)$ (since P_α's are upper classes).

Then

$$(\theta, 0) \leqslant (\alpha \oplus \beta, c) \leqslant (\alpha, a) \oplus (\beta, b) = (\alpha \oplus \beta, a + b + f(\alpha, \beta))$$

implies the existence of $a_1 \in P_\alpha$, $b_1 \in P_\beta$ such that $c = a_1 + b_1 + f(\alpha, \beta)$ by the Riesz property in A. Indeed, by the RDP in G, $(\alpha \oplus \beta, c) = (\alpha_1, a_1) \oplus (\beta_1, b_1)$ with $(\alpha_1, a_1) \leqslant (\alpha, a)$, $(\beta_1, b_1) \leqslant (\beta, b)$. Then there exist $(\delta, d) \geqslant (\theta, 0)$, $(\eta, e) \geqslant (\theta, 0)$ such that

$$(\alpha, a) = (\alpha_1, a_1) \oplus (\delta, d) = (\alpha_1 \oplus \delta, a_1 + d + f(\alpha_1, \delta),$$
$$(\beta, b) = (\beta_1, b_1) \oplus (\eta, e) = (\beta_1 \oplus \eta, b_1 + e + f(\beta_1, \eta)$$

which implies $\alpha_1 \oplus \delta = \alpha$, $\beta_1 \oplus \eta = \beta$, so that $\alpha_1 \leqslant \alpha$, $\beta_1 \leqslant \beta$ by (p1). Now

$$(\alpha \oplus \beta, c) = (\alpha_1, a_1) \oplus (\beta_1, b_1) = (\alpha_1 \oplus \beta_1, a_1 + b_1 + f(\alpha_1, \beta_1)$$

implies $\alpha \oplus \beta = \alpha_1 \oplus \beta_1$, which together with the above inequalities yields $\alpha_1 = \alpha$ and $\beta_1 = \beta$. Therefore

$$(\alpha \oplus \beta, c) = (\alpha \oplus \beta, a_1 + b_1 + f(\alpha, \beta)),$$

and $c = a_1 + b_1 = f(\alpha, \beta)$.

Thus

$$P_{\alpha\oplus\beta} \subset P_\alpha + P_\beta + f(\alpha,\beta),$$

and by (p3) we get (b).

Conversely, let A and Δ have the Riesz property and let f and P satisfy (a) and (b). Assume that

$$(\theta,0) \leqslant (\gamma,c) \leqslant (\alpha,a) \oplus (\beta,b) \tag{7}$$

with $a \in P_\alpha$, $b \in P_\beta$. We have to show that

$$(\gamma,c) = (\alpha',a') \oplus (\beta',b')$$

for some $a' \in P_{\alpha'}$, $b' \in P_{\beta'}$ with

$$(\alpha',a') \leqslant (\alpha,a);\ (\beta',b') \leqslant (\beta,b).$$

By the Riesz property of Δ, $\gamma = \alpha_0 \oplus \beta_0$ for some $\alpha_0, \beta_0 \in \Delta$, $\theta \leqslant \alpha_0 \leqslant \alpha$, $\theta \leqslant \beta_0 \leqslant \beta$. Then there is $a_0 \in P_{\alpha_0}$ such that

$$(\theta,0) \leqslant (\alpha_0,a_0) \leqslant (\alpha,a),$$

since $a \in P_\alpha = P_{\alpha\ominus\alpha_0} + P_\alpha + f(\alpha\ominus\alpha_0,\alpha_0)$, $\alpha = (\alpha\ominus\alpha_0)\oplus\alpha_0$, and so $a = a_1 + a_0$ with $a_1 \in P_{\alpha\ominus\alpha_0} + f(\alpha\ominus\alpha_0,\alpha_0)$ and $a_0 \in P_{\alpha_0}$. Similarly,

$$(\theta,0) \leqslant (\beta_0,b_0) \leqslant (\beta,b)$$

for some b_0. Since

$$c \in P_\gamma = P_{\alpha_0\oplus\beta_0} = P_{\alpha_0} + P_{\beta_0} + f(\alpha_0,\beta_0)$$

and $P_{\alpha_0}, P_{\beta_0}$ are lower directed, we may assume that a_0 and b_0 are such that

$$a_0 + b_0 + f(\alpha_0,\beta_0) \leqslant c.$$

Subtracting (α_0,a_0) and (β_0,b_0), we reduce (7) to the case $\gamma=\theta$.

