Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 55 (2005), No. 3, 653–663

Persistent URL: `http://dml.cz/dmlcz/128009`

Czechoslovak Mathematical Journal, 55 (130) (2005), 653–663

ON ITÔ-KURZWEIL-HENSTOCK INTEGRAL AND INTEGRATION-BY-PART FORMULA

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(Received October 10, 2002)

Abstract. In this paper we derive the Integration-by-Parts Formula using the generalized Riemann approach to stochastic integrals, which is called the Itô-Kurzweil-Henstock integral.

Keywords: generalized Riemann approach, stochastic integral, integration-by-parts

MSC 2000: 26A39, 60H05

1. Introduction

The classical stochastic integrals are defined through a non-explicit L^2-convergence procedure. A natural question arises: is it possible to define stochastic integrals as limits of Riemann-Stieltjes sums? In [1], [6], [7], [10], [11], [12], [13] a generalized Riemann approach using non-uniform meshes was used and it turned out that the integral defined by the generalized Riemann approach encompasses the classical stochastic integral. At this point, readers are reminded that it has always been emphasized in literature that it is impossible to define stochastic integrals using Riemann-Stieltjes sums with uniform mesh.

The generalized Riemann-Stieltjes approach using non-uniform meshes was first introduced independently by J. Kurzweil and R. Henstock in the 1950s to study the classical (non-stochastic) integral. It turns out that this integral encompasses the Riemann-Stieltjes integral and the more general Lebesgue-Stieltjes integral, see [2], [3], [4].

McShane's approach in using non-uniform meshes in [6], [7] assumed Vitali's covering property. In [12], the assumption of Vitali covering property was replaced by using any collection of partial divisions. This new definition, which was motivated by

Henstock's Lemma, turns out to be simpler and encompasses the classical stochastic integrals whenever they are defined.

Protter in [9, pp. 58, 76, 223] offered a proof of the Integration-by-Parts Formula for Fisk-Stratonovich stochastic integrals along the line of the classical stochastic integrals. In this note, using this generalized Riemann-Stieltjes approach along the definition given in [12], we offer an alternative proof of the Integration-by-Parts for our stochastic integral, based on the Kurzweil-Henstock approach.

2. Settings and definition of Itô-Kurzweil-Henstock stochastic integral

Let $(\Omega, \mathscr{F}, P)$ be a probability space, that is, a measure space with $P(\Omega) = 1$, and $\{\mathscr{F}_t\}$ an increasing family of σ-subfields of $\mathscr{F}$ for $t \in [0,1]$, that is, $\mathscr{F}_r \subset \mathscr{F}_s$ for $0 \leqslant r < s \leqslant 1$ with $\mathscr{F}_1 = \mathscr{F}$. The probability space together with its family of increasing σ-subfields is called a *filtering space* and denoted by $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$.

A process φ is a family of random variables φ_t on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$ for each $t \in [0,1]$. φ may be denoted by $\{\varphi_t \colon t \in [0,1]\}$.

A process φ is said to be adapted to the filtration $\{\mathscr{F}_t\}$ if φ_t is $\mathscr{F}_t$-measurable for all $t \in [0,1]$. For convenience, we consider the interval $[0,1]$, although any other $[a,b] \subset [0,\infty)$ would suffice.

Definition 1. Let $S\colon \Omega \to [0,\infty)$ be a random variable defined on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$. Then S is called a *stopping time* if

$$\{\omega \in \Omega\colon\ S(\omega) \leqslant t\} \in \mathscr{F}_t$$

for each $t \geqslant 0$.

Some standard properties of the stopping time include the following one: if S and T are two stopping times and $c > 0$ is a positive constant, then $S + T$, $\min\{S,T\}$, $S + c$ are stopping times. However, $S - T$ or $S - c$ need not be stopping times.

Definition 2. Let $\delta = \{\delta_t\colon\ t \in [0,1]\}$ be a positive process, i.e., $\delta_t(\cdot) > 0$ for all t. Then δ is called a stochastic gauge if $\xi + \delta_\xi(\cdot)$ is a stopping time for each $\xi \in [0,1]$.

