Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 55 (2005), No. 3, 773–780

Persistent URL: http://dml.cz/dmlcz/128020

Czechoslovak Mathematical Journal, 55 (130) (2005), 773–780

UNIVERSAL INTERPOLATING SEQUENCES ON SOME FUNCTION SPACES

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(Received November 26, 2002)

Abstract. Let $H(K)$ be the Hilbert space with reproducing kernel K. This paper characterizes some sufficient conditions for a sequence to be a universal interpolating sequence for $H(K)$.

Keywords: reproducing kernels, universal interpolating sequences, Bessel sequence, Riesz-Fischer sequence

MSC 2000: 47B38, 47B32, 46E20

1. Introduction

Let H be a Hilbert space of complex-valued analytic functions on the open unit disc $\mathbb{D}$ such that point evaluations are bounded linear functionals on H. Then for every $w \in \mathbb{D}$ there exists a function k_w in H such that $f(w) = \langle f, k_w \rangle$ for all $f \in H$. Now if we define $K\colon \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{C}$ by $K(z,w) = k_w(z)$, then K is a positive definite function with the reproducing property $f(w) = \langle f(\cdot), K(\cdot, w) \rangle$ for every $w \in \mathbb{D}$ and $f \in H$. The function K is called the *reproducing kernel* for H.

Recall that a function $K\colon \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{C}$ is *positive definite* (denoted $K \gg 0$) provided

$$\sum_{j,k=1}^{n} a_j \bar{a}_k K(w_j, w_k) \geqslant 0$$

for any finite set of complex numbers $a_1, \ldots, a_n$ and any finite subset $w_1, \ldots, w_n$ of $\mathbb{D}$. Conversely, if $K\colon \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{C}$ is positive definite then

$$\left\{ \sum_{j=1}^{n} a_j K(\cdot, w_j)\colon\ a_1, \ldots, a_n \in \mathbb{C} \ \text{ and } \ w_1, \ldots, w_n \in \mathbb{D} \right\}$$

This research was partially supported by the Shiraz University Research Council Grant.

has dense linear span in a Hilbert space $H(K)$ of functions with

$$\left\|\sum_{j=1}^{n} a_j K(\cdot, w_j)\right\|^2 = \sum_{j,k=0}^{n} a_j \bar{a}_k K(w_j, w_k)$$

and $f(w) = \langle f(\cdot), K(\cdot, w)\rangle$ for every w in $\mathbb{D}$ and f in $H(K)$. Thus evaluation at w is a bounded linear functional for each w in $\mathbb{D}$. Note also that convergence in $H(K)$ implies uniform convergence on compact subsets of $\mathbb{D}$.

Now if K is a kernel on $\mathbb{D}\times\mathbb{D}$ which is analytic in the first variable and consequently coanalytic in the second variable, then $K(z, \overline{w})$ is an analytic function on $\mathbb{D} \times \mathbb{D}$ in the two variables z and w. Hence $K(z, w)$ can be represented by the double power series $\sum_{j,k=0}^{\infty} a_{jk} z^j \overline{w}^k$. In this case K is called an analytic kernel. If C denotes the matrix $[a_{jk}]$, then such a K can be written more compactly in the form

$$K(z, w) = \overline{Z}^* C \overline{W} = \left\langle C\overline{W}, \overline{Z}\right\rangle_{\ell^2}$$

where Z denotes the column vector whose transpose is $(1, z, z^2, \ldots)$. (Here ℓ^2 denotes the usual space of all square summable sequences.) It is well known that $K \gg 0$ if and only if $C > 0$. Henceforth for positive matrices C, $H(C)$ will denote the space $H(K)$ where $K = \overline{Z}^* C \overline{W}$. For more information about reproducing kernels the reader is referred to [1], [2]. Some good sources on spaces of analytic functions are [4], [5], [6], [8], [9], [12], [13]. Throughout this paper, K will be an analytic kernel.

Following the interpolation theory for the Hardy space H^2 in [10] and for certain Banach spaces of analytic functions in [11], we call $\{w_n\}_n$ a universal interpolating sequence for $H(K)$ when the linear operator $T\colon H(K) \longrightarrow \ell^2$ defined by $Tf = \{f(w_n)/\|k_{w_n}\|\}_n$ is surjective. From this definition, we see that a universal interpolating sequence consists of distinct points and has no limit point in $\mathbb{D}$.

