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Janusz Jerzy Charatonik; Janusz R. Prajs

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AANR SPACES AND ABSOLUTE RETRACTS FOR TREE-LIKE CONTINUA

Janusz J. Charatonik†, Wrocław,
and Janusz R. Prajs, Sacramento, CA

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Abstract. Continua that are approximative absolute neighborhood retracts (AANR's) are characterized as absolute terminal retracts, i.e., retracts of continua in which they are embedded as terminal subcontinua. This implies that any AANR continuum has a dense arc component, and that any ANR continuum is an absolute terminal retract. It is proved that each absolute retract for any of the classes of: tree-like continua, λ-dendroids, dendroids, arc-like continua and arc-like λ-dendroids is an approximative absolute retract (so it is an AANR). Consequently, all these continua have the fixed point property, which is a new result for absolute retracts for tree-like continua. Related questions are asked.

Keywords: AANR, absolute retract, arc component, arc-like, continuum, decomposable, dendroid, hereditarily unicoherent, retraction, terminal continuum, tree-like

MSC 2000: 54F15, 54C15, 54C55, 54F50, 54H25

1. Introduction

The class of all absolute neighborhood retracts is a large domain for algebraically oriented topologists. For a general topologist, however, it is rather natural to seek an extension of this class such that many of the important properties of ANR's are still satisfied for this extended class. Such generalizations of Borsuk's absolute retracts are the concepts of an approximative absolute retract (abbreviated AAR) and an approximative absolute neighborhood retract (abbreviated AANR) defined by H. Noguchi in [27] and studied, for example, by A. Gmurczyk [13], A. Granas [14] and M. H. Clapp [8].

In this paper we continue the study of AARs and AANRs. First we prove that a continuum X is an AANR if and only if whenever X is a component of a com-

† J. J. Charatonik passed away in Mexico City in July 2004.

pactum Y, then X is a retract of Y. Next, a connection is shown between the study of absolute retracts for tree-like continua and Noguchi's generalization of absolute retracts by proving that each absolute retract for any of the classes of: tree-like, λ-dendroids, dendroids, arc-like, and hereditarily decomposable arc-like continua is an approximative absolute retract.

The concept of terminality of subcontinua, originated in studies of semigroups by A. D. Wallace [28], proved to be an important tool in the continuum theory, and was extensively studied (see e.g. [19], [20], [21] and [22]). The authors realized that most of absolute retracts for the considered classes of tree-like continua are absolute terminal retracts, i.e., retracts of any space in which they are embedded as terminal subcontinua. These investigations eventually led to the main result of the present paper which characterizes approximative absolute neighborhood retracts among continua as absolute terminal retracts.

The paper consists of five sections. After the Introduction and the Preliminaries, in the third section absolute retracts are studied for the classes of hereditarily unicoherent continua that were mentioned above. The leading theorem of this section states that each member of any of these classes is an approximative absolute retract. Consequences of this result are related to the fixed point property. It follows that if a continuum X is the inverse limit of trees with confluent bonding mappings, then it is an absolute retract for the class of tree-like continua, and thus it has the fixed point property. The main result of the paper is established in the fourth section. It states that a continuum is an approximative absolute neighborhood retract if and only if it is an absolute terminal retract. As a consequence of the result it is shown that these continua have a dense arc component. Finally, open questions related to the topic are collected in the last section.

2. Preliminaries

All spaces considered in this paper are assumed to be metric and separable. A *mapping* means a continuous function. The symbol $\mathbb{N}$ means the set of all positive integers, and $\mathbb{Q}$ always denotes the Hilbert cube.

A continuum X is said to be *unicoherent* if the intersection of every two of its subcontinua whose union is X is connected. X is said to be *hereditarily unicoherent* if all its subcontinua are unicoherent. A hereditarily unicoherent and arcwise connected continuum is called a *dendroid*.

A continuum is said to be *decomposable* provided that it can be represented as the union of two its proper subcontinua. A continuum is said to be *hereditarily decomposable* provided that each of its subcontinua is decomposable. A hereditarily unicoherent and hereditarily decomposable continuum is called a λ-*dendroid*.

A *tree* means a graph containing no simple closed curve. A continuum X is said to be *arc-like* (*tree-like*) provided that it is the inverse limit of an inverse sequence of arcs (of trees).

Let X be a metric space with a metric d. For a mapping $f\colon A \to B$, where A and B are subspaces of X, we define

$$d(f) = \sup\{d(x, f(x))\colon x \in A\}.$$

Given a space Y and its subspace $X \subset Y$, a mapping $r\colon Y \to X$ is called a *retraction* if the restriction $r|X$ is the identity. Then X is called a *retract* of Y. The reader is referred to [4] and [17] for needed information on these concepts.

A compactum X is called:

- an *absolute retract for a class $\mathcal{K}$ of spaces* (written AR($\mathcal{K}$)) provided that whenever X is embedded in a space $Y \in \mathcal{K}$ as a closed subset, the embedded copy of X is a retract of Y; if $\mathcal{K}$ is the class of all compacta, we simply write AR instead of AR($\mathcal{K}$);
- an *absolute neighborhood retract* (written ANR) provided that whenever X is embedded in a compactum Y the embedded copy, X', of X is a retract of some neighborhood of X' in Y;
- an *approximative absolute retract* (written AAR) provided that whenever X is embedded in a compactum (or, equivalently, in the Hilbert cube) Y, for each $\varepsilon > 0$ there is a mapping $f\colon Y \to X'$ such that $d(f|X') < \varepsilon$;
- an *approximative absolute neighborhood retract* (written AANR) provided that whenever X is embedded in a compactum (or, equivalently, in the Hilbert cube) Y, for each $\varepsilon > 0$ there are a neighborhood U of the embedded copy X' of X in Y and a mapping $f\colon U \to X'$ such that $d(f|X') < \varepsilon$.

