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# CIRCUIT AND COCIRCUIT PARTITIONS OF BINARY MATROIDS

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*Abstract.* We give an example of a class of binary matroids with a cocircuit partition and we give some characteristics of a set of cocircuits of such binary matroids which forms a partition of the ground set.

*Keywords*: binary matroid, affine matroid, cocircuit, Eulerian, circuit partition

*MSC 2000*: 05B35

## 1. Introduction

Matroid theory draws heavily on both graph theory and linear algebra for its notation and basic examples. Thus a matroid can be defined in several ways. For example we can define a matroid using the properties of its set of independent sets or its set of circuits. For further details we refer to [3]. Thus studying such type of sets as the set of circuits of a matroid enriches the field. The result of Welsh [4] below is an example of such a study. A matroid $M$ is *Eulerian* if its ground set $E(M)$ has a partition into circuits.

Furthermore, each matroid $M$ has a corresponding matroid $M^*$ called the dual. The circuits of the dual matroid $M^*$ are called the cocircuits of the matroid $M$. For a good introduction to duality theory, we refer to [5]. Thus studying the set of cocircuits gives information on both the matroid and its dual. But studying the set of cocircuits for a general matroid is not very easy, so in this paper we restrict our study to a well known class of binary matroids, see [3], [5].

In this paper we give a class of binary matroids whose ground set has a cocircuit partition and we give some characteristics of a set of cocircuits which forms a partition.

## 2. Circuit and cocircuit partitions of matroids

A matroid is a collection of objects with a certain function of rank defined just like in graphs and matrices. A *matroid* $M(E)$ is a set $E$ with a rank function $r$, for which the following properties hold

(R1) If $X \subseteq E$, then $0 \leqslant r(X) \leqslant |X|$.

(R2) If $X \subseteq Y \subseteq E$, then $r(X) \leqslant r(Y)$.

(R3) If $X$ and $Y$ are subsets of $E$, then

$$r(X \cup Y) + r(X \cap Y) \leqslant r(X) + r(Y).$$

The set $E$ is called the *ground set* of $M(E)$. A *circuit* of $M(E)$ is a non-empty subset $X$ of $E$ such that for all $x$ in $X$, $r(X - x) = |X| - 1 = r(X)$. For each matroid $M(E)$ there is another matroid associated with it. The *dual of a matroid* $M(E)$, denoted by $M^*(E)$, is a matroid with the rank function $r^*$ such that for all $X \subseteq E$,

$$r^*(X) = |X| - r(M) + r(E - X).$$

The circuits of $M^*$ are called the cocircuits of $M$. A function cl from $2^E$ into $2^E$ defined for all $X \subseteq E$ by $\mathrm{cl}(X) = \{x \in E\colon\ r(X \cup x) = r(X)\}$ is called the *closure operator* of $M$. Let $PG(r-1,q)$ be the projective space of rank $r$ over a finite field $GF(q)$ as described by Oxley [3, Chapter 6]. An affine space of rank $r$, denoted $AG(r-1,q)$, is obtained from the projective space $PG(r-1,q)$ by deleting all the points of a hyperplane. A simple matroid $M$ is *affine* over $GF(q)$ if it is isomorphic to a submatroid of $AG(r-1,q)$. In general, a loopless matroid $M$ is affine over $GF(q)$ if its associated simple matroid is affine over $GF(q)$. A *binary matroid* is a matroid that is representable over $GF(2)$. A *circuit partition* of a matroid $M$ is a partition of the ground set of $M$ into circuits. A *cocircuit partition* of a matroid $M$ is a partition of the ground set of $M$ into cocircuits.

If $M$ has a loop it is clear that the ground set of $M$ cannot be partitioned into cocircuits. Recall that $\mathrm{si}(M)$ denotes the simple matroid associated with the matroid $M$. Recall that a matroid $M$ is *Eulerian* if its ground set $E(M)$ has a partition into circuits. Also we say that a matroid $M$ is *bipartite* if every circuit has even cardinality. The next theorem summarizes the relationship between binary Eulerian matroids and binary bipartite matroids, see Welsh [4] and for other details see Brylawski [1] and Heron [2].

**Theorem 2.1** (Welsh 1969). *Let $M$ be a binary matroid. The following are equivalent:*

(i) *$M$ is Eulerian.*

(ii) *Every cocircuit of $M$ has even cardinality.*

(iii) *$M^*$ is bipartite.*

(iv) *$M^*$ has a partition into cocircuits.*

The next theorem is well known.

