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THE METHOD OF UPPER AND LOWER SOLUTIONS FOR A LIDSTONE BOUNDARY VALUE PROBLEM

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Abstract. In this paper we develop the monotone method in the presence of upper and lower solutions for the 2nd order Lidstone boundary value problem

$$u^{(2n)}(t) = f(t, u(t), u''(t), \ldots, u^{(2(n-1))}(t)), \quad 0 < t < 1,$$
$$u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad 0 \leqslant i \leqslant n-1,$$

where $f\colon [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is continuous. We obtain sufficient conditions on f to guarantee the existence of solutions between a lower solution and an upper solution for the higher order boundary value problem.

Keywords: n-parameter eigenvalue problem, Lidstone boundary value problem, lower solution, upper solution

MSC 2000: 34B15

1. Introduction

Consider the 2nd order Lidstone boundary value problem

$$u^{(2n)}(t) = f(t, u(t), u''(t), \ldots, u^{(2(n-1))}(t)), \quad 0 < t < 1, \tag{1.1}$$
$$u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad 0 \leqslant i \leqslant n-1, \tag{1.2}$$

where $f\colon [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is continuous.

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The fourth order boundary value problem

$$u^{(4)}(t) = f(t, u(t), u''(t)), \quad 0 < t < 1, \tag{1.3}$$

$$u(0) = u(1) = u''(0) = u''(1), \tag{1.4}$$

has been studied by many authors. In [1]–[5], the authors showed the existence of a positive solution to (1.3)–(1.4) under some growth conditions on f and a non-resonance condition involving a two parameter linear eigenvalue problem by using the Leray-Schauder continuation method and topological degree.

For an equation of the form

$$u^{(4)}(t) = f(t, u(t)),$$

the upper and lower solution method has been studied by several authors [6]–[10]. Recently, Ma and Bai [11], [12] developed the monotone method in the presence of upper and lower solutions for the problem (1.3)–(1.4).

For the 2nd order Lidstone boundary value problem (1.1)–(1.2), in [13]–[15], Davis et al. showed the existence of multiple positive solutions under some growth conditions by using the Leggett-Williams fixed point theorem and the five functionals fixed point theorem. Note that [14] and [15] are the only two works which have allowed f to depend on higher order derivatives of u. Motivated by Bai [11], in this paper we present an upper and lower solution type theorem for the boundary problem (1.1)–(1.2) without any growth restriction on f. The problem (1.1)–(1.2) is formulated without constants a_i (or r_i) which play a substantial role in Theorem 3.1. These constants specify a possible qualitative behavior of the function f. Our result relaxes the monotone conditions on f, and this approach is better than the simplest one—choosing $a_i = 0$, i.e., the monotone conditions on f (see Example).

2. Preliminary results

Lemma 2.1. *Given* $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, *the problem*

$$u^{(2n)} - a_1 u^{(2(n-1))} + \ldots + (-1)^{n-1} a_{n-1} u'' + (-1)^n a_n u = 0, \tag{2.1}$$

$$u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad 0 \leqslant i \leqslant n-1 \tag{2.2}$$

has a non-trivial solution if and only if

$$\frac{a_1}{(k\pi)^2} + \frac{a_2}{(k\pi)^4} + \ldots + \frac{a_n}{(k\pi)^{2n}} + 1 = 0 \tag{2.3}$$

for some $k \in \mathbb{N}$.

P r o o f. Let $Au = u''$. Then

$$u^{(2n)} - a_1 u^{(2(n-1))} + \ldots + (-1)^{n-1} a_{n-1} u'' + (-1)^n a_n u = \left(\prod_{i=1}^{n}(A - r_i)\right) u$$

for some $r_i \in C$, $1 \leqslant i \leqslant n$. It is easy to see that if (2.1)–(2.2) possesses a nontrivial solution, then one of the r_i $(1 \leqslant i \leqslant n)$ is equal to $-(k\pi)^2$ for some $k \in \mathbb{N}$, $k \neq 0$. So $\sin k\pi t$ is a nontrivial solution of (2.1)–(2.2). By substituting this solution into (2.1), (2.3) follows. Reciprocally, if (2.3) holds, then clearly $\sin k\pi t$ is a nontrivial solution of (2.1)–(2.2).

Lemma 2.2 [11]. *If $u(t)$ satisfies*

$$u''(t) + g(t)u'(t) + h(t)u(t) \geqslant 0, \quad t \in (a, b),$$

where $h(t) \leqslant 0$, g, h are bounded in any closed subset of (a, b), and there is $c \in (a, b)$ such that $M = u(c) = \max\limits_{a \leqslant t \leqslant b} u(t)$ is a nonnegative maximum, then $u(t) \equiv M$. Moreover, if $h(t) \leqslant 0$ and $h(t) \not\equiv 0$, then $M = 0$.