If $\gamma=\theta$, we have

$$(\theta,0) \leqslant (\theta,c) \leqslant (\alpha,a)\oplus(\beta,b) = (\alpha\oplus\beta, a+b+f(\alpha,\beta)),$$

which implies

$$a+b+f(\alpha,\beta)-c \in P_{\alpha\oplus\beta} = P_\alpha + P_\beta + f(\alpha,\beta).$$

So $a+b+f(\alpha,\beta)-c = a_1+b_1+f(\alpha,\beta)$ for some $a_1 \in P_\alpha$, $b_1 \in P_\beta$. Choose $a_2 \in P_\alpha$, $b_2 \in P_\beta$ such that $a_2 \leqslant a, a_1$, and $b_2 \leqslant b, b_1$. Then we have $a_2+b_2 \leqslant a_1+b_1 \leqslant a+b$. The Riesz property of A implies that $a_1+b_1 = a_3+b_3$ with $a_2 \leqslant a_3 \leqslant a$, $b_2 \leqslant b_3 \leqslant b$, and so $a_3 \in P_\alpha$, $b_3 \in P_\beta$. Setting $a' = a-a_3$, $b' = b-b_3$, we have

$$(\theta, a') \oplus (\theta, b') = (\theta, c),$$

where $(\theta,0) \leqslant (\theta,a') \leqslant (\alpha,a)$, $(\theta,0) \leqslant (\theta,b') \leqslant (\beta,b)$. The proof is complete. □

In what follows, $G(\Sigma)$ denotes the universal group of a PAM Σ if it exists. In the next two theorems, we show the relations between the extension of an ordered PAM and the extensions of its universal group by the same abelian p.o. group.

Theorem 5. *Let A be a directed abelian group, Δ an ordered PAM, and H a p.o. extension of Δ by A such that the universal groups $G(A)$ and $G(H)$ of A and H, respectively, exist. Then $G(H)$ is a p.o. extension of $G(\Delta)$ by A.*

P r o o f. We may assume that A is a g.o. ideal of H and that $H/A = \Delta$. We have $A \subset H \subset G(H)$ and $A^+ \subset H^+ \subset G(H)^+$, and it is easily seen that A is an ordered subgroup of $G(H)$. Let $\pi\colon H \to \Delta$ be the canonical mapping. By the properties of universal groups, π can be uniquely extended to a positive group homomorphism $\pi^*\colon G(H) \to G(\Delta)$. Since $\pi(H^+) = \Delta^+$, H^+ generates $G(H)$ and Δ^+ generates $G(\Delta)$, π^* is surjective. Since π^* is an order homomorphism, $\pi^*(G(H)^+) \subset G(\Delta)^+$. On the other hand, every $g \in G(\Delta)^+$ is of the form $g = g_1+\ldots+g_n$, $g_1,\ldots,g_n \in \Delta^+$, and so $\pi^*(g) = \pi(g_1)+\ldots+\pi(g_n) = \pi^*(g_1+\ldots+g_n) \in \pi^*(G(H)^+)$, whence $\pi^*(G(H)^+) = G(\Delta)^+$. Now let $\sigma\colon G(H) \to G(H)/A$ be the quotient mapping. If $h \in H$, then $\sigma(h) = h + A = \pi(h) \in \Delta$, so that σ restricted to H coincides with π. But $\sigma\colon H \to G(\Delta)$ has a unique extension to a positive group homomorphism from $G(H)$ to $G(\Delta)$, and therefore $\sigma = \pi^*$. This proves that $G(H)$ is an ordered extension of the group $G(\Delta)$ by the group A. □

Theorem 6. *Let A be a directed abelian group, Δ an ordered PAM such that its universal group $G(\Delta)$ exists, G a p.o. extension of $G(\Delta)$ by A. Put $H := \{g \in G\colon [g] \in \Delta\}$. Then H is an ordered PAM which is a p.o. extension of Δ by A.*

P r o o f. We may assume that A is an ordered subgroup of G and $G/A = G(\Delta)$. Put $H := \{g \in G\colon [g] \in \Delta\}$, where $g \mapsto [g]$ is the canonical mapping from G to $G(\Delta)$. We have to prove that

(1) (H, H^+) is an ordered PAM, where $H^+ := H \cap G^+$,
(2) A is a g.o. ideal of H,
(3) H/A is isomorphic to Δ.