In the next section we give some sufficient conditions for existence of a sequence $\{f_n\}_n$ of vectors in $H(K)$ such that $\{\langle f, f_n\rangle\}_n$ belongs to ℓ^2 for all f in $H(K)$. We also investigate some conditions on a sequence of points in $\mathbb{D}$ for being a universal interpolating sequence.

2. Main results

Related to the universal interpolating sequences, there are really two questions involved here: first, is the sequence always in ℓ^2 for every $f \in H(K)$, and second, is every ℓ^2 sequence obtained in this manner? Both of these questions can be formulated in an abstract Hilbert space as follows (see N. Bari [3]). We shall say that a sequence of elements in a Hilbert space H is a *Bessel sequence* with bound M if

$$\sum_n |\langle x, x_n\rangle|^2 \leqslant M\|x\|^2$$

for all $x \in H$. Also we shall say that $\{x_n\}_n$ is a *Riesz-Fischer sequence* with bound m if to each sequence $\{c_n\}_n \in \ell^2$ there corresponds at least one $x \in H$ for which

$$\langle x, x_n\rangle = c_n \ \ (n = 1, 2, \ldots) \qquad \text{and} \qquad \|x\|^2 \leqslant m \sum_n |c_n|^2.$$

If one merely assumes the existence of $x \in H$ such that $\langle x, x_n\rangle = c_n$ $(n = 1, 2, \ldots)$ for each sequence $\{c_n\} \in \ell^2$, then the existence of the constant m follows from the inverse mapping theorem.

So $\{w_n\}$ is a universal interpolating sequence for $H(K)$ if and only if the sequence $\{k_{w_n}/\|k_{w_n}\|\}_n$ is both Bessel and Riesz-Fischer.

We need the following two theorems in the proof of our main results.

Theorem 1. *Let $\{x_n\}$ be an infinite sequence of elements in a separable Hilbert space, and let A denotes the inner product matrix $[\langle x_i, x_j\rangle]_{i,j\in\mathbb{N}}$. Then*

a) *$\{x_n\}_n$ is a Bessel sequence with bound M if and only if the matrix A is a bounded operator on ℓ^2 with bound M.*

b) *$\{x_n\}_n$ is a Riesz-Fischer sequence with bound m if and only if the matrix A is bounded below on ℓ_2 with bound m.*

P r o o f. See [3]. □

Theorem 2. *Let $A = [a_{ij}]_{i,j\in\mathbb{N}}$ be given. If $\sum_i |a_{ij}| \leqslant M$ for all j, and if $\sum_j |a_{ij}| \leqslant N$ for all i, then*

$$\left|\sum_{i,j} a_{ij} x_i \bar{x}_j\right| \leqslant (MN)^{1/2} \sum_i |x_i|^2$$

for all $\{x_i\}_i$ in ℓ^2.

P r o o f. See [7]. □

Theorem 3. *Let $H = H(K)$ have a reproducing kernel of the form*

$$k_w(z) = \log \frac{1}{(1 - z\overline{w})^t}$$

for some $t \geqslant 1$. Also suppose that $\{w_n\}_n$ is a sequence of points in the open unit disc $\mathbb{D}$ which converges to a point in $\partial\mathbb{D}$ and

$$1 - |w_{n+1}| \leqslant (1 - |w_n|)^{1/\alpha}$$

for all n and some α such that $0 < \alpha < 1$. Then for each $\varepsilon > 0$ there exists a subsequence of $\{k_{w_n}/\|k_{w_n}\|\}$ that is a Bessel sequence with bound $(1+\varepsilon)^{1/2}(2+\varepsilon)(1+\alpha^{1/2})(1-\alpha^{1/2})^{-1}$.

Proof. Let $0 < \varepsilon < 1$ be given. We can choose an integer $j_0 = j_0(\varepsilon)$ such that if $m, n > j_0$, then

$$\begin{gathered} \operatorname{Arg}^2 \frac{1}{(1 - \overline{w}_n w_m)^t} \leqslant \varepsilon \log^2 \frac{1}{(1 - |w_n|)^t}, \\ \log(1 - |w_n|)/\log 2(1 - |w_n|) < 2 + \varepsilon, \\ |1 - w_n \overline{w}_m| < 1 \end{gathered}$$

and

$$\frac{1}{2} < |w_n| < 1.$$

Put $n_i = j_0 + i$ for $i = 1, 2, \ldots$. We prove that $\{k_{w_{n_i}}/\|k_{w_{n_i}}\|\}_{i=1}^{\infty}$ is a Bessel sequence with bound $(1+\varepsilon)^{1/2}(2+\varepsilon)(1+\alpha^{1/2})(1-\alpha^{1/2})^{-1}$. For this let $A_t = [a_{ij}]_{ij}$ be the infinite matrix defined by

$$a_{ij} = \frac{k_{w_{n_i}}(w_{n_j})}{\|k_{w_{n_i}}\|\|k_{w_{n_j}}\|}; \quad (i, j \in \mathbb{N}).$$