The reader is referred to [13, p. 9] and [8, Sections 1 and 2, pp. 117–119] for a discussion of variants of the last concept. Our definition agrees with the one given in [8, Definition 2.3, p. 118].

Observe the following statement.

Statement 2.1. *A compactum X is an* AANR *if and only if for each homeomorphic copy X' of X in a compactum (or, equivalently, in the Hilbert cube) Y and for each $\varepsilon > 0$, there are a neighborhood U of X' in Y and a mapping $g\colon U \to X'$ with $d(g) < \varepsilon$.*

Applying [8, Theorem 3.1, p. 120] to the case when $X_n \subset X$ we obtain the following proposition. The property assumed in (a) of Proposition 2.2 appeared in [10, Corollaries 2.4 and 2.9, p. 228 and 230, respectively].

Proposition 2.2. *Let X be a continuum.*

(a) *If, for each $\varepsilon > 0$, there exists a mapping $f\colon X \to X$ with $d(f) < \varepsilon$ such that $f(X) \in \mathrm{AR}$ ($f(X) \in \mathrm{ANR}$), then $X \in \mathrm{AAR}$ ($X \in \mathrm{AANR}$, respectively).*

(b) *If $X \in \mathrm{AANR}$, then for each $\varepsilon > 0$ there exists a mapping $f\colon X \to Y \subset X$ with $d(f) < \varepsilon$ such that Y is a locally connected continuum.*

Let Y be a compactum with a metric d. A closed subset A of Y is said to be a Z-*set in* Y provided that for each $\varepsilon > 0$ there is a mapping $f\colon Y \to Y \setminus A$ such that f is ε-near to the identity, i.e., $d(f) < \varepsilon$. See e.g. [18, Section 9, p. 76] or [24, Chapter 3, § 6, p. 262] for more information and bibliography. A continuum is said to have *trivial shape* if it is the intersection of a decreasing sequence of compact absolute retracts.

To prove the principal theorems of the paper we need several auxiliary results. We start with the following two known lemmas. The reader can supply a proof of the former applying [25, Theorem 6.4.8, p. 279] (compare also [9, Corollary 3.5.3, p. 38]). Using the fact that $\mathbb{Q}/A$ is a topological Hilbert cube for each closed Z-set $A \subset \mathbb{Q}$ having trivial shape, and then applying the theorem on extending homeomorphisms of Z-subsets of $\mathbb{Q}$ (see [25, Theorem 6.4.6, p. 278]; compare [9, Theorem 3.5.4, p. 39]), the proof of the latter lemma is straightforward.

Lemma 2.3. *If A and B are Z-sets in the Hilbert cube $\mathbb{Q}$ and $C \subset A \cap B$ is compact, then for each $\varepsilon > 0$ there is a mapping $f\colon A \cup B \to \mathbb{Q}$ such that $f(A \cup B)$ is a Z-set in $\mathbb{Q}$, $f|A = \mathrm{id}_A$, $f|B$ is a homeomorphism, $f(A) \cap f(B) = C$, and $d(f) < \varepsilon$.*

Lemma 2.4. *Let a subset L of the Hilbert cube $\mathbb{Q}$ be a Z-set in $\mathbb{Q}$ having trivial shape, and let a closed subset K of $\mathbb{Q}$ be disjoint with L. Then there exists a quotient mapping $q\colon \mathbb{Q} \to \mathbb{Q}/L = \mathbb{Q}$ such that $q|K = \mathrm{id}_K$, and $q|(\mathbb{Q} \setminus L)\colon \mathbb{Q} \setminus L \to \mathbb{Q} \setminus q(L)$ is a homeomorphism.*

The above lemmas will be applied to show the next one.

Lemma 2.5. *Let a subset A of the Hilbert cube $\mathbb{Q}$ be a Z-set in $\mathbb{Q}$, let U be an open connected subset of $\mathbb{Q}$, and let $a, b \in A \cap U$. Then there exists a quotient mapping $q\colon A \to q(A) \subset \mathbb{Q}$ such that*

(2.5.1) $$q(A) \text{ is a Z-set in } \mathbb{Q};$$

(2.5.2) $$q|(A \setminus U) = \mathrm{id}_{(A\setminus U)};$$

(2.5.3) $$q(A \cap U) \subset U;$$

(2.5.4) *a, b is the only pair of distinct points which are identified under q.*

Proof. Join the points a and b by an arc $ab \subset U$ which is a Z-set, and observe that $A \cup ab$ is a Z-set in $\mathbb{Q}$. Applying Lemma 2.3 with $B = ab$ and $C = \{a, b\}$ we get a new arc $L \subset U$ from a to b such that $A \cap L = \{a, b\}$. Apply now Lemma 2.4 with $K = \mathbb{Q} \setminus U$. The mapping q guaranteed by Lemma 2.4, when restricted to A, is the required mapping. □

