Czechoslovak Mathematical Journal

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*Czechoslovak Mathematical Journal*, Vol. 55 (2005), No. 3, 611–623

Persistent URL: http://dml.cz/dmlcz/128006

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# AN APPLICATION OF PÓLYA'S ENUMERATION THEOREM TO PARTITIONS OF SUBSETS OF POSITIVE INTEGERS

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(Received June 27, 2002)

*Abstract.* Let $S$ be a non-empty subset of positive integers. A partition of a positive integer $n$ into $S$ is a finite nondecreasing sequence of positive integers $a_1, a_2, \ldots, a_r$ in $S$ with repetitions allowed such that $\sum\limits_{i=1}^{r} a_i = n$. Here we apply Pólya's enumeration theorem to find the number $\mathrm{P}(n;S)$ of partitions of $n$ into $S$, and the number $\mathrm{DP}(n;S)$ of distinct partitions of $n$ into $S$. We also present recursive formulas for computing $\mathrm{P}(n;S)$ and $\mathrm{DP}(n;S)$.

*Keywords*: Pólya's enumeration theorem, partitions of a positive integer into a non-empty subset of positive integers, distinct partitions of a positive integer into a non-empty subset of positive integers, recursive formulas and algorithms

## 1. Introduction

Let $S$ be a non-empty subset of positive integers. A partition of a positive integer $n$ into $S$ is a finite nondecreasing sequence of positive integers $a_1, a_2, \ldots, a_r$ in $S$ with repetitions allowed such that $\sum\limits_{i=1}^{r} a_i = n$. The $a_i$'s are called the parts of a partition of $n$.

**Example 1.** (a) Let $S$ be the set of positive integers. Then the partition of positive integer $n$ into $S$ is the "usual" partition of $n$. For instance, there are 7 partitions of 5. Namely, 5, $1+4$, $2+3$, $1+1+3$, $1+2+2$, $1+1+1+2$ and $1+1+1+1+1$. (Usually, one writes the sequence as a series to indicate the sum is 5.) We note that each of the first three has distinct parts.

(b) Let $S$ be the set of all odd positive integers. Then there are 3 odd partitions of 5. Namely, 5, $1+1+3$ and $1+1+1+1+1$.

(c) Let $S$ be the set of all positive integers each of which is not a multiple of 3. Then there are 7 partitions of 6. Namely, $5+1$, $4+2$, $4+1+1$, $2+2+2$, $2+2+1+1$, $2+1+1+1+1$ and $1+1+1+1+1+1$.

(d) Let $S$ be the set of all positive integers each of which is not a multiple of 3, 4 or 5. Then there are 4 partitions of 6. Namely, the last 4 partitions in (c).

(e) Let $S = \{1, 2, 4\}$. Then there are 4 partitions of 5. Namely, $1+4$, $1+2+2$, $1+1+1+2$ and $1+1+1+1+1$. There are 3 partitions of 9 with 1, 2, 3, 4 or 5 parts. Namely, $1+4+4$, $1+2+2+4$ and $1+2+2+2+2$.

There are many results on the partitions of positive integers. (See [1].) Here we will apply Pólya's enumeration theorem ([5], [2], [4], [6]) to the partitions of positive integers into $S$ for any non-empty subset $S$ of positive integers. Based on this application, we obtain a recursive formula for the number $\mathrm{P}(n; S)$ of partitions of a positive integer $n$ into $S$ and a recursive formula for the number $\mathrm{DP}(n; S)$ of partitions of a positive integer $n$ into $S$ with distinct parts. Based on these recursive formulas, we present computer programs for computing $\mathrm{P}(n; S)$ and $\mathrm{DP}(n; S)$; in particular, $\mathrm{P}(n; I)$, $\mathrm{P}(n; 0)$, $\mathrm{DP}(n; I)$ and $\mathrm{DP}(n; 0)$ as well as some subsets of positive integers where $I$ is the set of positive integers and $O$ is the set of positive odd integers.

## 2. Pólya's Enumeration Theorem

We shall state Pólya's Enumeration Theorem. Let $G$ be a permutation group acting on a set $\{1, 2, \ldots, n\}$. Since every permutation can be uniquely written as a product of disjoint cycles, the cycle index $G$ is defined as the following polynomial in $Q[x_1, x_2, \ldots, x_n]$ where $Q$ is the field of rational numbers and $x_i x_j = x_j x_i$ for $i, j = 1, 2, \ldots, n$:

$$Z_G(x_1, x_2, \ldots, x_n) = \frac{1}{|G|} \sum_{\sigma \in G} x_1^{b_1} x_2^{b_2} \ldots x_n^{b_n}$$

where $|G|$ is the order of $G$ and $b_i$ is the number of cycles of length $i$ in the disjoint cycle decomposition of $\sigma$ for $i = 1, 2, \ldots, n$.