For example, if δ is (i) deterministic; or (ii) adapted to the space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$, then δ is a stochastic gauge.

Let S and T be two stopping times with $S \leqslant T \leqslant 1$, i.e. $S(\omega) \leqslant T(\omega) \leqslant 1$ for each $\omega \in \Omega$. Let $(S,T]$ be a stochastic interval, i.e. $(S,T] = \{(t,\omega)\colon\ S(\omega) < t \leqslant T(\omega)$ if $S(\omega) < T(\omega)$; $S(\omega) = t = T(\omega)$ if $S(\omega) = T(\omega)\}$. The definition of $(S,T]$ is slightly different from the standard definition in which $S < T$, see [9]. In this note, stochastic intervals $(\xi, T]$, where the left endpoint ξ is a real number, are used very often in the construction of our integral.

Definition 3. Let δ be a stochastic gauge. A finite collection of stochastic intervals $\{(\xi_i, T_i]\colon i = 1, 2, 3, \ldots, n\}$, where $\xi_i \in [0,1]$, is called a δ-fine belated partial stochastic division of $[0,1]$ if

1. for each i, $(\xi_i, T_i]$ is a stochastic interval and for each $\omega \in \Omega$, $(\xi_i, T_i(\omega)]$, $i = 1, 2, \ldots, n$, are disjoint left-open subintervals of $[0,1]$ and
2. each $(\xi_i, T_i]$ is δ-fine belated, i.e. for each $\omega \in \Omega$ we have $(\xi_i, T_i(\omega)] \subset [\xi_i, \xi_i + \delta_{\xi_i}(\omega)]$, i.e., $\xi_i \leqslant T_i(\omega) \leqslant \xi_i + \delta_{\xi_i}(\omega)$.

For the case $\xi_i = T_i(\omega)$ for all $\omega \in \Omega$, $(\xi_i, T_i(\omega)]$ is taken to be $\{\xi_i\}$.

Remark. Given any stochastic gauge and any $\xi \in [0,1]$, we can always find a stochastic interval within $[\xi, \xi + \delta_\xi(\cdot)]$, since we can simply take $(\xi, \xi + \delta_\xi(\cdot)]$ to be the half-open interval.

From now onwards a process $\varphi = \{\varphi_t\colon t \in [0,1]\}$ is denoted simply by φ. Suppose T is a stopping time. Then X_T denotes the random variable $X_T(\omega) = X(T(\omega), \omega)$ for all $\omega \in \Omega$. Let $L^2(\Omega)$ be the space of all square integrable functions on $(\Omega, \mathscr{F}, P)$.

Definition 4. A stochastic process u is said to be Itô-Kurzweil-Henstock integrable to a process A on $[0,1]$ (with respect to a stochastic process X) if for every $\varepsilon > 0$ there exists a stochastic gauge δ for which

$$E\left(\left|\sum_{i=1}^{n}\{u_{\xi_i}(X_{T_i} - X_{\xi_i}) - (A_{T_i} - A_{\xi_i})\}\right|^2\right) \leqslant \varepsilon$$

for every δ-fine belated partial stochastic division $D = \{((\xi_i, T_i], \xi_i)\colon i = 1, 2, \ldots, n\}$ of $[0,1]$.

To ensure that Definition 4 is meaningful, in this note we will always assume that for each $\xi \in [0,1]$, the random variable $u_\xi \in L^2(\Omega)$, and for each stopping time T, both X_T and A_T belong to $L^2(\Omega)$.

We remark that in the above definition, we use *any* collection of partial divisions of $[0,1]$. This was motivated by Henstock's Lemma for the classical (non-stochastic) integral. This type of integrals has been considered by Henstock [3, p. 54], [4, p. 61].

The integral of u over $[0,T]$ refers to $A_T - A_0$.