(1) Clearly, $0 \in H^+$. For $g, h \in H^+$, define $g \perp h$ iff $[g] \perp [h]$ in Δ. Define a partial binary operation on H^+ by $h \oplus g = h + g$ iff $h \perp g$, where $+$ is the group operation on G. It is easy to see that $\oplus$ is commutative and associative and 0 is a neutral element. If $g, h \in H^+$ and $g \oplus h = 0$, then $g + h = 0$ in G^+, whence $g = h = 0$. So H^+ is an ordering cone in H.

(2) For $g \in A$ we have $[g] = 0$, hence $A \subset H$ and $A \cap G^+ = A^+$, hence $A^+ \subset H^+$. If $a \in A$ and $g \in H$, then $[g + a] = [g] \in \Delta$, hence $g \oplus a = g + a \in H$. The remaining properties of a g.o. ideal are easy to check.

(3) Let $\varrho\colon H \to H/A$ be the canonical morphism. For any $g \in H$ we may write $\varrho(g) = h + A$, and since $[h] \in \Delta$ and $[h] = h + A$, we may define a mapping $\nu\colon H/A \to \Delta$, $\nu(\varrho(h)) = [h]$, and it is easy to see that ν is a surjective faithful morphism. Moreover, H/A endowed with the quotient ordering is an ordered PAM, and ν is a positive morphism. In addition, if $[h] \in \Delta^+$, then there is a representative $g \in G^+$ of $[h]$, hence $g \in H^+$. This shows that $\nu(H^+) = \Delta^+$.

From (1), (2), and (3) we may conclude that H is a p.o. extension of Δ by A. $\square$

Using Theorems 5, 6 we can derive the following statements.

Theorem 7. *Let A be a directed abelian group, Δ an upward directed PAM and H a p.o. extension of Δ by A such that each of A, Δ and H satisfies the RDP. Then $G(H)$ is a p.o. extension of $G(\Delta)$ by A possessing the RDP.*

Theorem 8. *Let A be a directed abelian group, Δ an upward directed PAM, G a p.o. extension of $G(\Delta)$ by A such that all of A, Δ and G possess the RDP. Then there is an ordered sub-PAM H of G which is a p.o. extension of Δ by A possessing the RDP.*

Let H be a partially ordered PAM. In agreement with [15, Definition p. 294], a *generating interval* in H^+ is any convex upward directed subset $D \subseteq H^+$ such that every element of H^+ is a sum of elements from D. If $D = [0, u]$, we will say that u is a *generating unit* for H^+. We will write (H, D) to denote an upward directed p.o. PAM where D is a generating interval for H^+. If $D = [0, u]$, we write (H, u) instead of (H, D). For example, if $(G(H), \varphi)$ is a universal group for an ordered PAM H, where $\varphi\colon H \to G(H)$ is the corresponding embedding, then $\varphi(H^+)$ is a generating interval for $G(H)^+$. The following lemma gives a characterization of universal groups for p.o. PAMs with the RDP.

Lemma 5. *Let P be an upward directed partially ordered PAM with the Riesz decomposition property. Then every directed group G with the RDP such that there is an injective faithful morphism $\varphi\colon P^+ \to G^+$ and $\varphi(P^+)$ is a generating interval in G^+, is a universal group for P.*

P r o o f. Let $\varphi\colon P \to G$ be an order isomorphism such that $\varphi(P)$ is a generating interval in G^+. To prove the universal property, assume that K is an abelian group and $\psi\colon P \to H$ a mapping which preserves the partial operation $\oplus$. By the properties of φ, we may identify P^+ with $\varphi(P^+)$. Since $\varphi(P^+)$ is a generating cone in G^+, every element $g \in G^+$ can be written in the form $g = \sum\limits_{i=1}^{n} g_i$ where $g_i \in \varphi(P^+), i = 1, \ldots, n$. Put $\psi_1(g) = \sum\limits_{i=1}^{n} \psi(g_i)$. To prove that $\psi_1\colon G^+ \to K$ is well defined, let $g = \sum\limits_{j=1}^{m} h_j$, $h_j \in \varphi(P^+)$ be any other decomposition of g. Owing to the Riesz decomposition properties, the equality

$$\sum_{i=1}^{n} g_i = \sum_{i=1}^{m} h_j$$

implies that there are elements w_{ij}, $i = 1, \ldots, n; j = 1, \ldots, m$ in G^+ such that $g_i = \sum\limits_{j=1}^{m} w_{ij}$, $h_j = \sum\limits_{i=1}^{n} w_{ij}$. This entails

$$\psi_1(g) = \sum_{i=1}^{n} \psi(g_i) = \sum_{i=1}^{n}\sum_{j=1}^{m} \psi(w_{ij}) = \sum_{j=1}^{m}\sum_{i=1}^{n} \psi(w_i j) = \sum_{j=1}^{m} \psi(h_j).$$