Let for

$$F = \{u \in C^{2n}[0,1] \colon (-1)^i u^{2i}(0) \geqslant 0,\ (-1)^i u^{(2i)}(1) \geqslant 0,\ 0 \leqslant i \leqslant n-1\}$$

the operator

$$L \colon F \to C[0,1]$$

be defined by $Lu = u^{(2n)} - a_1 u^{(2(n-1))} + \ldots + (-1)^{n-1} a_{n-1} u'' + (-1)^n a_n u$, $u \in F$. Here a_i $(1 \leqslant i \leqslant n)$ are such that the equation $x^n - a_1 x^{n-1} + \ldots + (-1)^{n-1} a_{n-1} x + (-1)^n a_n = 0$ has only nonnegative real roots.

Lemma 2.3. *If $u \in F$ satisfies $(-1)^n Lu \geqslant 0$, then $u \geqslant 0$ in $[0,1]$.*

P r o o f. Let $Au = u''$. Suppose r_i $(1 \leqslant i \leqslant n)$ are n nonnegative real roots of the equation $x^n - a_1 x^{n-1} + \ldots + (-1)^{n-1} a_{n-1} x + (-1)^n a_n = 0$; we have

$$(-1)^n Lu = (-1)^n (A - r_n)(A - r_{n-1}) \ldots (A - r_1) u \geqslant 0.$$

Let $y_i = (A - r_i) \ldots (A - r_1) u$, $1 \leqslant i \leqslant n-1$. Then $(-1)^n (A - r_n) y_{n-1} \geqslant 0$, i.e., $(-1)^n y''_{n-1} - (-1)^n r_n y_{n-1} \geqslant 0$. On the other hand, $r_i \geqslant 0$, $1 \leqslant i \leqslant n-1$, and

$u \in F$ yield

$$(-1)^n y_{n-1}(0) = (-1)^n \Big[u^{(2(n-1))}(0) - \sum_{i=1}^{n-1} r_i u^{(2(n-2))}(0) + \ldots + (-1)^{n-1} \prod_{i=1}^{n-1} r_i u(0)\Big]$$
$$\leqslant 0,$$
$$(-1)^n y_{n-1}(1) = (-1)^n \Big[u^{(2(n-1))}(1) - \sum_{i=1}^{n-1} r_i u^{(2(n-2))}(1) + \ldots + (-1)^{n-1} \prod_{i=1}^{n-1} r_i u(1)\Big]$$
$$\leqslant 0.$$

By Lemma 2.2, we have

$$(-1)^n y_{n-1} \leqslant 0 \quad \text{for } t \in [0,1],$$

i.e.,

$$(-1)^{n-1} y_{n-1} \geqslant 0 \quad \text{for } t \in [0,1].$$

By inductive method and using Lemma 2.2, the result follows. □

3. Main results

Definition 3.1. Suppose $\alpha \in C^{2n}[0,1]$. We say α is an upper solution for the problem (1.1)–(1.2) if α satisfies

$$(-1)^n \alpha^{(2n)}(t) \geqslant (-1)^n f(t, \alpha(t), \alpha''(t), \ldots, \alpha^{(2(n-1))}(t)), \quad 0 < t < 1,$$
$$(-1)^i \alpha^{(2i)}(0) \geqslant 0, \quad (-1)^i \alpha^{(2i)}(1) \geqslant 0, \quad 0 \leqslant i \leqslant n-1.$$

Definition 3.2. Suppose $\beta \in C^{2n}[0,1]$. We say β is a lower solution for the problem (1.1)–(1.2) if β satisfies

$$(-1)^n \beta^{(2n)}(t) \leqslant (-1)^n f(t, \beta(t), \beta''(t), \ldots, \beta^{(2(n-1))}(t)), \quad 0 < t < 1,$$
$$(-1)^i \beta^{(2i)}(0) \leqslant 0, \quad (-1)^i \beta^{(2i)}(1) \leqslant 0, \quad 0 \leqslant i \leqslant n-1.$$

If the equation $x^n - a_1 x^{n-1} + \ldots + (-1)^{n-1} a_{n-1} x + (-1)^n a_n = 0$ has only non-negative real roots, then $a_i \geqslant 0$, $1 \leqslant i \leqslant n$. Let

(3.1) $$f_1(t, u_0, \ldots, u_{n-1}) = f(t, u_0, \ldots, u_{n-1}) - a_1 u_{n-1} + \ldots + (-1)^{n-1} u_1 + (-1)^n u_0.$$

Then (1.1) is equivalent to

(3.2) $$Lu = f_1(t, u, u'', \ldots, u^{(2(n-1))}).$$

Remark 1. In Definition 3.1, we say α is an upper solution for the problem (3.2)–(1.2) if α satisfies

$$(-1)^n(L\alpha)(t) \geqslant (-1)^n f_1(t,\alpha(t),\alpha''(t),\dots,\alpha^{(2(n-1))}(t)), \quad 0<t<1,$$
$$(-1)^i\alpha^{(2i)}(0)\geqslant 0, \quad (-1)^i\alpha^{(2i)}(1)\geqslant 0, \quad 0\leqslant i\leqslant n-1.$$

Similarly, we may define a lower solution for the problem (3.2)–(1.2). Therefore, α, β are upper and lower solutions of the problem (1.1)–(1.2) if and only if α, β are upper and lower solutions of the problem (3.2)–(1.2).