Definition 5. Two processes F and G are said to be equal up to zero variation if given any $\varepsilon > 0$, there exists a stochastic gauge δ such that for any δ-fine belated partial stochastic division of $[0,1]$, denoted by $D = \{(\xi, V]\}$, we have

$$E\Big(\Big|(D)\sum\{(F_V - F_\xi) - (G_V - G_\xi)\}\Big|^2\Big) \leqslant \varepsilon.$$

It is clear that the integral, if it exists, is unique up to zero variation.

Uniqueness up to zero variation is an equivalence relation. Thus we let

$$\int_0^1 u_t \,\mathrm{d}X_t$$

denote a member in the equivalence class of the processes A which denote the Itô-Kurzweil-Henstock integral of u with respect to X over $[0,1]$.

It has been proved in [11] that if u is classical stochastic integrable with respect to an L^2-martingale on $[0,1]$, then u is Itô-Kurzweil-Henstock integrable on $[0,1]$ and the two integrals coincide.

3. Properties of Itô-Kurzweil-Henstock integral

We shall next state some standard properties of the stochastic integrals without proofs. The proofs are standard in the theory of Kurzweil-Henstock integration. However, we shall highlight Lemma 6, which forms the crucial part of the proofs of the other theorems in this section. The proof of Lemma 6 follows directly from the definition and the proof of Theorem 7 is standard of Kurzweil-Henstock integration theory, hence they are omitted.

Lemma 6. *Let δ_i, $i = 1,2$, be two locally stopping processes. Then $\delta = \min(\delta_1, \delta_2)$ is a locally stopping process.*

Theorem 7. *If φ^i is Itô-Kurzweil-Henstock integrable on $[0,1]$ for each $i = 1,2$, then $\varphi_t^1 + \varphi_t^2$ is Itô-Kurzweil-Henstock integrable on $[0,1]$ and*

$$\int_0^1 \varphi_t^1 + \varphi_t^2 \,\mathrm{d}X_t = \int_0^1 \varphi_t^1 \,\mathrm{d}X_t + \int_0^1 \varphi_t^2 \,\mathrm{d}X_t.$$

Remark. It is easy to see from the definition that if φ is Itô-Kurzweil-Henstock integrable with respect to X on $[0,1]$, then it is Itô-Kurzweil-Henstock integrable with respect to X on each $[a,b] \subset [0,1]$ since any δ-fine belated partial stochastic division of $[a,b]$ is also a division of $[0,1]$.

Theorem 8. *If φ is Itô-Kurzweil-Henstock integrable on $[0,c]$ and Itô-Kurzweil-Henstock integrable on $[c,1]$, where $0 \leqslant c \leqslant 1$, then φ is Itô-Kurzweil-Henstock integrable on $[0,1]$. Furthermore,*

$$\int_0^1 \varphi_t \,\mathrm{d}X_t = \int_0^c \varphi_t \,\mathrm{d}X_t + \int_c^1 \varphi_t \,\mathrm{d}X_t.$$

Theorem 9. *Let φ be Itô-Kurzweil-Henstock integrable with respect to X and Y, and let $\alpha, \beta \in \mathbb{R}$. Then φ is Itô-Kurzweil-Henstock integrable with respect to $\alpha X + \beta Y$ and moreover,*

$$\int_0^1 \varphi_t \,\mathrm{d}(\alpha X_t + \beta Y_t) = \alpha \int_0^1 \varphi_t \,\mathrm{d}X_t + \beta \int_0^1 \varphi_t \,\mathrm{d}Y_t.$$

4. Integration-by-parts formula

It is well-known that in the classical theory of Riemann-Stieltjes integration we have the integration-by-parts formula

$$(\mathrm{RS}) \int_a^b f \,\mathrm{d}g + (\mathrm{RS}) \int_a^b g \,\mathrm{d}f = f(b)g(b) - f(a)g(a)$$

where f and g are deterministic functions on $[a,b]$ and one of f and g is continuous while the other has bounded variation. So f is integrable with respect to g if and only if g is integrable with respect to f and the above formula holds true.