3. Absolute retracts for some classes of continua

The following notation will be used. Given an inverse sequence $\mathbf{S} = \{X_n, f_n\}$ of compact spaces X_n with bonding mappings $f_n\colon X_{n+1} \to X_n$, where the set of positive integers $\mathbb{N}$ is taken as the directed set of indices, we denote by $X = \varprojlim \mathbf{S}$ its inverse limit and by $\pi_n\colon X \to X_n$ the projections. Further, let $f^n_m\colon X_n \to X_m$ be the bonding mapping $f^n_m = f_m \circ f_{m+1} \circ \ldots \circ f_{n-1}$ of $\mathbf{S}$ for $m < n$, and $f^n_n = \mathrm{id}|X_n$. In particular, $f^{n+1}_n = f_n$.

The following construction will be useful in proofs of further results in this section.

Construction 3.1. Let $\mathbf{S} = \{X_n, f_n\}$ be an inverse sequence of continua X_n with surjective bonding mappings f_n, having a continuum $X = \varprojlim \mathbf{S}$ as its inverse limit. Consider the factor spaces X_n as embedded in the Hilbert cube $\mathbb{Q}$ in such a way that

$$s = (0, 0, \ldots) \in \mathbb{Q} \setminus \bigcup\{X_n\colon\ n \in \mathbb{N}\}.$$

Let

$$X_0 = \{s\} \times \left(\varprojlim\{X_n, f_n\}\right) \subset \mathbb{Q}^\omega. \tag{3.1.1}$$

In the Hilbert cube represented as the countable product $\mathbb{Q}^\omega$, for each $m \in \mathbb{N}$ define

$$Y_m = \{(s, x_1, \ldots, x_m, s, s, \ldots)\colon\ (s, x_1, \ldots, x_m, x_{m+1}, \ldots) \in X_0\}. \tag{3.1.2}$$

Then

(3.1.3) the mapping $e_n\colon\ X_n \to Y_n \subset \mathbb{Q}^\omega$ defined by $e_n(x_n) = (s, f^n_1(x_n), f^n_2(x_n), \ldots, f^n_n(x_n), s, s, \ldots)$ is an embedding;

(3.1.4) $X_0 \cap Y_n = \emptyset = Y_m \cap Y_n$ for $m \neq n$;

(3.1.5) the mappings $\alpha_n\colon\ X_0 \to Y_n$ defined by $(s, x_1, x_2, \ldots) \mapsto (s, x_1, \ldots, x_n, s, s, \ldots)$ correspond to the projections $\pi_n\colon\ X \to X_n$, and they approximate the identity id_{X_0}.

Finally, let

$$Y(\mathbf{S}) = X_0 \cup \bigcup\{Y_n\colon\ n \in \mathbb{N}\}. \tag{3.1.6}$$

Since $Y(\mathbf{S}) \subset \{s\} \times \mathbb{Q}^\omega \subset \mathbb{Q} \times \mathbb{Q}^\omega = \mathbb{Q}^\omega$, it follows that

$$Y(\mathbf{S}) \text{ is a Z-set in } \mathbb{Q}^\omega. \tag{3.1.7}$$

Construction 3.1 implies the following result.

Theorem 3.2. *Let* $\mathbf{S} = \{X_n, f_n\}$ *be an inverse sequence of continua* X_n *with surjective bonding mappings* f_n, *having a continuum* $X = \varprojlim \mathbf{S}$ *as its inverse limit, and let the compactum* $Y(\mathbf{S})$ *be defined by* (3.1.6). *If there exists a retraction* $r\colon Y(\mathbf{S}) \to X_0$ *and if the factor space* X_n *is an* AR *(an* ANR*) for each* $n \in \mathbb{N}$, *then the inverse limit space* X *is an* AAR *(an* AANR, *respectively).*

P r o o f. Take any $\varepsilon > 0$. For each $n \in \mathbb{N}$ let $\alpha_n\colon X_0 \to Y_n$ be as in (3.1.5), and define $g_n\colon Y_n \to X_0$ as $r|Y_n$. For sufficiently large n we have $d(\alpha_n) < \varepsilon$ and $d(g_n) < \varepsilon$. Therefore the sequence of continua Y_n converges to X_0 in the metric of continuity in the sense of [8, Definition 3.1, p. 119]. Hence X_0 is an AAR (an AANR, respectively) by [8, Theorem 3.1, p. 120]. □

We will use the following abbreviations for classes of continua:

- $\mathcal{D}$ for dendroids,
- $\lambda\mathcal{D}$ for λ-dendroids,
- $\mathcal{AL}$ for arc-like continua,
- $\lambda\mathcal{AL}$ for arc-like λ-dendroids, i.e., hereditarily decomposable arc-like continua,
- $\mathcal{TL}$ for tree-like continua.

Note that each member of any of these classes is a hereditarily unicoherent continuum.

Theorem 3.3. *Let* $\mathcal{K}$ *be any of the following classes of continua:* $\mathcal{TL}$, $\lambda\mathcal{D}$, $\mathcal{D}$, $\mathcal{AL}$ *and* $\lambda\mathcal{AL}$. *Then each member of* $\mathrm{AR}(\mathcal{K})$ *is an* AAR.