Definition 3.3. If $\ldots\leqslant\alpha_m\leqslant\ldots\leqslant\alpha_1\leqslant\alpha_0=\alpha$ are upper solutions converging uniformly to a solution u for the problem (1.1)–(1.2), we say u is an extremal solution for the problem (1.1)–(1.2).

Similarly, for an lower solutions $\beta=\beta_0\leqslant\beta_1\leqslant\ldots\leqslant\beta_m\leqslant\ldots$, we may define an extremal solution for the problem (1.1)–(1.2).

Let

$$\prod_{i=1}^{k}(A-r_i)u = u^{(2k)} - a_{kk}u^{(2(k-1))} + \ldots + (-1)^k a_{k1}u,$$

where $A=u''$, $r_i\geqslant 0$, $a_{ki}\geqslant 0$ ($i=1,2,\dots,k$; $k=1,2,\dots,n$). Set $b_{11}=a_{11}=r_1$, $b_{kk}=a_{kk}$, $b_{k,k-1}=a_{kk}b_{k-1,k-1}+a_{k,k-1}$, $b_{k,k-2}=a_{kk}b_{k-1,k-2}+a_{k,k-1}b_{k-2,k-2}+a_{k,k-2}$, …, $b_{k1}=a_{kk}b_{k-1,1}+a_{k,k-1}b_{k-2,1}+\ldots+a_{k2}b_{11}+a_{k1}$ ($k=2,3,\dots,n$).

Theorem 3.1. *Let there exist upper and lower solutions α and β respectively for the problem* (1.1)–(1.2) *which satisfy*

(1) $\beta\leqslant\alpha$, $\beta^{(2k)}\leqslant\alpha^{(2k)}+\sum\limits_{i=0}^{k-1}(-1)^i b_{k,i+1}(\alpha-\beta)^{(2i)}$ ($k=2,4,6,\dots$, *and* $k\leqslant n-1$), $\alpha^{(2k)}\leqslant\beta^{(2k)}+\sum\limits_{i=0}^{k-1}(-1)^i b_{k,i+1}(\alpha-\beta)^{(2i)}$ ($k=1,3,5,\dots$, *and* $k\leqslant n-1$); *and if* $f\colon[0,1]\times\mathbb{R}^n\to\mathbb{R}$ *is continuous and satisfies*

(2) $(-1)^n\big[f\big(t,y_0^{(2)},y_1,\dots,y_{n-1}\big)-f\big(t,y_0^{(1)},y_1,\dots,y_{n-1}\big)\big]\geqslant -a_n\big(y_0^{(2)}-y_0^{(1)}\big)$ *for* $\beta(t)\leqslant y_0^{(1)}\leqslant y_0^{(2)}\leqslant\alpha(t)$, $y_1,\dots,y_{n-1}\in\mathbb{R}$, *and* $t\in[0,1]$;

(3) $(-1)^{n-k}\big[f\big(t,y_0,\dots,y_k^{(2)},\dots,y_{n-1}\big)-f\big(t,y_0,\dots,y_k^{(1)},\dots,y_{n-1}\big)\big]\geqslant -a_{n-k}\times\big(y_k^{(2)}-y_k^{(1)}\big)$ *for* $y_k^{(1)}\leqslant y_k^{(2)}+\sum\limits_{i=0}^{k-1}(-1)^i b'_{k,i+1}(\alpha-\beta)^{(2i)}$ *and* $\alpha^{(2k)}-\sum\limits_{i=0}^{k-1}(-1)^i b'_{k,i+1}\times(\alpha-\beta)^{(2i)}\leqslant y_k^{(1)}$, $y_k^{(2)}\leqslant\beta^{(2k)}+\sum\limits_{i=0}^{k-1}(-1)^i b'_{k,i+1}(\alpha-\beta)^{(2i)}$ *if* $k=1,3,5,\dots$, $k\leqslant n-1$, $\beta^{(2k)}-\sum\limits_{i=0}^{k-1}(-1)^i b'_{k,i+1}(\alpha-\beta)^{(2i)}\leqslant y_k^{(1)}$, $y_k^{(2)}\leqslant\alpha^{(2k)}+\sum\limits_{i=0}^{k-1}(-1)^i b'_{k,i+1}(\alpha-\beta)^{(2i)}$ *if* $k=2,4,\dots$, $k\leqslant n-1$, $y_1,\dots,y_{k-1},y_{k+1},\dots,y_{n-1}\in\mathbb{R}$, *and* $t\in[0,1]$,

where $b'_{ki} = 2b_{ki} - a_{ki}$ ($k = 1, 2, \ldots, n-1$; $i \leqslant k$), $a_1, a_2, \ldots, a_n$ such that the equation $x^n - a_1x^{n-1} + \ldots + (-1)^{n-1}a_{n-1}x + (-1)^n a_n = 0$ has only nonnegative real roots, which are r_i ($i = 1, 2, \ldots, n$).

