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HOLLAND'S THEOREM FOR PSEUDO-EFFECT ALGEBRAS

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Abstract. We give two variations of the Holland representation theorem for ℓ-groups and of its generalization of Glass for directed interpolation po-groups as groups of automorphisms of a linearly ordered set or of an antilattice, respectively. We show that every pseudo-effect algebra with some kind of the Riesz decomposition property as well as any pseudo MV-algebra can be represented as a pseudo-effect algebra or as a pseudo MV-algebra of automorphisms of some antilattice or of some linearly ordered set.

Keywords: pseudo-effect algebra, pseudo MV-algebra, antilattice, prime ideal, automorphism, unital po-group, unital ℓ-group

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1. Introduction

A fundamental result of Holland [9] says that every ℓ-group G is an ℓ-subgroup of the ℓ-group $A(\Omega)$, the set of all automorphisms of a linearly ordered set Ω. This result was extended to directed interpolation po-groups[1] by Glass [7, Thm. 54] showing that G is isomorphic to a po-subgroup of the po-group $A(\Omega)$, the set of all automorphisms of an antilattice Ω.

Recently, partial algebraic structures, called pseudo-effect algebras and pseudo MV-algebras (as total algebraic structures), were introduced in [4], [5] and in [6], respectively. They can serve as models of quantum structures as well as of non-commutative logic, [8]. Under some natural conditions, it was proved, [4], [5] and [1], that they are precisely the intervals in unital po-groups or in unital ℓ-groups. Using these properties, we give an analogue of the Holland theorem showing that such a

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[1] A po-group G is an *interpolation group* if, for $g_1, g_2 \leqslant h_1, h_2$, there exists an element $s \in G$ such that $g_1, g_2 \leqslant s \leqslant h_1, h_2$, $g_1, g_2, h_1, h_2 \in G^+$.

pseudo-effect algebra is isomorphic to a pseudo-effect algebra of automorphisms of a $\wedge$-antilattice Ω. As a corollary, we show that every pseudo MV-algebra is isomorphic to a pseudo MV-algebra of automorphisms of a linearly ordered set Ω.

Such a representation is useful since it gives a visualization of some pseudo-effect algebras as a set of automorphisms.

The paper is organized as follows. Pseudo-effect algebras and pseudo MV-algebras are presented in Section 2. Ideals P and mainly prime ideals of a pseudo-effect algebra E, and their characterizations via $\wedge$-antilattice properties of cosets E/P are studied in Section 3. A connection among prime ideals and prime subgroups of unital po-groups is shown in Section 4. Finally, the main results, the Holland theorems for pseudo-effect algebras and pseudo MV-algebras, are presented in Section 5.

2. Pseudo-effect algebras

A partial algebra $(E;+,0,1)$, where $+$ is a partial binary operation and 0 and 1 are constants, is called a *pseudo-effect algebra*, [4], [5], if, for all $a,b,c \in E$, the following holds

(i) $a+b$ and $(a+b)+c$ exist if and only if $b+c$ and $a+(b+c)$ exist, and in this case $(a+b)+c = a+(b+c)$;
(ii) there is exactly one $d \in E$ and exactly one $e \in E$ such that $a+d = e+a = 1$;
(iii) if $a+b$ exists, there are elements $d,e \in E$ such that $a+b = d+a = b+e$;
(iv) if $1+a$ or $a+1$ exists, then $a=0$.

If we define $a \leqslant b$ if and only if there exists an element $c \in E$ such that $a+c=b$, then $\leqslant$ is a partial ordering on E such that $0 \leqslant a \leqslant 1$ for any $a \in E$. It is possible to show that $a \leqslant b$ if and only if $b = a+c = d+a$ for some $c,d \in E$. We write $c = a \mathbin{/} b$ and $d = b \setminus a$. Then

$$(b \setminus a) + a = a + (a \mathbin{/} b) = b,$$

and we write $a^- = 1 \setminus a$ and $a^\sim = a \mathbin{/} 1$ for any $a \in E$.

For basic properties of pseudo-effect algebras see [4], [5]. We recall that if $+$ is commutative, E is said to be an *effect algebra*, for properties of effect algebras see [3].

For example, if (G,u) is a unital (not necessary Abelian) po-group with a strong unit u (in fact it is sufficient to take a positive element u in G),[2] and

$$\Gamma(G,u) := \{g \in G\colon\ 0 \leqslant g \leqslant u\},$$

then $(\Gamma(G,u);+,0,u)$ is a pseudo-effect algebra if we restrict the group addition $+$ to $\Gamma(G,u)$.

[2] We say that a positive element u of a po-group G is a *strong unit* if, for any $g \in G$, there is an integer $n \geqslant 1$ such that $g \leqslant nu$.