P r o o f. Let $\mathcal{K}$ be any of the four classes $\mathcal{TL}$, $\lambda\mathcal{D}$, $\mathcal{AL}$ and $\lambda\mathcal{AL}$, and let $X \in \mathrm{AR}(\mathcal{K}) \subset \mathcal{K}$. Thus X can be represented as the inverse limit of an inverse sequence $\mathbf{S} = \{X_n, f_n\}$, where the factor spaces X_n are either trees (for the classes $\mathcal{TL}$ and $\lambda\mathcal{D}$), or arcs (for the classes $\mathcal{AL}$ and $\lambda\mathcal{AL}$).

We apply Construction 3.1. So, let X_0 be defined by (3.1.1). Denote by $\{W_n\colon n \in \mathbb{N}\}$ a basis of open connected neighborhoods of X_0 in $\mathbb{Q}^\omega$ such that $Y_k \subset W_n$ for $k \geqslant n$. For each $n \in \mathbb{N}$ choose two distinct end points a_n and b_n of Y_n, and take an arc $A_n \subset W_n$ being a Z-set in $\mathbb{Q}^\omega$ and having b_n and a_{n+1} as its end points. Applying Lemma 2.3 we may inductively modify these arcs to some arcs A'_n such that

$$b_n \text{ and } a_{n+1} \text{ are the end points of } A'_n; \tag{3.3.1}$$

$$A'_n \cap Y(\mathbf{S}) = \{b_n, a_{n+1}\}, \text{ and } A'_m \cap A'_n = \emptyset \text{ for } m \neq m; \tag{3.3.2}$$

$$A'_n \subset W_n. \tag{3.3.3}$$

Then

$$Y = Y(\mathbf{S}) \cup \bigcup \{A'_n \colon\ n \in \mathbb{N}\} \tag{3.3.4}$$

is a continuum in $\mathcal{K}$, and it contains the copy X_0 of X. Since $X \in \mathrm{AR}(\mathcal{K})$, there exists a retraction $r\colon\ Y \to X_0$. Since each X_n is a tree or an arc, thus an AR, the conclusion follows from Theorem 3.2.

In the case when $\mathcal{K} = \mathcal{D}$ we change the previous construction choosing a_n as an arbitrary point of X_0 (instead of it being an end point of Y_n) so that $a_m \neq a_n$ for $m \neq n$. Then the union Y obtained as in (3.3.4) is a dendroid, and the rest of the proof remains unchanged. The proof is thus complete. □

Remark 3.4. Actually, we can prove a stronger condition than the conclusion of Theorem 3.3, namely condition (5.2.3) of Proposition 5.2 below in the variant for AR's (see Observation 5.4). We do not know, however, if this condition is essentially stronger, see Question 5.3.

Remark 3.5. The conclusion of Theorem 3.3 for the case $\mathcal{K} = \mathcal{D}$ can also be deduced from [12, Theorem 2, p. 261] using Proposition 2.2. Indeed, each member X of $\mathrm{AR}(\mathcal{D})$ is a smooth dendroid according to [6, Corollary 3.6]. Hence for each $\varepsilon > 0$ there is a retraction $r\colon\ X \to T \subset X$ for some tree T with $d(r) < \varepsilon$. Thus the conclusion follows from Proposition 2.2. Observe that in this way we have actually proved condition (5.2.2) for $\mathrm{AR}(\mathcal{D})$. This condition is stronger than (5.2.3), see Remark 3.4 above, and stronger than the conclusion of Theorem 3.3. But we again do not know if this strengthening is essential, see Question 5.3.

One of important topics in the fixed point theory concerns fixed point property for tree-like continua. Namely, tree-like continua without fixed point property are known, [2], however the planar case remains open. Nevertheless, if a tree-like continuum is an AAR, we have the following result, see [14, Corollary, p. 19].

Theorem 3.6. *Each* AAR *continuum has the fixed point property.*

A consequence of Theorems 3.3 and 3.6 is the following.

Corollary 3.7. *Each member of* $\mathrm{AR}(\mathcal{TL})$ *has the fixed point property.*

If a tree-like continuum X is an absolute retract for the class of hereditarily unicoherent continua, then $X \in \mathrm{AR}(\mathcal{TL})$. Therefore, using Theorem 3.6 of [5], we get the following theorem.

Theorem 3.8. *Let* $\mathbf{S} = \{X_n, f_n\}$ *be an inverse sequence of trees* X_n *with confluent bonding mappings* f_n. *Then* $X = \varprojlim \mathbf{S} \in \mathrm{AR}(\mathcal{TL})$.

Theorem 3.8 and Corollary 3.7 imply the next corollary.

Corollary 3.9. *If a continuum* X *is the inverse limit of an inverse sequence of trees with confluent bonding mappings, then* X *has the fixed point property.*

Remark 3.10. For the classes $\lambda\mathcal{D}$, $\mathcal{D}$, $\mathcal{AL}$ and $\lambda\mathcal{AL}$ similar results to the one in Corollary 3.7 are known. Actually, all these classes are composed of continua having the fixed point property, see [23], [3] and [16], correspondingly. Compare [15].

4. Characterizations of AANR continua

We start this section with the following characterization of AANR compacta.

Theorem 4.1. *A compactum* X *is an* AANR *if and only if for any embedding* $e\colon X \to Y$ *into a compactum* Y *such that* $e(X)$ *is the union of some components of* Y, *the set* $e(X)$ *is a retract of* Y.