Then there exist two monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$, non-increasing and non-decreasing, with $\alpha_0 = \alpha$ and $\beta_0 = \beta$, which converge uniformly to the extremal solutions in $[\beta, \alpha]$ of the problem (1.1)–(1.2).

Proof. (1) implies $(-1)^k(\alpha-\beta)^{(2k)} + \sum\limits_{i=0}^{k-1}(-1)^i b_{k,i+1}(\alpha-\beta)^{(2i)} \geqslant 0$ for $1 \leqslant k \leqslant n-1$. Thus, for $1 \leqslant k \leqslant n-1$, we have

$$\sum_{i=0}^{k-1} a_{k,i+1}\Bigl[(-1)^i(\alpha-\beta)^{(2i)} + \sum_{j=0}^{i-1}(-1)^j b_{i,j+1}(\alpha-\beta)^{(2j)}\Bigr] \geqslant 0,$$

i.e.,

$$\sum_{i=0}^{k-1}(-1)^i\Bigl[a_{k,i+1} + \sum_{j=i+1}^{k-1} a_{k,j+1}b_{j,i+1}\Bigr](\alpha-\beta)^{(2i)} = \sum_{i=0}^{k-1}(-1)^i b_{k,i+1}(\alpha-\beta)^{(2i)} \geqslant 0,$$

and

$$(*)\qquad \begin{aligned}&\sum_{i=0}^{k-1}(-1)^i b'_{k,i+1}(\alpha-\beta)^{(2i)}\\ &\quad = \sum_{i=0}^{k-1}(-1)^i b_{k,i+1}(\alpha-\beta)^{(2i)} + a_{kk}\sum_{i=0}^{k-2}(-1)^i b_{k-1,i+1}(\alpha-\beta)^{(2i)}\\ &\qquad + a_{k,k-1}\sum_{i=0}^{k-3}(-1)^i b_{k-2,i+1}(\alpha-\beta)^{(2i)} + \ldots + a_{k2}b_{11}(\alpha-\beta)\\ &\quad \geqslant \sum_{i=0}^{k-1}(-1)^i b_{k,i+1}(\alpha-\beta)^{(2i)} \geqslant 0.\end{aligned}$$

Consider the problem

$$(3.3)\qquad \begin{aligned}u^{(2n)}(t) - a_1u^{(2(n-1))}(t) + \ldots + (-1)^{n-1}a_{n-1}u''(t) + (-1)^n a_n u(t)\\ = f_1(t, \eta(t), \eta''(t), \ldots, \eta^{(2(n-1))}(t)), \quad t \in (0,1),\end{aligned}$$

$$(3.4)\qquad u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad 0 \leqslant i \leqslant n-1,$$

where $\eta \in C^{2(n-1)}[0,1]$.

It is easy to see that if $x^n - a_1x^{n-1} + \ldots + (-1)^{n-1}a_{n-1}x + (-1)^n a_n = 0$ has only nonnegative real roots, then $a_i \geqslant 0$, $1 \leqslant i \leqslant n$. By Lemma 2.1 and the

Fredholm alternative [16], the problem (3.3)–(3.4) has a unique solution u. Define $T\colon C^{2(n-1)}[0,1] \to C^{2n}[0,1]$ by

$$T\eta = u. \tag{3.5}$$

We first prove

$$TC \subseteq C. \tag{3.6}$$

Here, $C = \{\eta \in C^{2(n-1)}[0,1]\colon \beta \leqslant \eta \leqslant \alpha,\ \alpha^{(2k)} - \sum_{i=0}^{k-1}(-1)^i b'_{k,i+1}(\alpha-\beta)^{(2i)} \leqslant \eta^{(2k)} \leqslant \beta^{(2k)} + \sum_{i=0}^{k-1}(-1)^i b'_{k,i+1}(\alpha-\beta)^{(2i)}$ if $k = 1,3,5,\ldots,\ k \leqslant n-1$ and $\beta^{(2k)} - \sum_{i=0}^{k-1}(-1)^i b'_{k,i+1}(\alpha-\beta)^{(2i)} \leqslant \eta^{(2k)} \leqslant \alpha^{(2k)} + \sum_{i=0}^{k-1}(-1)^i b'_{k,i+1}(\alpha-\beta)^{(2i)}$ if $k = 2,4,\ldots,$ $k \leqslant n-1\}$.

By $(*)$, it is easy to see that $\alpha, \beta \in C$. Therefore, C is a nonempty bounded closed subset of $C^{2(n-1)}[0,1]$.