Now let $g \in G$ be an arbitrary element. Since G is directed, $g = q_1 - g_2$ for some $g_1, g_2 \in G^+$. Put $\psi_1(g) := \psi_1(g_1) - \psi_1(g_2)$. If $g = h_1 - h_2$, $h_1, h_2 \in G^+$, is any other decomposition of g, then from $g_1 + h_2 = g_2 + h_1$ we obtain $\psi_1(g_1) + \psi_1(h_2) = \psi_1(h_1) + \psi_1(g_2)$. This proves that $\psi_1\colon G \to K$ is a well defined and unique extension of ψ. $\square$

The following lemma is analogous to [15, Lemma 17.8].

Lemma 6. *Let*

$$0 \longrightarrow H \xrightarrow{\tau} (P, u) \xrightarrow{\pi} Q \longrightarrow 0$$

be a short exact sequence where H is a directed group with the RDP, P and Q are partially ordered PAMs with the RDP, and u is a generating unit for P^+. Then $\tau^{-1}[0, u]$ is a generating interval in H^+, and $[0, \pi(u)]$ is a generating interval in Q^+.

P r o o f. Set $D := \tau^{-1}[0, u]$. It is clear that D is a convex subset of H^+. Given $x_1, x_2 \in D$, we may choose $y \in H$ such that $x_i \leqslant y$ for $i = 1, 2$. Then $\tau(x_1), \tau(x_2) \leqslant \tau(y), u$, and hence by the RDP there exists a $z \in P$ such that $\tau(x_1), \tau(x_2) \leqslant z \leqslant \tau(y), u$. As $\tau(H)$ is a g.o. ideal in P, we have $z = \tau(x)$ for some $x \in H$. Then $x \in D$ and $x_i \leqslant x$ for $i = 1, 2$, which proves that D is upward directed.

Given $x \in H^+$, we have $\tau(x) \in P^+$ and so $\tau(x) = \sum\limits_{i=1}^{n} y_i$ for some $y_i \in [0, u]$. As $0 \leqslant y_i \leqslant \tau(x)$, each $y_i = \tau(x_i)$ for some $x_i \in H^+$. Hence, $x_1, \ldots, x_n$ are elements of D whose sum equals x. Thus D is a generating interval in H^+.

Let $w \in Q^+$. By our hypothesis, there is an element $y \in P^+$ such that $\pi(y) = w$. Then $y = \sum_{i=1}^{n} y_i$ with $y_i \leqslant u$. So $w = \pi(y) = \sum_{i=1}^{n} \pi(y_i)$ with $\pi(y_i) \leqslant \pi(u)$. This proves that $\pi(u)$ is a generating unit in Q^+. $\square$

Given a lexicographic extension

$$0 \longrightarrow (H, D) \xrightarrow{\tau} (P, u) \xrightarrow{\pi} (Q, v) \longrightarrow 0$$

with Q nonzero, observe that $D = H^+$ because $\tau(x) \leqslant u$ for all $x \in H$.

Let (K, u) be a p.o. PAM with order unit u. We say that (K, u) is *simple* if every strictly positive element of K is an order unit.

We have the following generalization of the result [15, Prop. 17.9]. Recall that an abelian group G with the RDP is *unperforated* if $nx \geqslant 0$ for some $n \in \mathbb{N}$ implies $x \geqslant 0$.

Proposition 2. *Let H be a directed unperforated interpolation group and let (Q, v) be a simple p.o. PAM. Then all order-unit extensions (P, u) of (Q, v) by (H, H^+) are lexicographic.*

P r o o f. Let

$$0 \longrightarrow H \xrightarrow{\tau} (P, u) \xrightarrow{\pi} (Q, v) \longrightarrow 0$$

be an order-unit extension of (H, H^+) by (Q, v). Consider any element x in $\pi^{-1}(Q^+ \setminus \{0\})$. As $\pi(x)$ is a strictly positive element in the simple PAM Q, it is an order unit, and hence $v \leqslant \pi(nx)$ for some $n \in \mathbb{N}$. As $\pi(u) = v$, we get $\pi(nx - u) \geqslant 0$, whence $nx - u = (y + \tau(a))$ for some $y \in P^+$ and some $a \in H$. Choose $b \in H^+$ such that $-a \leqslant b$. Since $\tau^{-1}[0, u] = H^+$, we have $\tau(-a) \leqslant \tau(b) \leqslant u$ and so $nx = y + u - \tau(-a) \geqslant y \geqslant 0$, whence $x \geqslant 0$. Thus $\pi^{-1}(Q^+ \setminus \{0\}) \subseteq P^+$, so that the extension is lexicographic. $\square$