Extending pathwise, if $f, g\colon [a,b] \times \Omega \to \mathbb{R}$ are stochastic processes such that one of them has continuous paths while the other has paths which are of bounded variation, then we have the integration-by-parts formula. Then it is clear that f is integrable with respect to g if and only if g is integrable with respect to f and that

$$\int_a^b f_t \,\mathrm{d}g_t + \int_a^b g_t \,\mathrm{d}f_t = f_b g_b - f_a g_a.$$

However, general stochastic processes usually have paths of unbounded variation, so that the above integration-by-parts formula need not be true in general.

Definition 10. Let φ and X be stochastic processes such that there exists a non-decreasing process H with $E(H_1) < \infty$ such that for any $\varepsilon > 0$ there exists a stochastic gauge δ such that for any δ-fine belated partial stochastic division $D = \{(\xi, V]\}$ of $[0,1]$ we have

$$E\Bigl(\Bigl|(D)\sum\{[\varphi_V - \varphi_\xi][X_V - X_\xi] - [H_V - H_\xi]\}\Bigr|^2\Bigr) \leqslant \varepsilon.$$

Then H is called the *weak quadratic covariance process* of φ and X, which we shall denote by $\lfloor X, \varphi \rfloor$.

Remark. It is clear that $\lfloor X, Y \rfloor = \lfloor Y, X \rfloor$ for any stochastic processes X, Y. Also, we let $\lfloor X \rfloor$ denote $\lfloor X, X \rfloor$ if the latter exists.

Theorem 11. *Let φ and X be stochastic processes such that $\lfloor X, \varphi \rfloor$ exists. Then X is Itô-Kurzweil-Henstock integrable with respect to φ on $[0,1]$ if and only if φ is Itô-Kurzweil-Henstock integrable with respect to X on $[0,1]$. Furthermore,*

$$\int_0^1 \varphi_t \,\mathrm{d}X_t + \int_0^1 X_t \,\mathrm{d}\varphi_t = \varphi_1 X_1 - \varphi_0 X_0 - (\lfloor X, \varphi \rfloor_1 - \lfloor X, \varphi \rfloor_0).$$

P r o o f. Let φ be integrable with respect to X and $F_u = \int_0^u \varphi_t \,\mathrm{d}X_t$. Let

$$G_u = \varphi_u X_u - \int_0^u \varphi_t \,\mathrm{d}X_t - \lfloor \varphi, X \rfloor_u.$$

We shall prove that X is Itô-Kurzweil-Henstock integrable with respect to φ with $\int_0^1 X_t \,\mathrm{d}\varphi_t = G_1 - G_0$. Given $\varepsilon > 0$ there exists a stochastic gauge δ_1 such that for any δ_1-fine belated partial stochastic division $D_1 = \{(\xi, V]\}$ of $[0,1]$, we have

$$E\Bigl|(D_1)\sum\{\varphi_\xi[X_V - X_\xi] - (F_V - F_\xi)\}\Bigr|^2 < \frac{\varepsilon}{2}. \tag{1}$$

By assumption, $\lfloor X, \varphi \rfloor$ exists. Choose a stochastic gauge δ_2 which satisfies the condition that for every δ_2-fine belated partial stochastic division $D_2 = \{(\xi, V]\}$ we have

$$E\Bigl|(D_2)\sum\{[\varphi_V - \varphi_\xi][X_V - X_\xi] - (\lfloor \varphi, X \rfloor_V - \lfloor \varphi, X \rfloor_\xi)\}\Bigr|^2 \leqslant \frac{\varepsilon}{2}. \tag{2}$$

Take $\varepsilon > 0$ as above. Consider $\delta = \min(\delta_1, \delta_2)$, which is again a stochastic gauge by Lemma 6.