For $\eta \in C$, set $u = T\eta$. By conditions (2)–(3) and (3.3), we have

$$\begin{aligned}
(3.7)\quad & (-1)^n[(\alpha-u)^{(2n)}(t) - a_1(\alpha-u)^{(2(n-1))}(t) + \ldots + (-1)^n a_n(\alpha-u)(t)] \\
&\geqslant (-1)^n[f_1(t,\alpha(t),\alpha''(t),\ldots,\alpha^{(2(n-1))}(t)) \\
&\qquad - f_1(t,\eta(t),\eta''(t),\ldots,\eta^{(2(n-1))}(t))] \\
&= (-1)^n[f(t,\alpha(t),\alpha''(t),\ldots,\alpha^{(2(n-1))}(t)) \\
&\qquad - f(t,\eta(t),\eta''(t),\ldots,\eta^{(2(n-1))}(t)) - a_1(\alpha-\eta)^{(2(n-1))}(t) + \ldots \\
&\qquad + (-1)^{n-1}a_{n-1}(\alpha-\eta)''(t) + (-1)^n a_n(\alpha-\eta)(t)] \\
&= \sum_{k=0}^{n-1}(-1)^n[f(t,\eta(t),\ldots,\eta^{(2(k-1))},\alpha^{(2k)}(t),\ldots,\alpha^{(2(n-1))}(t)) \\
&\qquad - f(t,\eta(t),\ldots,\eta^{(2k)}(t),\alpha^{(2(k+1))}(t),\ldots,\alpha^{(2(n-1))}(t)) \\
&\qquad + (-1)^{n-k}a_{n-k}(\alpha-\eta)^{(2k)}(t)] \\
&= \sum_{k=0}^{n-1}(-1)^k\{(-1)^{n-k}[f(t,\eta(t),\ldots,\eta^{(2(k-1))},\alpha^{(2k)}(t),\ldots,\alpha^{(2(n-1))}(t)) \\
&\qquad - f(t,\eta(t),\ldots,\eta^{(2k)}(t),\alpha^{(2(k+1))}(t),\ldots,\alpha^{(2(n-1))}(t))] \\
&\qquad + a_{n-k}(\alpha-\eta)^{(2k)}(t)\} \geqslant 0,
\end{aligned}$$

$$
\begin{aligned}
(3.7)' \quad & (-1)^n[(u-\beta)^{(2n)}(t) - a_1(u-\beta)^{(2(n-1))}(t) + \ldots + (-1)^n a_n(u-\beta)(t)] \\
& \geqslant (-1)^n[f_1(t,\eta(t),\eta''(t),\ldots,\eta^{(2(n-1))}(t)) \\
& \qquad - f_1(t,\beta(t),\beta''(t),\ldots,\beta^{(2(n-1))}(t))] \\
& = (-1)^n[f(t,\eta(t),\eta''(t),\ldots,\eta^{(2(n-1))}(t)) \\
& \qquad - f(t,\beta(t),\beta''(t),\ldots,\beta^{(2(n-1))}(t)) - a_1(\eta-\beta)^{(2(n-1))}(t) + \ldots \\
& \qquad + (-1)^{n-1}a_{n-1}(\eta-\beta)''(t) + (-1)^n a_n(\eta-\beta)(t)] \\
& = \sum_{k=0}^{n-1}(-1)^n[f(t,\beta(t),\ldots,\beta^{(2(k-1))},\eta^{(2k)}(t),\ldots,\eta^{(2(n-1))}(t)) \\
& \qquad - f(t,\beta(t),\ldots,\beta^{(2k)}(t),\eta^{(2(k+1))}(t),\ldots,\eta^{(2(n-1))}(t)) \\
& \qquad + (-1)^{n-k}a_{n-k}(\eta-\beta)^{(2k)}(t)] \\
& = \sum_{k=0}^{n-1}(-1)^k\{(-1)^{n-k}[f(t,\beta(t),\ldots,\beta^{(2(k-1))},\eta^{(2k)}(t),\ldots,\eta^{(2(n-1))}(t)) \\
& \qquad - f(t,\beta(t),\ldots,\beta^{(2k)}(t),\eta^{(2(k+1))}(t),\ldots,\eta^{(2(n-1))}(t))] \\
& \qquad + a_{n-k}(\eta-\beta)^{(2k)}(t)\} \geqslant 0,
\end{aligned}
$$

$$
(3.8) \qquad (-1)^i(\alpha-u)^{(2i)}(0) \geqslant 0, \quad (-1)^i(\alpha-u)^{(2i)}(1) \geqslant 0, \quad 0 \leqslant i \leqslant n-1,
$$

$$
(3.8)' \qquad (-1)^i(u-\beta)^{(2i)}(0) \geqslant 0, \quad (-1)^i(u-\beta)^{(2i)}(1) \geqslant 0, \quad 0 \leqslant i \leqslant n-1.
$$

(3.7) and (3.8) imply $\alpha \geqslant u$ by Lemma 2.3. Similarly, (3.7)′ and (3.8)′ imply $u \geqslant \beta$.