**Theorem 2.2.** *A binary matroid $M$ is affine over $GF(2)$ if and only if $M$ is bipartite.*

Thus a binary affine matroid has a cocircuit partition.

## 3. A theorem on binary affine matroids

In this section we prove the main theorem of this paper. This theorem is on the characteristic of a binary affine matroid with a cocircuit partition.

Throughout this section $M$ denotes a binary affine matroid of rank $r$ represented over $GF(2)$ by a set $E$ of points of $PG(r-1,2)$ and $H$ denotes the unique hyperplane of $PG(r-1,2)$ such that $H \cap E = \emptyset$. Refer to the definitions in Section 2. For a subset $A$ of $E$, the closure of $A$ in $M$ is denoted by $\text{cl}_M(A)$ and the closure of $A$ in $PG(r-1,2)$ is denoted by $\text{cl}_P(A)$.

**Theorem 3.1.** *Let $M$ be a binary affine matroid of rank $r$ represented over $GF(2)$ by a set $E$ of points of $PG(r-1,2)$ and let $H$ denote the unique hyperplane of $PG(r-1,2)$ such that $H \cap E = \emptyset$. Let $\{C_1, C_2, \ldots, C_k\}$ be a set of pairwise disjoint cocircuits of $M$. Then $\{C_1, C_2, \ldots, C_k\}$ is a partition of $E$ if and only if*

$$r((\text{cl}_P(C_1) \cap H) \cup (\text{cl}_P(C_2) \cap H) \cup \ldots \cup (\text{cl}_P(C_k) \cap H)) = r - k.$$

To ease notation in what follows, for a subset $C$ of $E$, the set $\text{cl}_P(C) \cap H$ will be denoted by $C'$. Before proving Theorem 3.1 we will need the following propositions and lemmas. The next claim follows immediately from the fact that all flats in a projective geometry are modular.

**Claim 3.2.** *Let $F$ be a flat and $H'$ a hyperplane of $PG(n,q)$ such that $F \not\subseteq H'$. Then*

$$r(F \cap H') = r(F) - 1.$$

The next claim follows from the fact that intersections of flats in a matroid are flats.

**Claim 3.3.** *Let $M'$ be a binary matroid and suppose $X$ is the intersection of a three point line and a hyperplane in $M'$. Then $|X| \in \{1,3\}$.*

Recall that throughout this section $M$ denotes a binary affine matroid.

**Lemma 3.4.** *Let $C$ be a cocircuit of $M$. Then*

$$C' \subseteq \mathrm{cl}_P(E - C).$$

P r o o f. We can regard elements of $PG(r-1,2)$ as vectors in $V(r,2)$. Let $\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_m}$ be vectors that form a basis of $M|C$. First, let $\mathbf{b_1}$ be the only such vector. In this case $\mathrm{cl}_P(C)$ is a parallel class of $\mathbf{b_1}$. Hence $C'$ is empty. Thus $C' \subseteq \mathrm{cl}_P(E - C)$. Now consider $\mathbf{b_1} + \mathbf{b_i}$ for $i = 2, 3, \ldots, m$. It is obvious that $\mathbf{b_1} + \mathbf{b_i} \in \mathrm{cl}_{PG}(C)$. It follows that $\{\mathbf{b_1}, \mathbf{b_i}, \mathbf{b_1} + \mathbf{b_i}\}$ is a three point line. But $\mathbf{b_1}$ and $\mathbf{b_i}$ are not in $H$, hence by applying Proposition 3.3, $\mathbf{b_1} + \mathbf{b_i} \in H$. Hence $\mathbf{b_1} + \mathbf{b_i} \in C'$.

We now show that the set $B = \{\mathbf{b_1} + \mathbf{b_2}, \mathbf{b_1} + \mathbf{b_3}, \ldots, \mathbf{b_1} + \mathbf{b_m}\}$ is independent. Consider a sum of the form $\sum_{i=2}^{m} a_i(\mathbf{b_1} + \mathbf{b_i})$ where at least one $a_i$ is non-zero. Let $J = \{j \in \{2, 3, \ldots, m\} \colon\ a_j \neq 0\}$. Then

$$\sum_{j \in J} a_j(\mathbf{b_1} + \mathbf{b_j}) = \begin{cases} \mathbf{b_1} + \sum\limits_{j \in J} \mathbf{b_j} & \text{if } |J| \text{ is odd,} \\ \sum\limits_{j \in J} \mathbf{b_j} & \text{otherwise.} \end{cases}$$

Moreover, $\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_m}$ are independent. Hence both $\mathbf{b_1} + \sum_{j \in J} \mathbf{b_j} \neq 0$ and $\sum_{j \in J} \mathbf{b_j} \neq 0$. Therefore, we have $\sum_{j \in J} a_j(\mathbf{b_1} + \mathbf{b_j}) \neq 0$. Hence $B$ is an independent set. Furthermore, $|B| = r(C) - 1$. But by Proposition 3.2, $r(C') = r(\mathrm{cl}_P(C)) - 1 = r(C) - 1$. Hence $B$ is a maximal independent set of $C'$. Hence $B$ is a basis for $C'$.