Proof. Assume that X is an AANR and that X is the union of some components of a compactum Y. Then X is closed in Y, and thus there is a sequence of open and closed subsets U_n of Y such that $U_{n+1} \subset U_n$ and $\bigcap\{U_n\colon n \in \mathbb{N}\} = \operatorname{Lim} U_n = X$. Since X is an AANR, for each $n \in \mathbb{N}$ there is a mapping $f_n\colon U_n \to X$ such that $\lim d(f_n) = 0$. Fix a point x_0 in X. For any $x \in Y$ define

$$r(x) = \begin{cases} x_0 & \text{if } x \in Y \setminus U_1, \\ f_n(x) & \text{if } x \in U_n \setminus U_{n+1}, \\ x & \text{if } x \in X, \end{cases}$$

and observe that $r\colon Y \to X$ is a retraction.

Assume now that X is a retract of any compactum Y containing X as the union of some components of Y. To prove that X is an AANR assume that X is a subset of a metric space Z with a metric ϱ. Since Z can be compactified, we can assume that Z is compact. For $n \in \mathbb{N}$ let $V_n = \{z \in Z\colon \varrho(x, X) \leqslant 1/n\}$, and let $H = \bigl\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\bigr\}$. Define

$$Z_0 = X \times \{0\} \cup \bigcup\Bigl\{V_n \times \Bigl\{\frac{1}{n}\Bigr\}\colon n \in \mathbb{N}\Bigr\}$$

as a subspace of the product $Z \times H$. Then the copy $X \times \{0\}$ of X is the union of some components of Z_0. Therefore, by assumption, there exists a retraction $r\colon Z_0 \to X \times \{0\}$. Let $\pi\colon Z_0 \to X$ be the projection onto the first factor. For each $n \in \mathbb{N}$ define mappings $f_n\colon V_n \to X$ by letting $f_n(x) = (\pi \circ r)((x, 1/n))$ for any $x \in V_n$. Observe that $\lim d(f_n) = 0$. This implies that Z is an approximative retract of Z, and therefore X is an AANR. $\square$

Corollary 4.2. *A continuum is an* AANR *if and only if, whenever embedded into a space as its component, it is a retract of this space.*

Recall that a subcontinuum X of a continuum Y is called *terminal in* Y if every subcontinuum of Y that intersects X and $Y \setminus X$ contains X. For convenience let us accept the following definition.

Definition 4.3. A continuum X is said to be an *absolute terminal retract* provided that, if X is embedded in a continuum Y in such a way that the embedded copy, X', is a terminal subcontinuum of Y, then X' is a retract of Y.

The following notation will be used in the proof of the next lemma. Given a subset A of a space Y and a number $\varepsilon > 0$, we denote by $N(A, \varepsilon)$ the open ε-neighborhood about A in X. Further, given a sequence of subsets A_n of X, we denote by $\operatorname{Ls} A_n$ the upper limit of the sequence, i.e., the set of all points $p \in X$ with the property that each neighborhood of p intersects infinitely many sets A_n.

Lemma 4.4. *If X is a terminal subcontinuum of a continuum Y, then there exists a mapping $f\colon Y \to f(Y) = Z$ such that*

$$f|X = \mathrm{id}_X; \tag{4.4.1}$$

$$f(Y \setminus X) = Z \setminus X \tag{4.4.2}$$

$$X \text{ is a terminal subcontinuum of } Z; \tag{4.4.3}$$

$$\begin{gathered} \text{there exists a basis of neighborhoods } U_n \text{ of } X \text{ in } Z \\ \text{with } \operatorname{bd} U_n = \{p_n\} \text{ for some sequence} \\ \{p_1, p_2, \ldots\} \subset Z \setminus X \text{ such that } X \subset \operatorname{cl}\{p_1, p_2, \ldots\}. \end{gathered} \tag{4.4.4}$$

Further, if X is a retract of Z, then X is a retract of Y.

P r o o f. We start with showing two auxiliary facts.

Claim 1. *For each closed set $D \subset Y \setminus X$, for each point $a \in X$ and for each $\varepsilon > 0$ there exists an open set $V \subset Y \setminus X$ such that $D \subset V$ and $\operatorname{bd} V \subset N(a, \varepsilon) \setminus X$.*

To show Claim 1 let $A \subset N(a, \varepsilon/2)$ be an open set such that $A \cap X \neq \emptyset = D \cap \operatorname{cl} A$, and let G be the union of all components of $Y \setminus A$ that intersect D. Then G is closed in Y, and $G \cap X = \emptyset$ by the terminality of X. Choose $\delta > 0$ such that $N(G, \delta) \cap X = \emptyset$. Then there exists an open and closed subset H of $Y \setminus A$ such that $G \subset H \subset N(G, \delta/2)$. Define $V = \operatorname{int} H$ (with respect to Y). Thus $D \subset V$ and $\operatorname{bd} V \subset \operatorname{cl} A \cap N(G, \delta/2)$. Therefore $\operatorname{diam} \operatorname{bd} V < \varepsilon$ and $\operatorname{bd} V \subset N(X, \varepsilon) \setminus X$. The proof of Claim 1 is complete.