Next we prove

$$
(3.9) \qquad \alpha^{(2k)} - \sum_{i=0}^{k-1}(-1)^i b_{k,i+1}(\alpha-\beta)^{(2i)} \leqslant u^{(2k)} \leqslant \beta^{(2k)} + \sum_{i=0}^{k-1}(-1)^i b_{k,i+1}(\alpha-\beta)^{(2i)}
$$

for $k = 1, 3, 5, \ldots$, $k \leqslant n-1$, and

$$
(3.10) \qquad \beta^{(2k)} - \sum_{i=0}^{k-1}(-1)^i b_{k,i+1}(\alpha-\beta)^{(2i)} \leqslant u^{(2k)} \leqslant \alpha^{(2k)} + \sum_{i=0}^{k-1}(-1)^i b_{k,i+1}(\alpha-\beta)^{(2i)}
$$

for $k = 2, 4, 6, \ldots$, $k \leqslant n-1$.

By the proof of Lemma 2.3, combining (3.7) and (3.8), (3.7)′ and (3.8)′, for $1 \leqslant k \leqslant n-1$ we get

$$(3.11)\qquad (-1)^k[(\alpha-u)^{(2k)}(t)+\sum_{i=0}^{k-1}(-1)^{k-i}a_{k,i+1}(\alpha-u)^{(2i)}(t)] \geqslant 0, \quad t\in[0,1],$$

$$(3.11)'\qquad (-1)^k[(u-\beta)^{(2k)}(t)+\sum_{i=0}^{k-1}(-1)^{k-i}a_{k,i+1}(u-\beta)^{(2i)}(t)] \geqslant 0, \quad t\in[0,1].$$

Therefore,

$$u''(t) \geqslant \alpha''(t) - a_{11}(\alpha-u)(t) \geqslant \alpha''(t) - b_{11}(\alpha-\beta)(t), \quad t\in[0,1].$$

Similarly,

$$u''(t) \leqslant \beta''(t) + b_{11}(\alpha-\beta)(t), \quad t\in[0,1].$$

Thus,

$$\begin{aligned} u^{(4)}(t) &\leqslant \alpha^{(4)}(t) - a_{22}(\alpha-u)''(t) + a_{21}(\alpha-u)(t) \\ &= \alpha^{(4)}(t) - a_{22}\alpha''(t) + a_{22}u''(t) + a_{21}(\alpha-u)(t) \\ &\leqslant \alpha^{(4)}(t) - a_{22}(\alpha-\beta)''(t) + (a_{22}b_{11}+a_{21})(\alpha-\beta)(t) \\ &= \alpha^{(4)}(t) - b_{22}(\alpha-\beta)''(t) + b_{21}(\alpha-\beta)(t) \end{aligned}$$

for $t\in[0,1]$. Similarly,

$$u^{(4)}(t) \geqslant \beta^{(4)}(t) + b_{22}(\alpha-\beta)''(t) - b_{21}(\alpha-\beta)(t), \quad t\in[0,1].$$

Suppose (3.9)–(3.10) hold for i from 1 to $k-1$. When k is an odd number, using (3.11) we obtain

$$\begin{aligned} u^{(2k)}(t) &\geqslant \alpha^{(2k)}(t) + \sum_{i=0}^{k-1}(-1)^{k-i}a_{k,i+1}(\alpha-u)^{(2i)}(t) \\ &= \alpha^{(2k)}(t) + \sum_{i=0}^{k-1}(-1)^{k-i}a_{k,i+1}\alpha^{(2i)}(t) - \sum_{i=0}^{k-1}(-1)^{k-i}a_{k,i+1}u^{(2i)}(t) \end{aligned}$$

$$
\begin{aligned}
&\geqslant \alpha^{(2k)}(t) - \sum_{i=0}^{k-1}(-1)^i a_{k,i+1}\alpha^{(2i)}(t) \\
&\quad + \sum_{i=1}^{k-1}(-1)^i a_{k,i+1}\left[\beta^{(2i)}(t) + (-1)^{i-1}\sum_{j=0}^{i-1}(-1)^j b_{i,j+1}(\alpha-\beta)^{(2j)}(t)\right] + a_{k1}\beta(t) \\
&= \alpha^{(2k)}(t) - \sum_{i=0}^{k-1}(-1)^i (a_{k,i+1} + a_{k,i+2}b_{i+1,i+1} + \ldots + a_{kk}b_{k-1,i+1})(\alpha-\beta)^{(2i)}(t) \\
&= \alpha^{(2k)}(t) - \sum_{i=0}^{k-1}(-1)^i b_{k,i+1}(\alpha-\beta)^{(2i)}(t).
\end{aligned}
$$

Similarly,

$$
u^{(2k)}(t) \leqslant \beta^{(2k)}(t) + \sum_{i=0}^{k-1}(-1)^i b_{k,i+1}(\alpha-\beta)^{(2i)}(t).
$$