Let us recall that an effect algebra is a partial algebraic structure $(E; \oplus, 1, 0)$ where E is a nonempty set, $\oplus$ is a partial binary operation, 0 and 1 are constants such that the following conditions are satisfied [11]:

(E1) $a \oplus b = b \oplus a$,

(E2) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$,

(E3) for every $a \in E$ there is a unique $a' \in E$ with $a \oplus a' = 1$,

(E4) if $a \oplus 1$ is defined then $a = 0$,

where the equalities mean that if one side is defined, so is the other, and the equality holds. It has been shown that an effect algebra E can be viewed as a cancellative positive PAM, i.e., a p.o. PAM such that $E^+ = E$, and such that it contains a unit, that is, an element 1 such that for every $a \in E$ there is a $b \in E$ with $a \oplus b = 1$ [35], [20]. It turns out that in an effect algebra, 0 is a smallest and 1 is a greatest element.

We will say that a PAM P is *unital* if there is a distinguished element u in E such that for every $x \in P$ there is an element $y \in P$ such that $x \oplus y = u$. If P is cancellative, the element y is unique, and we may write $y = x'$. In general, there can be several units in a PAM. So, effect algebras can be identified with positive cancellative unital PAMs.

Theorem 9. *Let Δ be an effect algebra, A a p.o. abelian group. An extension H of Δ by A is a lexicographic extension if and only if any element in H of the form (α, a), $\alpha \in \Delta$, $\alpha \neq 1$, $a \in A$ is majorated by an element of the form $(1, d)$, and $(1, a) \leqslant (1, b)$ iff $a \leqslant b$.*

Proof. Let H be a lexicographic extension of Δ by A. Let $(\alpha, a) \in H$, $1 \neq \alpha \in \Delta$, $a \in A$, and $(1, d) \in H$, $d \in A$. We have to find an element $(\beta, b) \in H^+$ such that $(\alpha, a) \oplus (\beta, b) = (1, d)$. This yields $\alpha \oplus \beta = 1$, $a + b + f(\alpha, \beta) = d$, hence $\beta = \alpha'$, $b = d - a - f(\alpha, \beta)$. Since H is a lexicographic extension and $\alpha' \neq \theta$, we have $(\alpha', b) \in H^+$. Now assume that a p.o. extension H satisfies the conditions of the theorem. Let $x = (\alpha, a) \in H$, $\alpha \neq \theta$, $y = (1, d), d \in A$. Then x is the only element in H such that $(\alpha', d - a - f(\alpha, \alpha') \oplus x = y$, so that x must belong to H^+. It follows that $P_\alpha = A$. Further, $(\theta, a) \geqslant (\theta, 0)$ iff $(1, a) \ominus (1, 0) \geqslant (\theta, 0)$ iff $a \geqslant 0$, by the hypothesis. It follows that $P_\theta = A^+$, and hence H is a lexicographic extension. $\square$

5. Lattice ordered extensions

In this section we consider an ordered extension G of a lattice ordered partial abelian monoid Δ by a lattice ordered abelian group A. We will say that G is an *l-extension* if G is lattice ordered and the morphisms $\tau\colon A \to G$ and $\pi\colon G \to \Delta$ are lattice homomorphisms.

Let G be a p.o. extension of Δ by A. We have $(\beta, b) \leqslant (\alpha, a)$ iff there is a $(\gamma, c) \geqslant (\theta, 0)$ such that $(\beta, b) \oplus (\gamma, c) = (\alpha, a)$, that is, $(\beta \oplus \gamma, b + c + f(\beta, \gamma)) = (\alpha, a)$, which yields $\alpha = \beta \oplus \gamma$ and $a = b + c + f(\beta, \gamma)$, or $\gamma = \alpha \ominus \beta$, $c = a - b - f(\beta, \alpha \ominus \beta)$. From $(\gamma, c) \geqslant (\theta, 0)$ we get $c \in P_\gamma$, hence $a - b - f(\beta, \alpha \ominus \beta) \in P_{\alpha \ominus \beta}$.