Let $D = \{(\xi, V]\}$ be any δ-fine belated partial stochastic division of $[0,1]$. Then

$$\begin{aligned}
(D)\sum\{X_\xi(\varphi_V - \varphi_\xi) - (G_V - G_\xi)\} \\
&= (D)\sum\{\varphi_V X_V - \varphi_\xi X_\xi - (\varphi_V - \varphi_\xi)(X_V - X_\xi) \\
&\quad - \varphi_\xi(X_V - X_\xi) - (G_V - G_\xi)\} \\
&= (D)\sum\{\varphi_V X_V - \varphi_\xi X_\xi - (\varphi_V - \varphi_\xi)(X_V - X_\xi) - \varphi_\xi(X_V - X_\xi) \\
&\quad - (\varphi_V X_V - F_V - \lfloor X, \varphi\rfloor_V - \varphi_\xi X_\xi + F_\xi + \lfloor X, \varphi\rfloor_\xi)\} \\
&= (D)\sum\{-(\varphi_V - \varphi_\xi)(X_V - X_\xi) - \varphi_\xi(X_V - X_\xi) + F_V \\
&\quad - F_\xi + \lfloor X, \varphi\rfloor_V - \lfloor X, \varphi\rfloor_\xi\}.
\end{aligned}$$

Hence by equations (1) and (2), we get

$$E\Big|(D)\sum\{X_\xi(\varphi_V - \varphi_\xi) - (G_V - G_\xi)\}\Big|^2 < 2\frac{\varepsilon}{2} + 2\frac{\varepsilon}{2} = \varepsilon,$$

thus $\int_0^1 X_t \,\mathrm{d}\varphi_t = G_1 - G_0$, which completes the proof. □

Corollary 12. *Let φ and X be stochastic processes such that $\lfloor X, \varphi\rfloor = 0$. Then φ is Itô-Kurzweil-Henstock integrable with respect to X on $[0,1]$ if and only if X is Itô-Kurzweil-Henstock integrable with respect to φ on $[0,1]$. Furthermore,*

$$\int_0^1 \varphi_t \,\mathrm{d}X_t + \int_0^1 X_t \,\mathrm{d}\varphi_t = \varphi_1 X_1 - \varphi_0 X_0.$$

In fact, from the proof of Theorem 11 we have

Theorem 13. *Let X and φ be stochastic processes such that X is Itô-Kurzweil-Henstock integrable with respect to φ and φ is Itô-Kurzweil-Henstock integrable with respect to X. Then the weak quadratic covariance process $\lfloor X, \varphi\rfloor$ exists and*

$$\lfloor X, \varphi\rfloor_t = \varphi_t X_t - \int_0^t \varphi_s \,\mathrm{d}X_s - \int_0^t X_s \,\mathrm{d}\varphi_s.$$

Theorem 14. *Let X be a stochastic process such that $\lfloor X \rfloor$ exists. Then X is Itô-Kurzweil-Henstock integrable with respect to itself, and*

$$\int_0^1 X_t \, \mathrm{d}X_t = \frac{1}{2}X_1^2 - \frac{1}{2}X_0^2 - \frac{1}{2}(\lfloor X \rfloor_1 - \lfloor X \rfloor_0).$$

Proof. By assumption, $\lfloor X \rfloor$ exists. Hence given $\varepsilon > 0$, let δ be a stochastic gauge such that for any δ-fine belated partial stochastic division $D = \{(\xi, V]\}$ of $[0,1]$ we have

$$E\Big|(D)\sum\{(X_V - X_\xi)^2 - (\lfloor X \rfloor_V - \lfloor X \rfloor_\xi)]\}\Big|^2 \leqslant \varepsilon.$$

Then

$$E\Big|(D)\sum\{X_\xi(X_V - X_\xi) - \Big\{\frac{1}{2}(X_V^2 - X_\xi^2) - \frac{1}{2}(\lfloor X \rfloor_V - \lfloor X \rfloor_\xi)\Big\}\Big|^2$$
$$= \frac{1}{4}E\Big|(\mathrm{D})\sum\{-(X_V - X_\xi)^2 + (\lfloor X \rfloor_V - \lfloor X \rfloor_\xi)\}\Big|^2 \leqslant \frac{1}{4}\varepsilon,$$

which completes the proof. □

5. Classical stochastic integral

We shall show that we can derive some of the well-known formulae for classical stochastic integrals using our general results obtained in the previous sections.