Claim 2. *For each $\varepsilon > 0$ and for each neighborhood U of X in Y there exists a neighborhood W of X in Y such that $\operatorname{cl} W \subset U$ and that each continuum K intersecting $\operatorname{cl} W$ and $Y \setminus U$ satisfies $X \subset N(K, \varepsilon)$.*

To see Claim 2, take a basis of neighborhoods U_n of X in Y such that $U_{n+1} \subset U_n$ for each $n \in \mathbb{N}$, and suppose that there are continua $K_n \subset Y$ such that $K_n \setminus U \neq \emptyset \neq K_n \cap U_n$ with $X \setminus N(K_n, \varepsilon) \neq \emptyset$ for each n. Taking a convergent subsequence we may assume that the continua K_n converge to a continuum K (in the sense of the Hausdorff metric). Thus

$$K \cap X \neq \emptyset, \quad \emptyset \neq K \setminus U \subset K \setminus X \quad \text{and} \quad X \setminus K \neq \emptyset,$$

contrary to the terminality of X. This completes the argument for Claim 2.

Now we pass to the main part of the proof of Lemma 4.4. Choose a dense set $\{a_1, a_2, \ldots\}$ in X, $\varepsilon_n = 1/n$ and take a neighborhood W_1 of X in Y with $W_1 \subset N(X, \varepsilon_1)$ and $Y \setminus W_1 \neq \emptyset$. Let V_1 be the set guaranteed by Claim 1 for $D = Y \setminus W_1$, $a = a_1$ and $\varepsilon = \varepsilon_1$. Further, take $W_2 \subset N(X, \varepsilon_2)$ as the set guaranteed by Claim 2 for $\varepsilon = \varepsilon_1$ and $U = Y \setminus \operatorname{cl} V_1$.

Assume that the sets $V_1, \ldots, V_n$ and $W_1, \ldots, W_{n+1}$ are already defined. We take V_{n+1} as the set guaranteed by Claim 1 for $D = Y \setminus W_{n+1}$, $a = a_{n+1}$ and $\varepsilon = \varepsilon_{n+1}$. Finally, we choose $W_{n+2} \subset N(X, \varepsilon_{n+2})$ as the set guaranteed by Claim 2 for $\varepsilon = \varepsilon_{n+1}$ and $U = Y \setminus \operatorname{cl} V_{n+1}$. The inductive procedure is complete.

Consider the quotient mapping $q\colon Y \to q(Y) = Z$ which shrinks each set $\operatorname{bd} V_n$ to a point p_n. Since $X \cap \operatorname{bd} V_n = \emptyset$ and $\lim \operatorname{diam} \operatorname{bd} V_n = 0$, we infer that $q|X$ is a homeomorphism. Identifying X and $q(X)$ according to this homeomorphism we have $X \subset Z$. Let $U_n = Z \setminus q(\operatorname{cl} V_n)$. Thus $\operatorname{bd} U_n = \{p_n\}$, and we see that the family $\{U_n\colon n \in \mathbb{N}\}$ is a basis of neighborhoods of X in Z.

Let L be any continuum in Z such that $L \cap X \neq \emptyset \neq L \cap (Z \setminus X)$. Then L contains almost all points p_n. Consequently, $q^{-1}(L)$ contains continua L_n irreducibly intersecting $\operatorname{bd} V_n$ and $\operatorname{bd} V_{n+1}$ for almost all n. Such continua must intersect $\operatorname{cl} V_n$ and W_{n+1}, and thus $X \subset N(L_n, \varepsilon_n)$ by the choice of W_{n+1}. Therefore $X \subset \operatorname{Ls} L_n$, whence $q(X) = X \subset \operatorname{Ls} q(L_n)$ by the continuity of q. Since $q(L_n) \subset L$, we have $X \subset L$. So, X is terminal in Z. We have also $X = \operatorname{cl}\{a_1, a_2, \ldots\} \subset \operatorname{cl}\{p_1, p_2, \ldots\}$, and thus the proof is complete. □

The following theorem is the main result of the paper.

Theorem 4.5. *A continuum X is an* AANR *if and only if X is an absolute terminal retract.*

P r o o f. Let X be an absolute terminal retract contained in the Hilbert cube $\mathbb{Q}$, and let $\{U_n\colon n \in \mathbb{N}\}$ be a basis of closed connected neighborhoods of X in $\mathbb{Q}$. In

$\mathbb{Q} \times [0,1]$ take

$$Z = X \times \{0\} \cup U_1 \times \{1\} \cup U_2 \times \left\{\frac{1}{2}\right\} \cup \ldots,$$

and let $\{a_1, a_2, a_3, \ldots\}$ be a countable dense subset of X. Identify each pair $\langle a_n, 1/n\rangle$ with $\langle a_n, 1/(n+1)\rangle$ obtaining the quotient space T having the set $X \times \{0\}$ as a terminal subcontinuum. By assumption there is a retraction $r\colon T \to X \times \{0\}$. For each $n \in \mathbb{N}$ define a mapping $f_n\colon U_n \to X$ by

$$f_n(x) = y \iff r\Big(\Big\langle x, \frac{1}{n}\Big\rangle\Big) = \langle y, 0\rangle\,.$$

Then the mappings f_n satisfy the condition in Statement 2.1, so X is an AANR.