Assume that G is an l-extension. For $a \in A$, the mapping $\pi\colon (\alpha, a) \mapsto \alpha$ is an l-homomorphism so that $(\alpha, a) \vee (\beta, b) = (\alpha \vee \beta, d)$, where $d \in A$. According to the above condition, $d - a - f(\alpha, (\alpha \vee \beta) \ominus \alpha) \in P_{(\alpha \vee \beta) \ominus \alpha}$ and similarly, $d - b - f(\beta, (\alpha \vee \beta) \ominus \beta) \in P_{(\alpha \vee \beta) \ominus \beta}$. Hence

$$d \in [a + f(\alpha, (\alpha \vee \beta) \ominus \alpha) + P_{(\alpha \vee \beta) \ominus \alpha}] \cap [b + f(\beta, (\alpha \vee \beta) \ominus \beta) + P_{(\alpha \vee \beta) \ominus \beta}].$$

Since we have supposed that $(\alpha \vee \beta, d)$ is the supremum of (α, a) and (β, b), d should be a smallest element in the set

$$[a + f(\alpha, (\alpha \vee \beta) \ominus \alpha) + P_{(\alpha\vee\beta)\ominus\alpha}] \cap [b + f(\beta, (\alpha \vee \beta) \ominus \beta) + P_{(\alpha\vee\beta)\ominus\beta}].$$

So a necessary condition for preserving the suprema is the existence of a smallest element in the above set, for every $\alpha, \beta \in \Delta$ and $a, b \in A$. Similarly, we can derive that a necessary condition for preserving the infima is the existence of a greatest element in the set

$$[a - f(\alpha \wedge \beta, \alpha \ominus \alpha \wedge \beta) - P_{\alpha\ominus\alpha\wedge\beta}] \cap [b - f(\alpha \wedge \beta, \beta \ominus \alpha \wedge \beta) - P_{\beta\ominus\alpha\wedge\beta}]$$

for every $\alpha, \beta \in \Delta$ and $a, b \in A$.

6. Some applications to MV-algebras

An MV-algebra $A = (A; \dot{+}, \circ, {}^*, 0, 1)$ is an algebraic structure satisfying the following identities: $x \circ (y \circ z) = (x \circ y) \circ z$, $x \circ y = y \circ x$, $x \circ 0 = 0$, $x \circ 1 = x$, $0^* = 1$, $1^* = 0$, $(x^* \circ y)^* \circ y = (y^* \circ x)^* \circ x$, $x \dot{+} y = (x^* \circ y^*)^*$. In [25] it is shown that these identities are equivalent to Chang's original identities [2]. In that paper the author established a categorical equivalence Γ between the category $\mathscr{A}$ of abelian l-groups with a distinguished strong unit and unit-preserving l-homomorphisms, and the category $\mathscr{M}$ of Chang's MV-algebras and MV-homomorphisms. For every abelian l-group G with a distinguished ordering unit g, the structure $\Gamma(G, g)$ is the MV-algebra $A = ([0, g], \dot{+}, \circ, {}^*, 0, g)$ equipped with the following operations: $x \dot{+} y = g \wedge (x + y)$, $x \circ y = 0 \vee (x + y - g)$, $x^* = g - x$ for all x, y in the unit interval $[0, g]$ of G. We refer to [2], [3], [5], [8] for the background of MV-algebras.

Let $(A; \dot{+}, \circ, {}^*, 0, 1)$ be an MV-algebra. Let $\oplus$ denote the restriction of $\dot{+}$ to those pairs (a, b) in $A \times A$ for which $a \leqslant b^*$. Then $(A; \oplus, 0, 1)$ becomes a lattice ordered effect algebra with the RDP. Conversely, every such effect algebra can be organized into an MV-algebra if we put $a \dot{+} b = a \oplus a^* \wedge b$.

Recall that an ideal of an MV-algebra A is a subset I of A such that $a \in I$, $b \leqslant a$ implies $b \in A$, and $a, b \in I$ implies $a \dot{+} b \in I$. A subset I of A is an MV-algebra ideal iff it is an effect algebra ideal in the corresponding effect algebra. We will denote by $\mathscr{R}$ the *radical* of A, that is, the intersection of the maximal ideals of the MV-algebra A. If H is an ideal of A, then the congruence class of an element x of A will be denoted by x/H.

In the sequel, we will need the following theorem (see [6] and [8, Th. 5.247] for similar statements).