Next we show that for all $i \in \{1, 2, \ldots, m\}$ we have $\mathbf{b_1} + \mathbf{b_i} \in \mathrm{cl}_P(E - C)$. We know $E - C$ is a hyperplane of $M$, so $\mathrm{cl}_P(E - C)$ is a hyperplane of $PG(r-1,2)$. But $\mathrm{cl}_P(E - C) \cap E = E - C$. Hence $\mathbf{b_1}, \mathbf{b_i} \notin \mathrm{cl}_P(E - C)$. So by applying Proposition 3.3 we deduce that $\mathbf{b_1} + \mathbf{b_i} \in \mathrm{cl}_P(E - C)$. Hence $B \subseteq \mathrm{cl}_P(E - C)$. It follows that $\mathrm{cl}_P(B) \subseteq \mathrm{cl}_P(E - C)$. Hence $C' \subseteq \mathrm{cl}_P(E - C)$. □

**Lemma 3.5.** *Let $C_1, C_2, \ldots, C_k$ be a set of pairwise disjoint cocircuits of $M$, let $F$ be the set $C'_1 \cup C'_2 \cup \ldots \cup C'_k$ and let $M_1 = PG(r-1,2)\big|E \cup F$. Then $C_i$ is a cocircuit of $M_1$ for $i = 1, 2, \ldots, k$.*

P r o o f. Consider $\mathrm{cl}_P(E - C_i)$. For all $j \neq i$ and $C_j \subseteq (E - C_i)$ we have $\mathrm{cl}_P(C_j) \subseteq \mathrm{cl}_P(E - C_i)$. Hence $C'_j \subseteq \mathrm{cl}_P(E - C_i)$. But by Lemma 3.4, $C'_i \subseteq \mathrm{cl}_P(E - C_i)$. Thus $\mathrm{cl}_{M_1}(E - C_i) = (E - C_i) \cup F$. Therefore $C_i$ is a cocircuit of $M_1$. □

Evidently if $C$ is a circuit of $M$ and $A \cap C = \emptyset$, then $C$ is a circuit of $M \setminus A$. By duality we therefore have the following claim.

**Claim 3.6.** *Let $C$ be a cocircuit of a matroid $M$, let $H = E - C$, and $A \subseteq H$. Then $C$ is a cocircuit in $M/A$*

The next proposition is a direct application of Claim 3.6 in our context.

**Proposition 3.7.** *Let $C_1, C_2, \ldots, C_k$ be cocircuits of $M$, let $F$ be the set $C'_1 \cup C'_2 \cup \ldots \cup C'_k$ and let $M_1 = PG(r-1, 2)\big|E \cup F$. Then $C_i$ is a cocircuit of $M_1/F$ for $i = 1, 2, \ldots, k$.*

Proof. By Lemma 3.5, $C_i$ is a cocircuit of $M_1$. Thus $(E - C_i) \cup F$ is a hyperplane of $M_1$. But $F \subseteq (E - C_i) \cup F$. Hence the result follows by applying Proposition 3.6. □

**Lemma 3.8.** *Let $C_1, C_2, \ldots, C_k$ be cocircuits of $M$ such that $r(C'_1 \cup C'_2 \cup \ldots \cup C'_k) = r - k$. Let $M_1 = PG(r-1, 2)\big|(E \cup C'_1 \cup C'_2 \cup \ldots \cup C'_k)$. Then*
(i) *$r(M_1/(C'_1 \cup C'_2 \cup \ldots \cup C'_k)) = k$;*
(ii) *$C_i$ is a parallel class of $M_1/(C'_1 \cup C'_2 \cup \ldots \cup C'_k)$ for $i \in \{1, 2, \ldots, k\}$.*

Proof. Let $F = C'_1 \cup C'_2 \cup \ldots \cup C'_k$. Consider (i). $r(M_1)/F = r(M_1) - r(F)$. But $r(M_1) = r(M)$ and $r(F) = r(M) - k$. Hence $r(M_1/F) = k$.