By Theorem 1 it is sufficient to show that A_t is a bounded operator on ℓ^2 with bound $(1+\varepsilon)^{1/2}(2+\varepsilon)(1+\alpha^{1/2})(1-\alpha^{1/2})^{-1}$. Note that $a_{ii} = 1$ and $a_{ij} = \overline{a_{ji}}$ for all i and j in $\mathbb{N}$. We have

$$a_{ij} = \left(\log \frac{1}{(1 - |w_{n_i}|^2)^t}\right)^{-1/2} \left(\log \frac{1}{(1 - |w_{n_j}|^2)^t}\right)^{-1/2} \log \frac{1}{(1 - \overline{w}_{n_i} w_{n_j})^t}.$$

If follows from the hypothesis that

$$\begin{gathered} 1 - |w_{n_i+p}| \leqslant (1 - |w_{n_i}|)^{1/\alpha^p}, \\ |1 - \overline{w}_{n_i} w_{n_j}| \geqslant 1 - |w_{\min(n_i, n_j)}| \end{gathered}$$

and

$$1 - |w_{n_i}|^2 \leqslant 2(1 - |w_{n_i}|)$$

for all i, j and p in $\mathbb{N}$. Therefore

$$\log \frac{1}{2^t(1-|w_{n_i}|)^t} \leqslant \log \frac{1}{(1-|w_{n_i}|^2)^t},$$
$$\log \frac{1}{|1-\overline{w}_{n_i} w_{n_j}|^t} \leqslant \log \frac{1}{(1-|w_{n_i}|)^t}$$

and

$$\log(1-|w_{n_{i+p}}|)^t \leqslant \frac{1}{\alpha^p} \log(1-|w_{n_i}|)^t$$

for all i, j and p in $\mathbb{N}$. If $i \leqslant j$, by using the last inequality for $p = n_j - n_i$ we get

$$\log 2^t(1-|w_{n_j}|)^t \leqslant \frac{1}{\alpha^p} \log 2^t(1-|w_{n_i}|)^t.$$

Thus for $i \leqslant j$, we have

$$\begin{aligned}
|a_{ij}| &\leqslant \left|\log \frac{1}{(1-|w_{n_i}|^2)^t}\right|^{-1/2} \left|\log \frac{1}{(1-|w_{n_j}|^2)^t}\right|^{-1/2} \\
&\quad \times \left[\log^2 \frac{1}{(1-|w_{n_i}|)^t} + \operatorname{Arg}^2 \frac{1}{(1-\overline{w}_{n_i} w_{n_j})^t}\right]^{1/2} \\
&\leqslant \left|\log \frac{1}{2^t(1-|w_{n_i}|)^t}\right|^{-1/2} \left|\log \frac{1}{2^t(1-|w_{n_j}|)^t}\right|^{-1/2} \left|\log \frac{1}{(1-|w_{n_i}|)^t}\right| (1+\varepsilon)^{1/2} \\
&\leqslant (1+\varepsilon)^{1/2} \alpha^{(n_j-n_i)/2} \left|\log \frac{1}{2^t(1-|w_{n_i}|)^t}\right|^{-1} \left|\log \frac{1}{(1-|w_{n_i}|)^t}\right| \\
&= (1+\varepsilon)^{1/2} \alpha^{(j-i)/2} (2+\varepsilon).
\end{aligned}$$

Also if $i > j$, in the same way we obtain

$$|a_{ij}| \leqslant (1+\varepsilon)^{1/2}(2+\varepsilon)\alpha^{(i-j)/2}.$$

Thus for all i and j we get

$$|a_{ij}| \leqslant (1+\varepsilon)^{1/2}(2+\varepsilon)\alpha^{|i-j|/2}$$

and so

$$\begin{aligned}
\sum_i |a_{ij}| &= \sum_{i=1}^{j} |a_{ij}| + \sum_{i=j+1}^{\infty} |a_{ij}| \\
&\leqslant (1+\varepsilon)^{1/2}(2+\varepsilon)\left[\sum_{i=1}^{j} \alpha^{(j-i)/2} + \sum_{i=j+1}^{\infty} \alpha^{(i-j)/2}\right] \\
&< (1+\varepsilon)^{1/2}(2+\varepsilon)\left(\frac{1}{1-\alpha^{1/2}} + \frac{\alpha^{1/2}}{1-\alpha^{1/2}}\right) \\
&= (1+\varepsilon)^{1/2}(2+\varepsilon)(1+\alpha^{1/2})(1-\alpha^{1/2})^{-1}.
\end{aligned}$$