According to [4], we introduce for pseudo-effect algebras the following forms of the Riesz decomposition properties:

(a) For $a, b \in E$, we write $a\,\mathbf{com}\,b$ to mean that for all $a_1 \leqslant a$ and $b_1 \leqslant b$, a_1 and b_1 commute.

(b) We say that E fulfils the *Riesz interpolation property*, (RIP) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1, a_2 \leqslant b_1, b_2$ there is a $c \in E$ such that $a_1, a_2 \leqslant c \leqslant b_1, b_2$.

(c) We say that E fulfils the *weak Riesz decomposition property*, (RDP_0) for short, if for any $a, b_1, b_2 \in E$ such that $a \leqslant b_1 + b_2$ there are $d_1, d_2 \in E$ such that $d_1 \leqslant b_1$, $d_2 \leqslant b_2$ and $a = d_1 + d_2$.

(d) We say that E fulfils the *Riesz decomposition property*, (RDP) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \in E$ such that $d_1 + d_2 = a_1$, $d_3 + d_4 = a_2$, $d_1 + d_3 = b_1$, $d_2 + d_4 = b_2$.

(e) We say that E fulfils the *commutational Riesz decomposition property*, (RDP_1) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \in E$ such that
   (i) $d_1 + d_2 = a_1$, $d_3 + d_4 = a_2$, $d_1 + d_3 = b_1$, $d_2 + d_4 = b_2$, and
   (ii) $d_2\,\mathbf{com}\,d_3$.

(f) We say that E fulfils the *strong Riesz decomposition property*, (RDP_2) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \in E$ such that
   (i) $d_1 + d_2 = a_1$, $d_3 + d_4 = a_2$, $d_1 + d_3 = b_1$, $d_2 + d_4 = b_2$, and
   (ii) $d_2 \wedge d_3 = 0$.

We introduce analogous notions for po-groups. Let G be a po-group and for $a, b \in G^+$, we write $a\,\mathbf{com}\,b$ iff, for all $a_1, b_1 \in G^+$ such that $a_1 \leqslant a$ and $b_1 \leqslant b$, we have $a_1 + b_1 = b_1 + a_1$.

Let $(G; +, 0, \leqslant)$ be a directed po-group. According to [4], [5], we say that G fulfills (RIP), (RDP_0), (RDP), (RDP_1), and (RDP_2), respectively, if analogous properties as those for pseudo-effect algebras hold also for the positive cone G^+ of G.

A mapping $h\colon E \to F$, where E and F are pseudo-effect algebras, is said to be a *homomorphism* if (i) $h(0) = 0$ and $h(1) = 1$, and (ii) $h(a + b) = h(a) + h(b)$ whenever $a + b$ is defined in E. If h is injective and surjective such that also h^{-1} is a homomorphism, then h is said to be an *isomorphism*, and E and F are *isomorphic*. It is clear that a one-to-one homomorphism f from E onto F is an isomorphism iff $f(a) \leqslant f(b)$ implies $a \leqslant b$.

According to [6], a *pseudo MV-algebra* is an algebra $(M; \oplus, ^-, ^\sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ such that the following axioms hold for all $x, y, z \in M$ with an additional binary operation $\odot$ defined via

$$y \odot x = (x^- \oplus y^-)^\sim$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = 0 \oplus x = x$;
(A3) $x \oplus 1 = 1 \oplus x = 1$;
(A4) $1^\sim = 0$; $1^- = 0$;
(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;
(A6) $x \oplus x^\sim \odot y = y \oplus y^\sim \odot x = x \odot y^- \oplus y = y \odot x^- \oplus x$;[3]
(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;
(A8) $(x^-)^\sim = x$.

If we define $x \leqslant y$ iff $x^- \oplus y = 1$, then $\leqslant$ is a partial order such that M is a distributive lattice with $x \vee y = x \oplus x^\sim \odot y$ and $x \wedge y = x \odot (x^- \oplus y)$. For basic properties of pseudo MV-algebras see [6].

If we define a partial binary operation $+$ on M via: $x + y$ is defined iff $x \leqslant y^-$, and in this case $x + y := x \oplus y$, then $(M; +, 0, 1)$ is a pseudo-effect algebra, and a pseudo-effect algebra E can be converted into a pseudo MV-algebra such that the $+$ derived from $\oplus$ and the original $+$ coincide iff E satisfies (RDP$_2$) [5].

For example, if u is a strong unit of a (not necessarily Abelian) ℓ-group G,

$$\Gamma(G, u) := [0, u]$$

and

$$\begin{aligned} x \oplus y &:= (x + y) \wedge u, \\ x^- &:= u - x, \\ x^\sim &:= -x + u, \\ x \odot y &:= (x - u + y) \vee 0, \end{aligned}$$

then $(\Gamma(G, u); \oplus, ^-, ^\sim, 0, u)$ is a pseudo MV-algebra [6].

The basic representation theorem for pseudo effect-algebras is the following result [4], [5], and for pseudo MV-algebras see also [1].