Now consider (ii). For any cocircuit, $C_i$ of $M_1$, we have $r_{M_1/C_i}(C_i) = r_{M_1}(C_i \cup C'_i) - r_{M_1}(C'_i)$. But by definition we know that $C'_i \subseteq \mathrm{cl}_{M_1}(C_i)$. Hence $r_{M_1}(C_i \cup C'_i) = r_{M_1}(C_i)$. We also know that $r_{M_1}(C'_i) = r_{M_1}(C_i) - 1$ by applying Proposition 3.2. Hence $r_{M_1/C_i}(C_i) = 1$. It follows that $r_{M_1/F}(C_i) \leqslant 1$. But no element of $C_i$ is in $\mathrm{cl}_{PG}(F)$. Thus no element of $C_i$ is a loop of $M_1/F$. Therefore $r_{M_1/F}(C_i) = 1$. Thus $C_i$ is a parallel class of $M_1/F$. □

**Lemma 3.9.** *Let $M$ be a rank-$k$ loopless matroid containing $k$ parallel classes $P_1, P_2, \ldots, P_k$ of which each $P_i$ is a cocircuit of $M$. Then $E(M) = P_1 \cup P_2 \cup \ldots \cup P_k$.*

Proof. For each $P_i$ there is a corresponding hyperplane $H_i$ such that $E(M) - H_i = P_i$. Thus $\bigcup_i (E(M) - H_i) = P_1 \cup P_2 \cup \ldots \cup P_k$. But $\bigcup_i (E(M) - H_i) = E(M) - \bigcap_i (H_i)$.

Next we show that $\bigcap_i (H_i) = \emptyset$. Consider a set of hyperplanes $\{H_1, H_2, \ldots, H_k\}$. The cocircuit $P_k$ is contained in all hyperplanes, $H_j$, such that $k \neq j$. Hence $P_k \subseteq (H_1 \cap H_2 \cap \ldots \cap H_{k-1})$. Moreover, we know that $P_k \not\subseteq H_k$. Hence

$$(H_1 \cap H_2 \cap \ldots \cap H_{k-1} \cap H_k) \subset (H_1 \cap H_2 \cap \ldots \cap H_{k-1}).$$

But we know that any set which is an intersection of flats is a flat. Thus $(H_1 \cap H_2 \cap \ldots \cap H_{k-1} \cap H_k)$ and $(H_1 \cap H_2 \cap \ldots \cap H_{k-1})$ are flats. We also know that for any two flats $F_1$ and $F_2$ such that $F_1 \subset F_2$ we have $r(F_1) < r(F_2)$. Without loss of generality, it follows that $r(H_1 \cap H_2) < r(H_1)$. Hence $r(H_1 \cap H_2) < k-1 \leqslant k-2$. It follows by recursion that $r(H_1 \cap H_2 \cap \ldots \cap H_{k-1} \cap H_k) \leqslant k-k = 0$. But the rank of a set can not be negative. Thus $r(H_1 \cap H_2 \cap \ldots \cap H_{k-1} \cap H_k) = 0$. Since $M$ is loopless $(H_1 \cap H_2 \cap \ldots \cap H_{k-1} \cap H_k) = \emptyset$. Hence $E(M) = P_1 \cup P_2 \cup \ldots \cup P_k$. □

**Proposition 3.10.** *Let $M$ be a binary affine matroid with a partition $C_1, C_2, \ldots, C_k$ of $E(M)$ into cocircuits. Then $C_i$ is a flat of $M$ for all $i \in \{1, 2, \ldots, k\}$.*

Proof. Consider the following set of hyperplanes, $\{H_i\colon H_i = E(M) - C_i, i \in 1, 2, \ldots, k\}$, of $M$. Without loss of generality let $i = k$. Then $H_k = C_1 \cup C_2 \cup \ldots \cup C_{k-1}$. Moreover for any $k \neq j$, we have $C_k \subseteq H_j$. Thus $C_k \subseteq H_1 \cap H_2 \cap \ldots \cap H_{k-1}$. Furthermore, we know by definition that no element of $C_j$ is contained in $H_j$. Hence $H_1 \cap H_2 \cap \ldots \cap H_{k-1} = C_k$. Hence $C_k$ is a flat since it is an intersection of flats. □

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Let $C_1, C_2, \ldots, C_k$ be a set of pairwise disjoint cocircuits of $M$ and let $M_1 = PG(r-1, 2)\big|E \cup (C_1' \cup C_2' \cup \ldots \cup C_k')$.