Similarly

$$\sum_j |a_{ij}| < (1+\varepsilon)^{1/2}(2+\varepsilon)(1+\alpha^{1/2})(1-\alpha^{1/2})^{-1}.$$

So by Theorem (2), the matrix $A_t = [a_{ij}]_{i,j}$ is bounded above by

$$(1+\varepsilon)^{1/2}(2+\varepsilon)(1+\alpha^{1/2})(1-\alpha^{1/2})^{-1}.$$

Now Theorem 1 implies that $\{k_{w_{n_i}}/\|k_{w_{n_i}}\|\}_i$ is a Bessel sequence and the proof is complete. □

The following corollary is an immediate consequence of Theorem 3.

Corollary 4. *Under the conditions of the theorem,* $\{f(w_n)/\|k_{w_n}\|\}_n \in \ell^2$ *for all* f *in* $H(K)$.

Theorem 5. *Let* $H = H(K)$ *have a reproducing kernel of the form*

$$k_w(z) = \log \frac{1}{(1-z\overline{w})^t}$$

for some $t \geqslant 1$. *If* $\{w_n\}$ *converges to a point in* $\partial\mathbb{D}$ *and*

$$1 - |w_{n+1}| \leqslant (1-|w_n|)^{1/\alpha}$$

for all n *and some* α *such that* $0 < \alpha < \frac{1}{25}$, *then there exists a subsequence of* $\{k_{w_n}/\|k_{w_n}\|\}_n$ *that is a universal interpolating sequence for* $H(K)$.

P r o o f. The function $f\colon [0,1] \longrightarrow \mathbb{R}^+$ defined by $f(t) = \big(1+2(1+t)^{1/2}(2+t)\big)^{-2}$ is a nonnegative decreasing function on $[0,1]$ and $\lim\limits_{t\to 0^+} f(t) = \frac{1}{25}$. So there exists $0 < \varepsilon < 1$ such that $\alpha < f(\varepsilon) \leqslant \frac{1}{25}$. By Theorem 3 there exists a subsequence $\{k_{w_{n_i}}/\|k_{w_{n_i}}\|\}_i$ that is a Bessel sequence and as we saw in the proof of Theorem 3, if $A_t = [a_{ij}]_{i,j}$ where

$$a_{ij} = \frac{k_{w_{n_i}}(w_{n_j})}{\|k_{w_{n_i}}\|\|k_{w_{n_j}}\|} \quad (i,j \in \mathbb{N}),$$

then for all i and j we get

$$|a_{ij}| \leqslant (1+\varepsilon)^{1/2}(2+\varepsilon)\alpha^{|i-j|/2}.$$

Now we estimate the operator norm of the difference of A_t and the identity operator I:

$$\begin{aligned}\|A_t - I\|_{\mathrm{op}} &\leqslant \sup_j \sum_{i\neq j} |a_{ij}| \\ &= (1+\varepsilon)^{1/2}(2+\varepsilon)\sup_j \Big[\sum_{i<j} \alpha^{(j-i)/2} + \sum_{i=j+1}^{\infty} \alpha^{(i-j)/2}\Big] \\ &\leqslant (1+\varepsilon)^{1/2}(2+\varepsilon)\frac{2\alpha^{1/2}}{1-\alpha^{1/2}} < 1,\end{aligned}$$

since $f(\varepsilon) > \alpha$. Hence A_t is invertible and so by Theorem 1, the proof is complete. $\square$

Remark. Note that in Theorems 3 and 5, the reproducing kernels k_w are analytic on the unit disc $\mathbb{D}$ when $t \in [1,2]$ and, in the special case, when $t = 1$ we have the Bloch space.

The constant $\frac{1}{25}$ in the Theorem may not be sharp. We conclude this paper by raising the following question.

Question. Can we replace $\frac{1}{25}$ by 1 in Theorem 5?

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