[3] $\odot$ has a higher priority than $\oplus$.

Theorem 2.1. *For a pseudo-effect algebra E fulfilling* (RDP$_1$), *there is a unique (up to isomorphism of unital po-groups) unital po-group (G,u) fulfilling* (RDP$_1$) *such that $E\cong\Gamma(G,u)$.*

If M is a pseudo MV-algebra, there is a unique (up to isomorphism of unital ℓ-groups) unital ℓ-group (G,u) such that $M\cong\Gamma(G,u)$.

3. Ideals of pseudo-effect algebras

A non-empty subset I of a pseudo-effect algebra E is said to be an *ideal* of E if (i) $x+y\in I$ whenever $x,y\in I$ and if $x+y$ is defined in E, and (ii) if $x\leqslant y$ for $x\in E$ and $y\in I$, then $x\in I$. Then E as well as $\{0\}$ are ideals of E. We denote by $\mathscr{I}(E)$ the set of all ideals of E.

Let $a\in E$, then by $I_0(a)$ we denote the ideal of E generated by a. If E satisfies (RDP$_0$), then by [2, Prop. 3.1],

$$I_0(a)=\{x\in E\colon\ x=a_1+\ldots+a_n,\ a_i\leqslant a,\ i=1,\ldots,n,\ n\geqslant 1\}.$$

An ideal I of E is (i) *normal* if $a+I=I+a$,[4] (ii) *maximal* if I is a proper subset of E and it is not included in any proper ideal of E as a proper subset, and (iii) *prime* if $I_0(a)\cap I_0(b)\subseteq I$ implies $a\in I$ or $b\in I$. We denote by $\mathscr{N}(E)$, $\mathscr{M}(E)$, and $\mathscr{P}(E)$ the set of all normal ideals, maximal ideals, and prime ideals, respectively, of E. Using the Zorn lemma, we see that $\mathscr{M}(E)$ is non-void. Under some conditions on E, [2], we can prove that $\mathscr{M}(E)\subseteq\mathscr{P}(E)$.

We recall that $\{0\},E\in\mathscr{N}(E)$ and if f is a homomorphism from a pseudo-effect algebra E into another one F, then

$$\mathrm{Ker}(f):=\{x\in E\colon\ f(x)=0\}$$

is a normal ideal of E.

The following result was proved in [2, Prop. 3.5].

Proposition 3.1.

(1) *An ideal P of a pseudo-effect algebra E is prime if and only if, for all $I,J\in\mathscr{I}(E)$ with $I\cap J\subseteq P$, we have $I\subseteq P$ or $J\subseteq P$.*

(2) *If P is prime, then $I\cap J=P$ implies $I=P$ or $J=P$. If E satisfies* (RDP), *then an ideal P is prime if and only if, for all $I,J\in\mathscr{I}(E)$ with $I\cap J=P$, we have $I=P$ or $J=P$.*

[4] If A is a non-empty subset of E, then $a+A:=\{a+x:\ x\in A$ and $a+x$ is defined in $E\}$. In a similar way we define $A+a$.

Let P be an ideal of a pseudo-effect algebra E. For $a, b \in E$, we write

$$a \sim_P b$$

iff there are two elements $e, f \in P$ such that $a \setminus e = b \setminus f$. We note that in Remark 3.6, we define another relation, symmetric to $\sim_P$, which coincides with $\sim_P$ in the case of a normal ideal P.

Proposition 3.2. *Let E be a pseudo-effect algebra with* (RDP)*. If P is an ideal of E, then $\sim_P$ is an equivalence on E, and on $E/P = \{a/P \colon\ a \in P\}$, where $a/P := \{b \in E \colon\ b \sim_P a\}$, we can define a partial ordering $a/P \leqslant b/P$ if and only if there is an element $e \in P$ such that $a \setminus e \leqslant b$. If $a \wedge b$ is defined in E, then $(a \wedge b)/P = a/P \wedge b/P$.*

In addition, if P is a normal ideal, then E/P can be organized into a pseudo-effect algebra $(E/P; +, 0/P, 1/P)$, where the partial addition $+$ is defined by $a/P + b/P = c/P$ if and only if there are $a_1 \in a/P$, $b_1 \in b/P$ and $c_1 \in c/P$ such that $a_1 + b_1 = c_1$. Moreover, if P is a normal ideal of an E satisfying (RDP)*, or* (RDP_1)*, or* (RDP_2)*, then so satisfies E/P.*