Theorem 10. *Let I be an ideal of an MV-algebra $(X; \dot{+}, ^*, 0, 1)$ and let $(G(X); g)$ be the universal group of X. Let $\gamma\colon X \to [0, g]$, $\gamma(1) = g$, be the corresponding universal embedding. Let $\langle I\rangle$ denote the ideal of $G(X)$ generated by the image $\gamma(I)$ of I in $G(X)$. Then $(G(X)/\langle I\rangle)$ is a universal p.o. group for X/I, where $\gamma_I(x/I) := \gamma(x)/\langle I\rangle$ is the universal embedding of X/I into $G(X)/\langle I\rangle$.*

Proof. Since X/I is an MV-algebra, it has a universal group which is lattice ordered. Similarly, $G(X)/\langle I\rangle$ is a lattice ordered group. We will prove that γ_I is a well defined injective order preserving homomorphism.

To prove that γ_I is well defined, assume that $x/I = y/I$, $x, y \in X$. Then $x \wedge y \in X$, $x \ominus x \wedge y \in I$ and $y \ominus x \wedge y \in I$ so that $\gamma(x \ominus x \wedge y) = \gamma(x) - \gamma(x \wedge y) \in \gamma(I) \subset \langle I\rangle$ and similarly $\gamma(y) - \gamma(x \wedge y) \in \langle I\rangle$. Hence $\gamma(x)/\langle I\rangle = \gamma(y)/\langle I\rangle$. In addition, $\gamma_I(x/I)$ is a positive element of $G(X)/\langle I\rangle$.

We will show further that γ_I is injective. Suppose that $x, y \in X$ and $\gamma_I(x/I) = \gamma_I(y/I)$. Then $\gamma(x)/\langle I\rangle = \gamma(y)/\langle I\rangle$ so that $\gamma(x) - \gamma(y) \in \langle I\rangle$, and therefore $(\gamma(x) - \gamma(x \wedge y)) - (\gamma(y) - \gamma(x \wedge y)) \in \langle I\rangle$. As $\langle I\rangle$ is directed, there is a positive element $g \in \langle I\rangle^+$ such that

$$(\gamma(x) - \gamma(x \wedge y)) - (\gamma(y) - \gamma(x \wedge y)) \leqslant g$$

and, since $\gamma(I)$ is a generating interval for $\langle I\rangle$, there are elements $x_1, x_2, \ldots, x_n$ in I such that $g = \sum\limits_{i=1}^{n} \gamma(x_i)$. Then

$$0 \leqslant \gamma(x) - \gamma(x \wedge y) \leqslant \sum_{i=1}^{n} \gamma(x_i) + (\gamma(y) - \gamma(x \wedge y)).$$

By the Riesz decomposition property we can find elements $g_1, \ldots, g_n$, g' in $G^+(X)$ such that $g_i \leqslant \gamma(x_i)$, $i = 1, 2, \ldots, n$, $g' \leqslant \gamma(y) - \gamma(x \wedge y)$ and $\gamma(x) - \gamma(x \wedge y) = \sum\limits_{i=1}^{n} g_i + g'$. Since X is a generating interval for $G^+(X)$, there are elements $x_1', \ldots, x_n'$, x' in X such that $g_i = \gamma(x_i')$, $i = 1, \ldots, n$, $g' = \gamma(x')$. By virtue of the Riesz decomposition property it can be shown that these elements are uniquely defined. So $x_1', \ldots, x_n' \in I$ and $x' \leqslant y - x \wedge y$, and simultaneously $x' \leqslant x - x \wedge y$, which implies that $x' = 0$. It follows that $\gamma(x - x \wedge y) = \sum\limits_{i=1}^{n} \gamma(x_i')$, which implies, by injectivity of γ,

$$x - x \wedge y = x_1' + \ldots + x_n' = x_1' \oplus \ldots \oplus x_n' \in I.$$

By symmetry, $y - x \wedge y \in I$, hence $x/I = y/I$.

For every $x \in X$ we have $x^* = 1 \ominus x$ and

$$\gamma_I(x^*) = \gamma(x^*)/\langle I\rangle = \gamma(1 \ominus x)/\langle I\rangle = \gamma(1)/\langle I\rangle - \gamma(x)/\langle I\rangle = \gamma_I(1) - \gamma_I(x).$$

Let now $z/I = x/I \oplus y/I = (x_1 \oplus y_1)/I$, where $x_1 \in x/I$, $y_1 \in y/I$, $x_1 \perp y_1$. Then

$$\begin{aligned}\gamma_I(z/I) &= \gamma_I((x_1 \oplus y_1)/I) = \gamma(x_1)/\langle I\rangle + \gamma(y_1)/\langle I\rangle\\ &= \gamma(x)/\langle I\rangle + \gamma(y)/\langle I\rangle = \gamma_I(x/I) + \gamma_I(y/I).\end{aligned}$$

This proves that γ_I preserves $\oplus$ on X. In a similar way we prove that γ_I is faithful.