Definition 15. A process $X = \{X_t \colon t \in [0,1]\}$ is called a *Martingale* on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$ if

1. X is adapted to $\{\mathscr{F}_t\}$, that is, X_t is $\mathscr{F}_t$-measurable for each $t \in [0,1]$;
2. $\int_\Omega |X_t| \, \mathrm{d}P$ is finite for almost all $t \in [0,1]$; and
3. $E(X_t|\mathscr{F}_s) = X_s$ for all $t \geqslant s$, where $E(X_t|\mathscr{F}_s)$ is the conditional expectation of X_t given $\mathscr{F}_s$. By the Radon-Nikodym Theorem, $E(X_t|\mathscr{F}_s)$ exists and is well-defined.

If, in addition,

$$\sup_{t\in[0,1]} \int_\Omega |X_t|^2 \, \mathrm{d}P$$

is finite, we say that X is a L_2-martingale.

Definition 16. A process $W = \{W_t(\omega)\colon t \in [0,1]\}$ defined on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$ is called a canonical *Brownian motion* (or *Wiener Process*) if it has the following properties:

1. $W_0(\omega) = 0$ for all $\omega \in \Omega$;
2. (normal increments): $W_t - W_s$ has a normal distribution with mean 0 and variance $t - s$ for all $t > s$. This implies that W_t has a normal distribution with mean 0 and variance t;
3. (independence of increments): $W_t - W_s$ is independent of the past, that is, W_u, $0 \leqslant u \leqslant s < t$; and
4. (continuity of paths): W is continuous.

It is well-known that a canonical Brownian motion is in fact a continuous martingale.

Lemma 17. *Let φ be an adapted process on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$ such that $\varphi_t(\omega)$, as a function of t, is right-continuous on $[0,1]$ for all $\omega \in \Omega$. Then given any $\varepsilon > 0$ there exists a stochastic gauge δ such that*

$$|\varphi(u(\omega), \omega) - \varphi(\xi, \omega)| < \varepsilon$$

for all $u(\omega)$ with $0 < u(\omega) - \xi < \delta(\xi, \omega)$, where $\varphi_t(\omega)$ is denoted by $\varphi(t, \omega)$.

Proof. Let $\varepsilon > 0$ be given and let $\xi \in [0,1]$. For any $(\xi, \omega) \in [0,1] \times \Omega$ define a process δ as

$$\delta(\xi, \omega) = \inf\{u \in [0, \infty)\colon |\varphi(u + \xi, \omega) - \varphi(\xi, \omega)| \geqslant \varepsilon\}.$$

Then δ is a stochastic gauge since for $t \geqslant 0$,

$$\begin{aligned}\{\omega \in \Omega\colon \xi + \delta(\xi, \omega) \leqslant t\} &= \{\omega \in \Omega\colon \delta(\xi, \omega) \leqslant t - \xi\} \\ &= \bigcup_{u \in \mathbb{Q} \cap [0, t-\xi]} \{\omega \in \Omega\colon |\varphi(\xi + u, \omega) - \varphi(\xi, \omega)| \geqslant \varepsilon\} \\ &\in \mathscr{F}_{\xi + (t - \xi)} = \mathscr{F}_t\end{aligned}$$

so that $\xi + \delta(\xi, \cdot)$ is a stopping time for all $\xi \in [0,1]$. Hence for any u with $u - \xi < \delta(\xi, \omega)$ we have $|\varphi(\xi + u, \omega) - \varphi(\xi, \omega)| < \varepsilon$. $\square$

Let X be a right-continuous L_2-martingale on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$. Then there exists a unique predictable (for a definition see, for example, [9, p. 117]) non-decreasing process $\langle X \rangle$ such that $X^2 - \langle X \rangle$ is a right-continuous L_2-martingale. The process $\langle X \rangle$ is called the quadratic variation process of X.