Proof. (i) $\sim_P$ is an equivalence. It is clear that $a \sim_P a$, and $a \sim_P b$ implies $b \sim_P a$. Assume now $a \sim_P b$ and $b \sim_P c$. There are four elements $e, f, u, v \in P$ such that $a \setminus e = b \setminus f$ and $b \setminus u = c \setminus v$. Therefore, $b = (a \setminus e) + f = (c \setminus v) + u$. Due to (RDP), we can find $c_{11}, c_{12}, c_{21}, c_{22}$ in E such that $a \setminus e = c_{11} + c_{12}$, $c \setminus v = c_{11} + c_{21}$, $f = c_{21} + c_{22}$, and $u = c_{12} + c_{22}$. It is clear that $c_{12}, c_{21}, c_{22} \in P$. Hence, $a = c_{11} + c_{12} + e$ and $c = (b \setminus u) + v = (c_{11} + c_{12} + c_{21} + c_{22}) \setminus (c_{12} + c_{22}) + v = c_{11} + c_{21} + v$. Putting $s = c_{12} + e \in P$ and $t = c_{21} + v \in P$, we have $a \setminus s = c_{11} = c \setminus t$, i.e., $a \sim_P c$.

(ii) We show that $\leqslant$ is a well-defined relation. Assume $a/P = a_1/P$ and $b/P = b_1/P$ and let $a \setminus e \leqslant b$ for some $e \in P$. There are $u, v, s, t \in P$ such that $a \setminus u = a_1 \setminus v$ and $b \setminus s = b_1 \setminus t$. Then $a = (a_1 \setminus v) + u$ and $b = (b_1 \setminus t) + s$, and there is an element $x \in E$ such that $(a \setminus e) + x = b$. Then $s = s_1 + s_2 + s_3$, where $s_1 \leqslant a_1 \setminus v$, $s_2 \leqslant u$, and $s_3 \leqslant x$ Hence

$$\begin{gathered}(a \setminus v) + u + x = (b_1 \setminus t) + s,\\ ((a \setminus v) \setminus s_1)) + s_1 + (u \setminus s_2) + s_2 + (x \setminus s_3) + s_3 = (b_1 \setminus t) + s_1 + s_2 + s_3,\\ (a \setminus (s_1 + v)) + x_1 + x_2 + s_1 + s_2 + s_3 = (b_1 \setminus t) + s_1 + s_2 + s_3,\end{gathered}$$

where $x_1, x_2 \in E$, which gives $(a \setminus (s_1 + v)) \leqslant b_1 \setminus t \leqslant b_1$.

(iii) We now show that $\leqslant$ is a partial order on E/P. It is clear that $a/P \leqslant a/P$. Assume $a/P \leqslant b/P$ and $b/P \leqslant a/P$. There are two elements $a_1, b_1 \in P$ such that $a \setminus a_1 \leqslant b$ and $b \setminus b_1 \leqslant a$. Hence, there exists $x \in E$ such that $(a \setminus a_1) + x = b = (b \setminus b_1) + b_1$. Then $b_1 = b' + b''$, where $b' \leqslant a \setminus a_1$ and $b'' \leqslant x$, which gives

$((a \setminus a_1) \setminus b') + b' + (x \setminus b'') + b'' = (b \setminus b_1) + b' + b''$, i.e., $((a \setminus a_1) \setminus b') + x_1 + b' + b'' = (b \setminus b_1) + b' + b''$, where $x_1 \in E$. Hence, $a \setminus (b' + a_1) + x_1 = b \setminus b_1$, and there exists an element $y \in E$ such that

$$(*) \qquad (a \setminus (b' + a_1)) + x_1 + y = (b \setminus b_1) + y = a = (a \setminus (b' + a_1)) + (b' + a_1),$$

which yields $x_1 + y = b' + a_1 \in P$, and, consequently, $x_1, y \in P$. Using $(*)$, we have $a \setminus (b' + a_1) = b \setminus (x_1 + b_1)$ which proves $a/P = b/P$.

Finally, assume $a/P \leqslant b/P$ and $b/P \leqslant c/P$. There are $a_1, b_1 \in P$ such that $a \setminus a_1 \leqslant b$ and $b \setminus b_1 \leqslant c$. Hence, $(a \setminus a_1) + x = b = (b \setminus b_1) + b_1$ for some $x \in E$. Then $b_1 = b' + b''$, where $b' \leqslant a \setminus a_1$ and $b'' \leqslant x$. Therefore, $((a \setminus a_1) \setminus b') + b' + (x \setminus b'') + b'' = (b \setminus b_1) + b' + b''$, i.e., $(a \setminus (b' + a_1)) + x_1 \leqslant b \setminus b_1 \leqslant c$ for some $x_1 \in E$, which implies $a \setminus (b' + a_1) \leqslant c$ and, consequently $a/P \leqslant c/P$.

(iv) It is clear that $(a \wedge b)/P \leqslant a/P, b/P$. Assume $x/P \leqslant a/P$ and $x/P \leqslant b/P$. There are $x_1, x_2 \in x/P$ such that $x_1 \leqslant a$ and $x_2 \leqslant b$. Since $x_1 \sim_P x_2$, there are $e, f \in P$ with $x_1 \setminus e = x_2 \setminus f$. Hence, for $x_0 = x_1 \setminus e$, we have $x_0 \in x/P$, and $x_0 \leqslant x_1$, $x_0 \leqslant x_2$. Consequently, $x_0 \leqslant a, b$ which yields $x_0 \leqslant a \wedge b$, i.e., $x/P = x_0/P \leqslant (a \wedge b)/P$.