If X is an AANR, assume that X is a terminal subcontinuum of a continuum Y. Without loss of generality we assume, according to Lemma 4.4, that X has a basis $\{U_i\colon i \in \mathbb{N}\}$ of neighborhoods U_i each having a singleton $\{p_i\}$ as its boundary. Fix an embedding of Y in the Hilbert cube $\mathbb{Q}$ such that Y is a Z-set in $\mathbb{Q}$. Let

$$\mathcal{F} = \{Y' \subset \mathbb{Q}\colon \text{ there is a homeomorphism } h\colon Y \to Y' \text{ such that } h|X = \mathrm{id}_X \text{ and } Y' \text{ is a Z-set in } \mathbb{Q}\},$$

and let Φ be the family of all homeomorphisms $h\colon Y_1 \to Y_2$ such that $Y_1, Y_2 \in \mathcal{F}$ and $h|X = \mathrm{id}_X$. For $Y_1, Y_2 \in \mathcal{F}$ put

$$\varrho(Y_1, Y_2) = \inf\{d(h)\colon h \in \Phi \text{ and } h\colon Y_1 \to Y_2\}.$$

Fix a basis $\{V_n\colon n \in \mathbb{N}\}$ of open neighborhoods of the set X in $\mathbb{Q}$ such that $V_{n+1} \subset V_n$ for each n, and let $\mathcal{F}_n = \{Y' \in \mathcal{F}\colon Y' \subset V_n\}$.

Claim 1. $\mathcal{F}_n \neq \emptyset$ *for each* $n \in \mathbb{N}$.

Indeed, for sufficiently large index $j \in \mathbb{N}$ we embed $W = \mathrm{cl}(Y \setminus U_j)$ in a sufficiently small open neighborhood of p_j by a homeomorphism h so that $h(p_j) = p_j$ and $h(W)$ is a Z-set. Using Lemma 2.3 we can slightly modify this homeomorphism to a homeomorphism h' such that $h'(W) \cap \mathrm{cl}\, U_j = \{p_j\}$. Finally, we extend h' to Y taking the identity on U_j. In this way we obtain a homeomorphism $h''\colon Y \to h''(Y) = U_j \cup h'(W) \subset V_j$ belonging to Φ, which completes the proof of Claim 1.

Claim 2. *For each $\varepsilon > 0$ there exists an index $k \in \mathbb{N}$ such that for every m, n with $k < m < n$ and for each $Y' \in \mathcal{F}_m$ there is $Y'' \in \mathcal{F}_n$ satisfying $\varrho(Y', Y'') < \varepsilon$.*

To prove Claim 2 apply Statement 2.1 taking k so large that there exists a mapping $g_k\colon V_k \to X$ satisfying $d(g_k) < \varepsilon/2$. Let $m > k$ and $Y' \in \mathcal{F}_m$. Let $\{U'_i\colon i \in \mathbb{N}\}$ be a basis of neighborhoods of X in Y' such that $\operatorname{bd} U'_i = \{p'_i\}$. Fix j so large that the points p'_j and $g_k(p'_j)$ can be joined by an open connected set $V \subset V_n$ with $\operatorname{diam} V < \varepsilon/2$. Let $W = \operatorname{cl}(Y' \setminus U'_j)$. According to [25, Theorem 6.4.8, p. 279] (compare also [9, Corollary 3.5.3, p. 38]) there is a homeomorphism $h\colon W \to h(W) \subset \mathbb{Q}$ which is arbitrarily near to the mapping $g_k|W$ and such that $h(W)$ is a Z-set in $\mathbb{Q}$. So, we may assume that

$$d(h) < \frac{\varepsilon}{2}, \quad h(p'_j) \in V \quad \text{and} \quad h(W) \subset V_n.$$

Since $h(W) \cup Y'$ is a Z-set in $\mathbb{Q}$, we may assume without loss of generality that $h(W) \cap Y' = \emptyset$. Further, since X is a Z-set in $\mathbb{Q}$, there exists an open connected set $U' \subset V \setminus X$ containing both p'_j and $h(p'_j)$.

Now we apply Lemma 2.5 to the set $A = \operatorname{cl} U'_i \cup h(W)$ and points $a = p'_j$ and $b = h(p'_j)$ obtaining a quotient mapping $q\colon A \to q(A) \subset \mathbb{Q}$ such that conditions (2.5.1)–(2.5.4) are satisfied. Define a mapping $f\colon Y' \to f(Y') = Y''$ by

$$f(x) = \begin{cases} q(h(x)) & \text{if } x \in W, \\ q(x) & \text{if } x \notin W. \end{cases}$$

Observe that f is a homeomorphism in Φ, $d(f) < \varepsilon$, and $f(Y') \subset V_n$. Thus $\varrho(Y', Y'') < \varepsilon$, so the proof of Claim 2 is complete.