Proposition 18. *Let X be a (right-continuous) L_2-martingale on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$ and $\langle X\rangle$ the associated quadratic variation process. Then $\lfloor X\rfloor = \langle X\rangle$, where $\lfloor X\rfloor$ is the weak quadratic covariance process of X and itself, see Definition* 10.

Proof. Given $\varepsilon > 0$, by Lemma 17 there exists a locally stopping process δ such that whenever $0 < v(\omega) - \xi < \delta(\xi, \omega)$ then we have

$$|\langle X\rangle(v(\omega), \omega) - \langle X\rangle(\xi, \omega)| < \frac{\varepsilon}{2} \quad \text{and} \quad |X(v(\omega), \omega) - X(\xi, \omega)| < \frac{\sqrt{\varepsilon}}{\sqrt{2}}.$$

So if $(\xi, V]$ is δ-fine, then

$$E(X_V - X_\xi)^4 = E[(X_V - X_\xi)^2(X_V - X_\xi)^2] \leqslant \frac{\varepsilon}{2} E(X_V - X_\xi)^2 \tag{3}$$

and

$$E(\langle X\rangle_V - \langle X\rangle_\xi)^2 = E[(\langle X\rangle_V - \langle X\rangle_\xi)(\langle X\rangle_V - \langle X\rangle_\xi)] \leqslant \frac{\varepsilon}{2} E(\langle X\rangle_V - \langle X\rangle_\xi). \tag{4}$$

Also, if $(\xi_j, V_j]$ and $(\xi_i, V_i]$ are disjoint intervals, it can be shown by direct computation that

$$E[[(X_{V_i} - X_{\xi_i})^2 - (\langle X\rangle_{V_i} - \langle X\rangle_{\xi_i})][(X_{V_j} - X_{\xi_j})^2 - (\langle X\rangle_{V_j} - \langle X\rangle_{\xi_j})]] = 0. \tag{5}$$

Choose a δ-fine belated partial stochastic division $D = \{((\xi, V], \xi)\}$. Then

$$\begin{aligned}
&E\Big|\sum((X_V - X_\xi)^2 - (\langle X\rangle_V - \langle X\rangle_\xi))\Big|^2 \\
&\quad = E\Big\{\sum\{(X_V - X_\xi)^2 - (\langle X\rangle_V - \langle X\rangle_\xi)\}^2\Big\} \text{ by (5)} \\
&\quad \leqslant 2\sum E(X_V - X_\xi)^4 + 2\sum E(\langle X\rangle_V - \langle X\rangle_\xi)^2 \text{ by (3) and (4)} \\
&\quad \leqslant \varepsilon \sum E(X_V - X_\xi)^2 + \varepsilon \sum E(\langle X\rangle_V - \langle X\rangle_\xi) \\
&\quad \leqslant 4\varepsilon(\langle X\rangle_1 - \langle X\rangle_0),
\end{aligned}$$

which fits Definition 10 with $\varphi = X$ and $H = \langle X\rangle$, thereby completing the proof. □

Example 19. Let X be an L^2-martingale with quadratic variation denoted by $\langle X\rangle$. By Theorem 14 and Proposition 18 we have

$$\int_0^1 X_t \, \mathrm{d}X_t = \frac{1}{2}X_1^2 - \frac{1}{2}X_0^2 - \frac{1}{2}(\langle X\rangle_1 - \langle X\rangle_0).$$

As a special case, if X is a Brownian motion, then $\langle X\rangle_t \equiv t$ (see, for example, [9, p. 71]). So

$$\int_0^1 X_t \, \mathrm{d}X_t = \frac{1}{2}X_1^2 - \frac{1}{2}X_0^2 - \frac{1}{2}.$$

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