If P is a normal ideal, the assertion was proved in [2, Prop. 4.1]. □

We recall that a poset $(E; \leqslant)$ is (i) an *antilattice* if only comparable elements of E have an infimum or a supremum, (ii) a $\wedge$-antilattice if only comparable elements of E have an infimum. It is clear that any linearly ordered poset is an antilattice. Let E be a pseudo-effect algebra. Then E is an antilattice iff $a \wedge b = 0$ implies $a = 0$ or $b = 0$, while $(a \setminus (a \wedge b)) \wedge (b \setminus (a \wedge b)) = 0$, see [2].

Proposition 3.3. *Let P be an ideal of a pseudo-effect algebra E with* (RDP) *and let $a \leqslant b$, $a, b \in E$. Then $a/P = b/P$ if and only if $a = b \setminus s$ for some $s \in P$.*

Proof. One direction is clear. Assume $a/P = b/P$. There are $e, f \in P$ such that $a \setminus e = b \setminus f$. Then $a = (b \setminus f) + e \leqslant b = (b \setminus f) + f$ which entails $e \leqslant f$. Hence $a = (b \setminus f) + e = (b \setminus (e + (e \mathbin{/} f))) + e = (b \setminus (e \mathbin{/} f)) \setminus e + e = b \setminus (e \mathbin{/} f) = b \setminus s$, where $s = e \mathbin{/} f \in P$. □

Proposition 3.4. *Let P be an ideal of a pseudo-effect algebra E with* (RDP). *Then E/P is a $\wedge$-antilattice if and only if $x/P \wedge y/P = 0/P$ implies $x/P = 0/P$ or $y/P = 0/P$.*

Proof. One direction is evident. Assume $x/P \wedge y/P = 0/P$ implies $x/P = 0/P$ or $y/P = 0/P$. Suppose $a/P \wedge b/P = c/P$. We claim there exists an element $c_0 \in c/P$ such that $c_0 \leqslant a$, $c_0 \leqslant b$ and $(c_0 \mathbin{/} a)/P \wedge (c_0 \mathbin{/} b)/P = 0/P$. Indeed, there are $c_1, c_2 \in c/P$ such that $c_1 \leqslant a$ and $c_2 \leqslant b$. Since $c_1 \setminus e = c_2 \setminus f$, for $c_0 := c_1 \setminus e$, we have $c_0/P = a/P \wedge b/P$.

Assume now $x/P \leqslant (c_0 / a)/P$ and $x/P \leqslant (c_0 / b)/P$. There are two elements $x_1, x_2 \in x/P$ such that $x_1 \leqslant c_0 / a$ and $x_2 \leqslant c_0 / a$. Since $x_1 \sim_P x_2$, there are $e_1, f_1 \in P$ such that $x_0 := x_1 \setminus e_1 = x_2 \setminus f_1 \leqslant c_0 / a, c_0 / b$. Then $c_0 \leqslant c_0 + x_0 \leqslant a, b$, which proves $c_0/P \leqslant (c_0 + x_0)/P \leqslant c/P = c_0/P$, i.e., $c_0/P = (c_0 + x_0)/P$. By Proposition 3.3, there is an element $s \in P$ such that $c_0 = (c_0 + x_0) \setminus s$ which yields $c_0 + s = c_0 + x_0$, i.e., $x_0 = s \in P$, and consequently, $(c_0 / a)/P \wedge (c_0 / b)/P = 0/P$.

By the assumptions, $c_0 / a \in P$ or $c_0 / b \in P$. In the first case, there is $t \in P$ such that $c_0 / a = t$, i.e., $a = c_0 + t$ which by Proposition 3.3 gives $a/P = c_0/P = c/P$, i.e., E/P is an $\wedge$-antilattice. □

Theorem 3.5. *An ideal P of a pseudo-effect algebra E with* (RDP) *is prime if and only if E/P is a $\wedge$-antilattice.*

P r o o f. Assume that P is prime and let $a/P \wedge b/P = 0/P$. We assert that $I_0(a) \cap I_0(b) \subseteq P$. Take $x \in I_0(a) \cap I_0(b)$. Then $x = a_1 + \ldots + a_m = b_1 + \ldots + b_n$, where $a_i \leqslant a$ and $b_j \leqslant b$ for all i and all j. (RDP) implies that there is a system $\{c_{ij}\}$ of elements of E such that $a_i = \sum_j c_{ij}$ and $b_j = \sum_i c_{ij}$. Since $c_{ij} \leqslant a, b$, we have $c_{ij}/P = 0/P$, i.e., $c_{ij} \in P$, which yields $a_i \in P$ and $x \in P$. Since P is prime, then $a \in P$ or $b \in P$, i.e., $a/P = 0/P$ or $b/P = 0/P$, which proves by Proposition 3.4 that E/P is a $\wedge$-antilattice.