When k is an even number, using (3.11) we get

$$
\begin{aligned}
u^{(2k)}(t) &\leqslant \alpha^{(2k)}(t) + \sum_{i=0}^{k-1}(-1)^{k-i} a_{k,i+1}(\alpha-u)^{(2i)}(t) \\
&= \alpha^{(2k)}(t) + \sum_{i=0}^{k-1}(-1)^i a_{k,i+1}\alpha^{(2i)}(t) - \sum_{i=0}^{k-1}(-1)^i a_{k,i+1}u^{(2i)}(t) \\
&\leqslant \alpha^{(2k)}(t) + \sum_{i=0}^{k-1}(-1)^i a_{k,i+1}\alpha^{(2i)}(t) \\
&\quad - \sum_{i=1}^{k-1}(-1)^i a_{k,i+1}\left[\beta^{(2i)}(t) + (-1)^{i-1}\sum_{j=0}^{i-1}(-1)^j b_{i,j+1}(\alpha-\beta)^{(2j)}(t)\right] - a_{k1}\beta(t) \\
&= \alpha^{(2k)}(t) + \sum_{i=0}^{k-1}(-1)^i b_{k,i+1}(\alpha-\beta)^{(2i)}(t).
\end{aligned}
$$

Similarly,

$$
u^{(2k)}(t) \geqslant \beta^{(2k)}(t) - \sum_{i=0}^{k-1}(-1)^i b_{k,i+1}(\alpha-\beta)^{(2i)}(t).
$$

By inductive method, (3.9)–(3.10) hold. Thus, (3.6) holds.

Let $u_1 = T\eta_1$, $u_2 = T\eta_2$, where $\eta_1, \eta_2 \in C$ satisfy

$$\eta_1 \leqslant \eta_2,$$

$$\eta_2^{(2k)} \leqslant \eta_1^{(2k)} + \sum_{i=0}^{k-1} (-1)^i b'_{k,i+1} (\alpha - \beta)^{(2i)} \quad (k = 1, 3, 5, \ldots, k \leqslant n-1),$$

$$\eta_1^{(2k)} \leqslant \eta_2^{(2k)} + \sum_{i=0}^{k-1} (-1)^i b'_{k,i+1} (\alpha - \beta)^{(2i)} \quad (k = 2, 4, 6, \ldots, k \leqslant n-1).$$

Next we show

(3.12) $$u_1 \leqslant u_2,$$

$$u_2^{(2k)} \leqslant u_1^{(2k)} + \sum_{i=0}^{k-1} (-1)^i b'_{k,i+1} (\alpha - \beta)^{(2i)} \quad (k = 1, 3, 5, \ldots, k \leqslant n-1),$$

$$u_1^{(2k)} \leqslant u_2^{(2k)} + \sum_{i=0}^{k-1} (-1)^i b'_{k,i+1} (\alpha - \beta)^{(2i)} \quad (k = 2, 4, 6, \ldots, k \leqslant n-1).$$

In fact, by conditions (2)–(3),

$$\begin{aligned}
&(-1)^n L(u_2 - u_1)(t) \\
&\quad = (-1)^n \bigl[f_1(t, \eta_2(t), \ldots, \eta_2^{(2(n-1))}(t)) - f_1(t, \eta_1(t), \ldots, \eta_1^{(2(n-1))}(t))\bigr] \geqslant 0, \\
&\qquad (u_2 - u_1)^{(2i)}(0) = (u_2 - u_1)^{(2i)}(1) = 0, \quad 1 \leqslant i \leqslant n-1.
\end{aligned}$$

By virtue of Lemma 2.3, we obtain $u_1 \leqslant u_2$, and

$$(-1)^k \Biggl[(u_2 - u_1)^{(2k)} + \sum_{i=0}^{k-1} (-1)^{k-i} a_{k,i+1} (u_2 - u_1)^{(2i)}\Biggr] \geqslant 0, \quad 1 \leqslant i \leqslant n-1.$$

When k is an odd number, we have

$$\begin{aligned}
u_2^{(2k)} &\leqslant u_1^{(2k)} - \sum_{i=0}^{k-1} (-1)^{k-i} a_{k,i+1} (u_2 - u_1)^{(2i)} \\
&\leqslant u_1^{(2k)} + \sum_{i=0}^{k-1} (-1)^i a_{k,i+1} \Biggl[(\alpha - \beta)^{(2i)} + 2(-1)^i \sum_{j=0}^{i-1} (-1)^j b_{i,j+1} (\alpha - \beta)^{(2j)}\Biggr] \\
&= u_1^{(2k)} + \sum_{i=0}^{k-1} (-1)^i a_{k,i+1} (\alpha - \beta)^{(2i)} + 2 \sum_{i=0}^{k-1} \sum_{j=0}^{i-1} (-1)^j a_{k,i+1} b_{i,j+1} (\alpha - \beta)^{(2j)} \\
&= u_1^{(2k)} + \sum_{i=0}^{k-1} (-1)^i (a_{k,i+1} + 2(a_{kk} b_{k-1,i+1} + \ldots + a_{k,i+2} b_{i+1,i+1}) (\alpha - \beta)^{(2i)} \\
&= u_1^{(2k)} + \sum_{i=0}^{k-1} (-1)^i b'_{k,i+1} (\alpha - \beta)^{(2i)}.
\end{aligned}$$

Similarly, when k is an even number, we have

$$u_1^{(2k)} \leqslant u_2^{(2k)} + \sum_{i=0}^{k-1}(-1)^i b'_{k,i+1}(\alpha-\beta)^{(2i)}.$$

Therefore, (3.12) holds.