Assume that $r(C_1' \cup C_2' \cup \ldots \cup C_k') = r(M) - k$. By Lemma 3.7, each $C_i$ for $i \in \{1, 2, \ldots, k\}$ is a cocircuit of $M_1/(C_1' \cup C_2' \cup \ldots \cup C_k')$. By Lemma 3.8, each $C_i$ for $i \in \{1, 2, \ldots, k\}$ is a parallel class of $M_1/(C_1' \cup C_2' \cup \ldots \cup C_k')$. Moreover, the ground set of $M_1/(C_1' \cup C_2' \cup \ldots \cup C_k')$ is $E$. Thus $E = C_1 \cup C_2 \cup \ldots \cup C_k$ by Lemma 3.9. But we know that $C_1, C_2, \ldots, C_k$ are pairwise disjoint cocircuits of $M$ and $E$ is the ground set of $M$. Hence $C_1, C_2, \ldots, C_k$ is a partition of $E(M)$ into cocircuits.

Assume that $C_1, C_2, \ldots, C_k$ is a partition of $E(M)$ into cocircuits. Then $M$ is a loopless matroid since a cocircuit does not contain loops. We use induction on the number of cocircuits of $M$. Let $M$ be a matroid with one cocircuit $C_1$. Then $M$ is a matroid with a single parallel class. Thus $H = \emptyset$. Therefore $C_1' = \emptyset$. Hence $r(C_1') = 0 = r - 1$. Therefore it holds for a matroid with a single cocircuit.

Assume that for every binary matroid $M$ with a cocircuit partition $C_1, C_2, \ldots, C_n$ of $E(M)$ we have $r(C_1' \cup C_2' \cup \ldots \cup C_n') = r - n$ for $n \geqslant 1$.

Now let $M$ be a binary affine matroid with a partition $C_1, C_2, \ldots, C_n, C_{n+1}$ of $E(M)$. Then by Claim 3.6, $C_1, C_2, \ldots, C_n$ is a cocircuit partition of $M/C_{n+1}$. Let $M_1 = PG(r-1, 2)\big|E \cup C_1' \cup C_2' \cup \ldots \cup C_{n+1}'$. Hence by the induction assumption,

$$r_{M_1/C_{n+1}}(C_1' \cup C_2' \cup \ldots \cup C_n') = r(M_1/C_{n+1}) - n.$$

It is clear that $C'_{n+1}$ is a collection of loops of $M_1/C_{n+1}$. Let $X$ denote $C'_1 \cup C'_2 \cup \ldots \cup C'_{n+1}$. Hence

$$r_{M_1/C_{n+1}}(X) = r(M_1/C_{n+1}) - n. \tag{3.1}$$

But

$$r_{M_1/C_{n+1}}(X) = r_{M_1}(X \cup C_{n+1}) - r_{M_1}(C_{n+1}). \tag{3.2}$$

We now show that

$$r_{M_1}(X \cup C_{n+1}) = r_{M_1}(X) + 1. \tag{3.3}$$

Certainly, $r_{M_1}(X \cup C_{n+1}) > r_{M_1}(X)$ because $C_{n+1} \not\subseteq \mathrm{cl}_{M_1}(X)$. But $r_{M_1}(C_{n+1} \cup C'_{n+1}) = r_{M_1}(C'_{n+1}) + 1$ thus $C_{n+1}$ and $C'_{n+1}$ is a modular pair of flats. Thus

$$r_{M_1}(X \cup C_{n+1}) + r_{M_1}(X \cap C_{n+1}) \leqslant r_{M_1}(X) + r_{M_1}(C_{n+1}).$$

Hence

$$r_{M_1}(X \cup C_{n+1}) \leqslant r_{M_1}(X) + 1.$$

Hence Equation 3.3 holds. Using Equations 3.1, 3.2, 3.3, we see that

$$\begin{aligned} r(C'_1 \cup C'_2 \cup \ldots \cup C'_{n+1}) &= r_{M_1}(X \cup C_{n+1}) - 1 \\ &= r_{M_1/C_{n+1}}(X) + r_{M_1}(C_{n+1}) - 1 \\ &= r(M_1/C_{n+1}) - n + r_{M_1}(C_{n+1}) - 1 \\ &= r(M_1) - n - 1 = r - 1 - n = r - (n+1). \end{aligned}$$

Therefore any binary affine matroid $M$ with a cocircuit partition $C_1, C_2, \ldots, C_k$ has $r(C'_1 \cup C'_2 \cup \ldots \cup C'_k) = r - k$. □

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