Conversely, let E/P be a $\wedge$-antilattice and assume $I_0(a) \cap I_0(b) \subseteq P$. We assert $a/P \wedge b/P = 0/P$. Assume $x/P \leqslant a/P$ and $x/P \leqslant b/P$. As before, there exists an element $x_0 \in x/P$ such that $x_0 \leqslant a, b$. Hence, $x_0 \in I_0(a) \wedge I_0(b) \subseteq P$ which proves $x_0 \in P$, and therefore, $x/P = x_0/P = 0/P$, which implies $a/P = 0/P$ or $b/P = 0/P$, i.e., $a \in P$ or $b \in P$. □

Remark 3.6. Let P be an ideal of a pseudo-effect algebra E with (RDP). We define a new relation ${}_P\!\sim$ on E defined via $a \;{}_P\!\sim b$ iff there are two elements $e, f \in P$ such that $e / a = f / b$. In fact, $\sim_P$ and ${}_P\!\sim$ induce two orderings. Then all previous results can be rewritten also for this relation. In addition, if P is normal, then both orderings induced by $\sim_P$ and ${}_P\!\sim$ coincide.

4. Prime aubgroups of po-groups

Let G be a directed po-group written additively, and let $\mathscr{C}(G)$ denote the set of all convex directed subgroups of G.

In analogy with pseudo-effect algebras, we say that a directed convex subgroup P of a po-group G is a *prime subgroup* of G if, for all directed convex subgroups I and J of G, $I \cap J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. We denote by $\mathscr{P}(G)$ the set of all prime subgroups of a unital po-group G. An equivalent definition is (see [2, Prop. 6.2]): $C \in \mathscr{C}(G)$ is prime iff, for $a, b \in G$, $G_0(a) \cap G_0(b) \subseteq C$ implies $a \in P$ or $b \in P$.

Let $C \in \mathscr{C}(G)$ and define $x/C := \{y \in G\colon\ x - y \in C\}$, and $G/C := \{x/C\colon\ x \in G\}$. We order the set G/C with the usual order of left cosets of G/C via $x/C \leqslant y/C$ iff $x \leqslant y + c$ for some $c \in C$.

The following result has been proved in [7, Lemma 22].

Theorem 4.1. *A convex directed subgroup C of a directed po-group G with* (RIP) *is prime if and only if G/C is a $\wedge$-antilattice.*

If a pseudo-effect algebra $E = \Gamma(G, u)$ satisfies (RDP_1), then there exists a one-to-one correspondence between the sets $\mathscr{I}(E)$ of all ideals or $\mathscr{P}(E)$ of all prime ideals of E and the sets $\mathscr{C}(G)$ and $\mathscr{P}(G)$, respectively, established in [2].

Theorem 4.2. *Let $E = \Gamma(G, u)$, where (G, u) is a unital po-group satisfying* (RDP_1)*. Let I be an ideal of E. Set*

$$\varphi(I) = \{x \in G\colon\ \exists\, x_i, y_j \in I,\ \ x = x_1 + \ldots + x_n - y_1 - \ldots - y_m\}.$$

Then $\varphi(I)$ is an o-ideal of (G, u) if and only if I is a normal ideal of E. In that case

$$(E/I, u/I) = \Gamma(G/\varphi(I), u/\varphi(I)).$$

In addition, if K is an o-ideal of (G, u), then its restriction to E, denoted by $\psi(K)$, gives a normal ideal of E, i.e.,

$$\psi(K) := K \cap E \in \mathscr{I}(E), \quad K \in \mathscr{I}(G, u).$$

Moreover, both mappings, φ and ψ, are mutually bijective and preserve the set-theoretical inclusion.

Theorem 4.3. *Let $E = \Gamma(G,u)$, where (G,u) is a unital po-group satisfying* (RDP$_1$). *Let I be an ideal of E. Set*

$$\begin{aligned}\delta(I) &= \{x \in G\colon\ x = x_1 - y_1 + \ldots + x_n - y_n,\ x_i, y_i \in I\},\\ \delta_c(I) &= \{h \in G\colon\ h = x + p_1 = y - p_2,\ x, y \in \delta(I),\ p_1, p_2 \in G^+\},\\ \delta_0(I) &= \{h_1 - h_2\colon\ h_1, h_2 \in \delta_c(I) \cap G^+\}.\end{aligned}$$

Then $\delta(I)$ and $\delta_c(I)$ is the subgroup and the convex subgroup, respectively, of G generated by I, and $\delta_0(I)$ is the largest directed convex subgroup of G that is contained in $\delta_c(I)$.