Using Claim 2 we take an increasing sequence k_n such that k_n is the number guaranteed by Claim 2 for $\varepsilon = \frac{1}{2^n}$. Then we find $Y_1 \in \mathcal{F}_{k_1}$ by Claim 1, and inductively choose a sequence of continua $Y_n \in \mathcal{F}_{k_n}$ and a sequence $f_n\colon Y_n \to Y_{n+1}$ of mappings $f_n \in \Phi$ satisfying $d(f_n) < \frac{1}{2^n}$. Setting $g_n = f_n \circ \ldots \circ f_1$ we see that the mappings g_n uniformly converge to a retraction $r\colon Y_1 \to X$. Hence X is a retract of Y. □

Corollary 4.6. *If a terminal subcontinuum X of a continuum Y is an* ANR, *then X is a retract of Y.*

Corollary 4.7. *Each* AANR *continuum contains a dense arc component.*

P r o o f. Let a continuum X be an AANR. Take a compactification Y of the half line H (i.e., a homeomorphic image of $[0, \infty)$) with X as the remainder, see [1, Theorem, p. 35]. It can easily be seen that then X is terminal in Y. By Theorem 4.5 there is a retraction $r\colon Y \to X$. Thus X contains a dense arcwise connected subset $r(H)$. The proof is then complete. □

5. Problems

In view of Theorem 3.3 the following question naturally arises.

Question 5.1. Is every absolute retract for the class of hereditarily unicoherent continua an AAR?

Proposition 5.2. *Consider the following four conditions that a continuum X may satisfy:*

$$X = \varprojlim\{X_n, r_n\}, \text{ where each factor space } X_n \text{ is an AR (an ANR),} \quad (5.2.1)$$
$$X_n \subset X_{n+1}, \text{ and the bonding mappings } r_n \text{ are retractions;}$$

$$\text{for each } \varepsilon > 0 \text{ there is a retraction } r\colon X \to X' \subset X \text{ such that} \quad (5.2.2)$$
$$X' \text{ is an AR (an ANR) and } d(r) < \varepsilon;$$

$$\text{for each } \varepsilon > 0 \text{ there is a mapping } f\colon X \to X' \subset X \text{ such that} \quad (5.2.3)$$
$$X' \text{ is an AR (an ANR) and } d(f) < \varepsilon;$$

$$X \text{ is an AAR (an AANR).} \quad (5.2.4)$$

Then the following implications hold:

$$(5.2.1) \Longrightarrow (5.2.2) \Longrightarrow (5.2.3) \Longrightarrow (5.2.4).$$

Proof. To show the first implication it is enough to take the projections $\pi_n\colon X \to X_n$ which are r-mappings in the sense of Borsuk, see [4, Chapter I, 1, p. 7] with the right inverse mappings $g_n\colon X_n \to X$ defined by (see notation at the beginning of Section 3) $g_n(x) = (r_1^n(x), \ldots, r_{n-1}^n(x), x, x, \ldots) \in X$. The required retractions are $g_n \circ \pi_n$.

The second implication is obvious; the last one is a consequence of Proposition 2.2 (a). □

Question 5.3. What implications of those in Proposition 5.2 can be reversed?

If a continuum $X \in$ AANR is hereditarily unicoherent, then $f(X) \subset Y \subset X$ in Proposition 2.2 (b) is a dendrite. Thus we answer the last question in a particular case.

Observation 5.4. The implication (5.2.4) $\Longrightarrow$ (5.2.3) holds if X is a hereditarily unicoherent continuum.

Corollary 5.5. *Let* $\mathcal{K} \in \{\mathcal{TL}, \lambda\mathcal{D}, \mathcal{D}, \mathcal{AL}, \lambda\mathcal{AL}\}$. *If a continuum* X *is in* $\mathrm{AR}(\mathcal{K})$, *then for each* $\varepsilon > 0$ *there are a tree* $T \subset X$ *and a mapping* $f\colon X \to T$ *such that* $d(f) < \varepsilon$.

P r o o f. By Theorem 3.3 the continuum X is an AAR, so by Observation 5.4 it satisfies (5.2.3). Any AR in X is a dendrite D which admits mappings g onto trees contained in D with arbitrarily small $d(g)$. □

Let us recall the following question, related to the topic of this paper. The question was asked in early sixties of the previous century on Prof. B. Knaster's Topology Seminar in Wrocław, Poland, and became a classic question in the continuum theory (for example see [26, Exercise 10.58, p. 192], and compare [12, p. 261] and [7, Problem 5, p. 34]).

Question 5.6. Let X be a dendroid. Do there exist, for each $\varepsilon > 0$, a tree $T \subset X$ and a retraction $r\colon X \to T$ with $d(r) < \varepsilon$?

Some partial positive answers to this question are known under additional assumptions on X: (a) if X is smooth, [12, Theorem 2, p. 261], and (b) if X is a fan (i.e., has only one ramification point), [11, Theorem 1, p. 120].

Observe that in Question 5.6 one asks whether condition (5.2.2) holds if X is a dendroid. Along the same lines one can ask if the other conditions of Proposition 5.2 hold for dendroids. More precisely, we have the following three questions.

Question 5.7. Is each dendroid the inverse limit of an inverse sequence of (nested) trees with retractions as bonding mappings?

Question 5.8. Let X be a dendroid. Do there exist, for each $\varepsilon > 0$, a tree $T \subset X$ and a mapping $f\colon X \to T$ with $d(f) < \varepsilon$?

Question 5.9. Is every dendroid an AAR?

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Authors' addresses: J. J. Charatonik, Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland, and Instituto de Matemáticas, UNAM, Circuito Exterior, Ciudad Universitaria, 04510 México, D.F., México; J. R. Prajs, Department of Mathematics and Statistics, California State University, Sacramento, CA 95819, USA, e-mail: prajs@csus.edu, and University of Opole, Institute of Mathematics, ul. Oleska 48, 45-052 Opole, Poland.