Let $\alpha_0 = \alpha$, $\beta_0 = \beta$, $\alpha_m = T\alpha_{m-1}$, $\beta_m = T\beta_{m-1}$, $m \in \mathbb{N}$. By (3.6) and (3.12), we have

$$\beta = \beta_0 \leqslant \beta_1 \leqslant \ldots \leqslant \beta_m \leqslant \ldots \leqslant \alpha_m \leqslant \ldots \leqslant \alpha_1 \leqslant \alpha_0 = \alpha, \tag{3.13}$$

$$\alpha^{(2k)} - \sum_{i=0}^{k-1}(-1)^i b'_{k,i+1}(\alpha-\beta)^{(2i)} \leqslant \alpha_m^{(2k)}, \tag{3.14}$$

$$\beta_m^{(2k)} \leqslant \beta^{(2k)} + \sum_{i=0}^{k-1}(-1)^i b'_{k,i+1}(\alpha-\beta)^{(2i)}$$

for $k = 1, 3, 5, \ldots$, $k \leqslant n-1$, and

$$\beta^{(2k)} - \sum_{i=0}^{k-1}(-1)^i b'_{k,i+1}(\alpha-\beta)^{(2i)} \leqslant \alpha_m^{(2k)}, \tag{3.15}$$

$$\beta_m^{(2k)} \leqslant \alpha^{(2k)} + \sum_{i=0}^{k-1}(-1)^i b'_{k,i+1}(\alpha-\beta)^{(2i)}$$

for $k = 2, 4, \ldots$, $k \leqslant n-1$. From the definition of T we get

$$\begin{aligned} \alpha_m^{(2n)}(t) &= f_1(t, \alpha_{m-1}(t), \alpha''_{m-1}(t), \ldots, \alpha_{m-1}^{(2(n-1))}(t)) + a_1\alpha_m^{(2(n-1))}(t) - \ldots \\ &\quad - (-1)^{n-1}a_{n-1}\alpha''_m(t) - (-1)^n a_n \alpha_m(t), \end{aligned} \tag{3.16}$$

$$\alpha_m^{(2i)}(0) = \alpha_m^{(2i)}(1) = 0, \quad 1 \leqslant i \leqslant n-1. \tag{3.17}$$

From (3.13)–(3.16), we have that there exists $M_n > 0$ depending only on α and β (but not on m or t) such that

$$|\alpha_m^{(2n)}(t)| \leqslant M_n \quad \text{for all } t \in [0,1]. \tag{3.18}$$

Using the boundary condition (3.17), we get that there exists $\xi_m \in (0,1)$ such that $\alpha_m^{(2n-1)}(\xi_m) = 0$ for each $m \in \mathbb{N}$. This together with (3.18) yields

$$|\alpha_m^{(2n-1)}(t)| = \left|\alpha_m^{(2n-1)}(\xi_m) + \int_{\xi_m}^t \alpha_m^{(2n)}(s)\,\mathrm{d}s\right| \leqslant M_n \quad \text{for all } t \in [0,1]. \tag{3.19}$$

By (3.14) and (3.15), we can similarly get that there are $M_i > 0$, $1 \leqslant i \leqslant n-1$, depending only on α and β (but not on m or t) such that

$$|\alpha_m^{(2i)}(t)| \leqslant M_i, \quad |\alpha_m^{(2i-1)}(t)| \leqslant M_i \quad \text{for all } t \in [0,1]. \tag{3.20}$$

Thus, from (3.13) and (3.18)–(3.20) we know that $\{\alpha_m\}$ is bounded in $C^{2n}[0,1]$. Similarly, $\{\beta_m\}$ is bounded in $C^{2n}[0,1]$. Therefore, $\{\alpha_m\}$, $\{\beta_m\}$ converge uniformly to the extremal solutions in $[\beta, \alpha]$ of the problem (3.2)–(1.2), i.e., $\{\alpha_m\}$, $\{\beta_m\}$ converge uniformly to the extremal solutions in $[\beta, \alpha]$ of the problem (1.1)–(1.2).

Example. Consider the boundary value problem

$$u^{(6)}(t) = 5u^{(4)}(t) - 8u''(t) + (u(t)+1)^2 - (\sin \pi t + 1)^2, \tag{3.21}$$

$$u(0) = u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = 0. \tag{3.22}$$

It is easy to check that $\alpha = \sin \pi t$, $\beta = 0$ are respectively upper and lower solutions of (3.21)–(3.22). Let $a_1 = 5$, $a_2 = 8$, $a_3 = 4$, $r_1 = 1$, $r_2 = r_3 = 2$. Clearly, all conditions of Theorem 3.1 are fulfilled. Hence the problem (3.21)–(3.22) has at least one solution u which satisfies $0 \leqslant u \leqslant \sin \pi t$.

Remark 2. If $a_1 = a_2 = a_3 = 0$, we can not conclude that the above problem has at least one solution. Thus, our result is better than the approach $a_i = 0$.

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