Let I and J be two ideals of E. Then $I \subseteq J$ if and only if $\delta(I) \subseteq \delta(J)$ if and only if $\delta_c(I) \subseteq \delta_c(J)$ if and only if $\delta_0(I) \subseteq \delta_0(J)$.

Let K be a convex subgroup of (G,u). Then

$$\gamma(K) := K \cap E$$

is an ideal of E, and $\delta_c(\gamma(K)) \subseteq K$. If K is directed, then $\delta_0(\gamma(K)) = K$, and $\gamma(\delta_0(I)) = I$ for any ideal I of E. In addition, if K_1 and K_2 are two directed convex subgroups of (G,u), then $\gamma(K_1) \subseteq \gamma(K_2)$ if and only if $K_1 \subseteq K_2$.

If K is a prime subgroup of (G,u), then $\gamma(K) := K \cap E$ is a prime ideal of E, and if P is a prime ideal of E, then $\delta_0(P)$ is a prime subgroup of (G,u). In addition, both mappings, γ and δ_0, are mutually bijective and preserve the set-theoretical inclusion.

Moreover, the mappings γ and δ_0 restricted to normal prime ideals and prime o-ideals are mutually bijective.

We recall that if $a, b \in E$ and if I is an ideal of E, then $a/I \leqslant b/I$ iff $a/\delta_0(I) \leqslant b/\delta_0(I)$.

5. Holland theorem and pseudo-effect algebras

Let $(\Omega, \leqslant)$ be a nonvoid $\wedge$-antilattice, and let $A(\Omega)$ be the set of all automorphisms $\alpha\colon \Omega \to \Omega$ which preserve the partial order $\leqslant$. Then $A(\Omega)$ can be converted into a po-group such that the group-addition is the composition of automorphisms, the order on $A(\Omega)$ is defined via $\alpha \leqslant \beta$ iff $(\omega)\alpha \leqslant (\omega)\beta$ for all $\omega \in \Omega$, and the neutral element is the identity function on Ω. If α is a positive element from $A(\Omega)$, then $\Gamma(G,\alpha)$ is a pseudo-effect algebra of automorphisms of an $\wedge$-antilattice set Ω.

Holland [9] proved the basic result that every ℓ-group can be injectively embedded into the ℓ-group $A(\Omega)$ for some linearly ordered set Ω, and Glass [7, Thm. 54] generalized this result to directed po-groups satisfying (RIP) showing that every such a po-group can be embedded into the po-group $A(\Omega)$ for some antilattice Ω.

We show that a similar result can be proved also for pseudo-effect algebras by proving that every pseudo-effect algebra E satisfying (RDP_1) can be embedded into some $\Gamma(A(\Omega),\alpha)$.

Theorem 5.1. *Every pseudo-effect algebra E with* (RDP_1) *can be represented as a pseudo-effect algebra of automorphisms from $A(\Omega)$ for some $\wedge$-antilattice set Ω such that all finite infima and suprema existing in E are preserved.*

Proof. Without loss of generality, by Theorem 2.1, we can assume that $E = \Gamma(G,u)$, where (G,u) is a unital po-group satisfying (RDP_1). The proof will follow the following steps.

Step 1. Let P be a prime ideal of E. According to Theorem 4.3, $\delta_0(P)$ is a prime subgroup of G, and consider the mapping $\varphi_P\colon E \to A(\Omega_P)$, where $\Omega_P = G/\delta_0(P)$, defined by $(x/\delta_0(P))\varphi_P(a) := (x+a)/\delta_0(P)$, $a \in E$ $(x \in G)$. Then, for $a,b \in E$, (i) $a \leqslant b$, implies $\varphi_P(a) \leqslant \varphi_P(b)$, (ii) $\varphi_P(a+b) = \varphi_P(a)\circ\varphi_P(b)$, (iii) $\varphi_P(a\wedge b) = \varphi_P(a)\wedge\varphi_P(b)$ if $a\wedge b$ is defined in E, (iv) $\varphi_P(a\vee b) = \varphi_P(a)\vee\varphi_P(b)$ if $a\vee b$ is defined in E, and (v) $\{a\in E\colon \varphi_P(a)=0\} = \bigcap\{-x+\delta_0(P)+x\colon x\in G\}\cap E = P$. Moreover, we have $E(P) := \varphi_P(E) \subseteq \Gamma(A(\Omega_P),\varphi_P(u))$.