It remains to prove the universal property of $(G(X/I), \gamma_I)$. Suppose that φ is a mapping from X/I into an abelian group K which preserves the $\oplus$ operation. Since $(G(X), \gamma)$ is a universal group for X, there is a unique group homomorphism $\psi\colon G(X) \to K$ such that $\varphi(x/I) = \psi(\gamma(x))$, $x \in X$. Define $\psi_I\colon G(X)/\langle I\rangle \to K$ by $\psi_I(g/\langle I\rangle) = \psi(g)$, $g \in G(X)$. We shall show that ψ_I is well defined. Assume that $g_1 - g_2 \in \langle I\rangle$, where $g_1, g_2 \in G(X)$. Since $\langle I\rangle$ is directed, there are $u, v \in \langle I\rangle^+$ such that $g_1 - g_2 = u - v$. Since I generates $\langle I\rangle$, we have $u = u_1 + \ldots + u_m$, $v = v_1 + \ldots + v_k$ with $u_i, v_j \in I$, $i = 1, \ldots, m, j = 1, \ldots, k$. Then $\psi(u) = \psi(v) = 0$, whence $\psi_I(g_1/\langle I\rangle) = \psi_I(g_2/\langle I\rangle)$. It is easy to see that ψ_I is a group homomorphism and that $\psi_I \circ \gamma_I = \varphi$. $\square$

Theorem 11. *Let M be an MV-algebra, I an ideal of M, and M/I the quotient of M by I. We have the sequence*

$$0 \longrightarrow I \xrightarrow{\tau_0} M \xrightarrow{\pi_0} M/I \longrightarrow 0$$

where τ_0 is the identity mapping and π_0 is the quotient mapping with $\ker(\pi_0) = \tau_0(I)$. Let A be a universal group for I, $G(M)$ a universal group for M, and $G(M/I)$ a universal group for G/M. Then

$$0 \longrightarrow A \xrightarrow{\tau} G(M) \xrightarrow{\pi} G(M/I) \longrightarrow 0$$

where τ and π are the extensions of τ_0 and π_0, respectively, is a short exact sequence of partially ordered abelian l-groups.

Proof. By the properties of universal groups, τ_0 uniquely extends to an injective l-group homomorphism $\tau\colon A \to G(M)$. Similarly, π_0 can be uniquely extended to an l-group homomorphism $\pi\colon G(M) \to G(M/I)$ and since M generates $G(M)$ while M/I generates $G(M/I)$, π is surjective.

It is straightforward to see that $\tau^{-1}(G(M)^+) = A^+$ and $\pi(G(M)^+) = G(M/I)^+$.

It remains to prove that the kernel of π is $\tau(A)$. Clearly, $\tau(A) \subseteq \ker \pi$. Assume that $\pi(g) = 0$, $g \in G(M)$. Then $g = g_1 - g_2$ with $g_1, g_2 \in G(M)^+$, and $\pi(g_1) = \pi(g_2) = \pi(g_1 \wedge g_2)$. So we have $0 \leqslant g_1 - g_1 \wedge g_2 = \sum\limits_{i=1}^{n} z_i$, where $z_i \in M$. From $\pi(g_1 - g_1 \wedge g_2) = 0$ we obtain that $\pi(z_i) = \pi_0(z_i) = 0$, so that $z_i \in I$. By Lemma 5, A can be identified with the l-subgroup $\langle I\rangle$ of $G(M)$ generated by I. It follows that $g_1 - g_1 \wedge g_2 \in A$ and similarly $g_2 - g_1 \wedge g_2 \in A$, so that $g \in A$. $\square$

Using Proposition 2, the following result can be derived.

Theorem 12. *Let I be an ideal of an MV-algebra M such that the MV-algebra M/I is simple. Then $G(M)$ is a lexicographic extension of $G(I)$ by $G(M/I)$.*

The case when $I = R$, the radical of M, and M/R is a subgroup of $\mathbb{R}$ containing 1, has been treated in detail in [28].

Extensions of MV-algebras will be studied in more detail in a subsequent paper.

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