Step 2. Let $g \in G$ and let $g \not\leqslant 0$ and set $U(g) := \{h \in G\colon h \geqslant g\}$, where $E = \Gamma(G,u)$. We denote by $A(g)$ an ideal of E which is maximal with respect to the property $U(g)\cap A(g) = \emptyset$. Since $0 \notin U(g)$, $A(g)$ exists due to the Zorn lemma. We assert $A(g)$ is a prime ideal of E. Let $I\cap J = A(g)$, where I and J are ideals of E. Assume (ad absurdum hypothesis) $A(g)$ that is a proper subset of I as well as of J. Take $a \in I\cap U(g)$ and $b \in J\cap U(g)$. We have $0,g \leqslant a,b$. By (RIP) holding in (G,u), there is an element $c \in G$ such that $0,g \leqslant c \leqslant a,b$. Since $0\leqslant c\leqslant a$, we have $c\in E$, and $g \leqslant c \in I\cap J = A(g)$ which gives $c \in U(g)\cap A(g)$, a contradiction.

Step 3. We define the Cartesian product $E_0 = \prod\{A(\Omega_g)\colon g\in G,\ g\not\leqslant 0\}$ of the system of $\wedge$-antilattices $\{A(\Omega_g)\}_g$, where $\Omega_g = G/\delta_0(A(g))$, and we order E_0 by coordinates. Define a mapping $f\colon E\to E_0$ by $f(a) = \{\varphi_g(a)\}_g$ $(a\in E)$, where $\varphi_g := \varphi_{A(g)}$, and let us set $C_g = \delta_0(A(g))$.

We claim that f is injective. Assume $f(a) = f(b)$. Then $(x+a)/C_g = (x+b)/C_g$ for all $x\in G$ and $g\not\leqslant 0$. In particular, for $x=0$ this gives $a/C_g = b/C_g$. Hence, $a-b = c_g$ for some $c_g \in A(g)$ ($a-b$ is taken in the group G), consequently, $a-b \in \bigcap\limits_{g\not\leqslant 0} C_g = \{0\}$. This proves that f is an injective homomorphism of E onto $f(E)\subseteq E_0$.

Assume $f(a)\leqslant f(b)$. If $g = -b+a \not\leqslant 0$, then $(x+a)/C_g \leqslant (x+b)/C_g$ for all $x\in G$ and $g\not\leqslant 0$. Consequently, this holds also for $x=0$, i.e., $a/C_g \leqslant b/C_g$ which means $a \leqslant b+c'_g$ for some $c'_g \in A(g)$. Therefore, $-b+a\leqslant c'_g$, and $c'_g \in A(g)\cap U(g)$, a contradiction according to Step 2. The set $f(E)$ can be converted into a pseudo-effect algebra, i.e., $(f(E);\circ, f(0), f(1))$ is a pseudo-effect algebra isomorphic to E, where $\circ$ is the composition of automorphisms defined by coordinates.

According to Step 1, f preserves all finite infima and suprema existing in E.

Step 4. Totally order the nonnegative elements of G, say $\{g_t : t \in T\}$, where T is a linearly ordered set. Set $\Omega_t := G/C_{g_t}$, and without loss of generality we can assume $\Omega_s \cap \Omega_t = \emptyset$ for all $s, t \in T$ such that $s \neq t$. Let $\Omega = \bigcup_{t\in T} \Omega_t$, and define a partial order $\preccurlyeq$ on Ω by $\omega_1 \preccurlyeq \omega_2$ iff $\omega_1 \in \Omega_s$ and $\omega_2 \in \Omega_t$ and $s < t$ or $s = t$ and $\omega_1 \leqslant \omega_2$ in Ω_s. Then Ω is a $\wedge$-antilattice with respect to $\preccurlyeq$.

Define a mapping $f_0\colon E \to A(\Omega)$ via: let $\omega \in \Omega$, and $\omega \in \Omega_t$ for a unique $t \in T$. Let $(\omega)f_0(a) = (\omega)(\varphi_{g_t})(a) \in \Omega_t$, where φ_{g_t} is defined in Step 1 and Step 3. Hence, if $a \in E$, then $f_0(a) \mid \Omega_t$ maps Ω_t onto Ω_t for all $t \in T$. Similarly as in Step 3, f_0 is injective from E onto $f_0(E)$, and $f_0(E)$ is a pseudo-effect algebra of automorphisms of Ω (indeed, f_0 practically coincides with the function f defined in Step 3), which finishes the proof. $\square$

As a direct consequence of Theorem 5.1, we show that every pseudo MV-algebra is isomorphic to a pseudo MV-algebra of automorphisms of a linearly ordered set Ω.

Corollary 5.2. *Every pseudo MV-algebra M can be represented as a pseudo MV-algebra of automorphisms from $A(\Omega)$ for some linearly ordered set Ω.*

Proof. Since a pseudo MV-algebra is a distributive lattice, an ideal of a pseudo MV-algebra M (considered as a pseudo-effect algebra) is prime iff M/P is a linearly ordered set. Consequently, $M = \Gamma(G, u)$ for some unital ℓ-group (G, u), M satisfies (RDP$_2$), hence also (RDP$_1$). Hence, the set Ω from the proof of Theorem 5.1 is linearly ordered, which by Theorem 5.1 gives the assertion in question. $\square$

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