Pólya's Enumeration Theorem. Let $D$ be a finite set and $S$ a countable set, $S^D$ the set of all functions from the domain $D$ into the codomain $S$, $G$ a permutation group acting on $D$, $w$ a function, called the weight function, from $S$ into $R$ where $R$ is a commutative ring with an identity containing the field of rational numbers $Q$, and let a relation be defined on $S^D$ such that for $f, g \in S^D$, $f \sim g$ if and only if there exists a $\sigma \in G$ with

$$f(\sigma d) = g(d) \quad \text{for every } d \in D.$$

(Since $G$ is a group, the relation $\sim$ is an equivalence relation. Consequently, $S^D$ is partitioned into disjoint equivalence classes each of which is called a pattern.) Then the total pattern or the counting series is

$$Z_G\Bigl(\sum_{s\in S} w(s), \sum_{s\in S}(w(s))^2, \ldots, \sum_{s\in S}(w(s))^t, \ldots\Bigr). \tag{1}$$

## 3. Counting partitions of a positive integer

**Theorem 1.** (a) *For any positive integer $k$, let $D_k = \{1, 2, \ldots, k\}$, let $S$ be a non-empty subset of positive integers, $S^{D_k}$ the set of all functions from $D_k$ into $S$, let the symmetric group $S_k$ act on $D_k$, let the weight function $w\colon S \to Q[x]$ be defined as $w(i) = x^i$ for all $i$ in $S$, and for $f, g \in S^{D_k}$, $f \sim g$ if and only if there exists a $\sigma \in S_k$ such that $f(\sigma d) = g(d)$ for every $d$ in $D_k$. Then the number of partitions of a positive integer $n$ with $k$ parts into $S$ is the coefficient of $x^n$ in the counting series*

$$Z_{S_k}\Bigl(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \ldots, \sum_{i\in S} x^{ki}\Bigr); \tag{2}$$

(b) *the number $\mathrm{P}(n; S)$ of partitions of $n$ into $S$ is the coefficient of $x^n$ in the counting series*

$$\sum_{k=1}^{\infty} Z_{S_k}\Bigl(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \ldots, \sum_{i\in S} x^{ki}\Bigr). \tag{3}$$

Proof. (a) We claim that each equivalence class in $S^{D_k}$ with weight $n$ (i.e., every function in the equivalence class has weight $n$) determines a partition of $n$ into $S$ with $k$ parts. Let $E$ be an equivalence class with weight $n$ in $S^{D_k}$, and let $f$ be a function in $E$. Then $f$ has $k$ values with repetition allowed in $S$ such that $w(f) = x^n$. Since $S$ is a subset of positive integers, we may arrange the $k$ values of $f$ in a nondecreasing order, say, $j_1 \leqslant j_2 \leqslant \ldots \leqslant j_k$. Since $w(f) = x^n$ and $w(f) = \prod_{i+1}^{k} w(f(i)) = x^{j_1+j_2+\ldots+j_k}$, we have $j_1 + j_2 + \ldots + j_k = n$. Thus, $f$ corresponds to a partition of $n$ into $S$ with $k$ parts. Since $S_k$ acts on $D_k$ and $f(\sigma d) = g(d)$ for some $\sigma \in S_k$ and all $d \in D_k$, the equivalence class containing $f$ consists of all functions in $S^{D_k}$ such that each has the function values $\{j_1, j_2, \ldots, j_k\}$. Thus, each equivalence class with weight $n$ corresponds to a partition of $n$ into $S$ with $k$ parts.

Conversely, each partition $t_1 \leqslant t_2 \leqslant \ldots \leqslant t_k$ of $n$ into $S$ with $k$ parts determines an equivalence class with weight $n$ in $S^{D_k}$. Clearly, $h(i) = t_i$ for $i = 1, 2, \ldots, k$ is a function in $S^{D_k}$, and $w(h) = \prod_{i=1}^{k} x^{t_i} = x^{t_1+t_2+\ldots+t_k} = x^n$. Thus, the partition of $n$ into $S$ with $k$ parts determines the equivalence class containing $h$ in $S^{D_k}$.

By Pólya's enumeration theorem, the coefficient of $x^n$ in the counting series

$$Z_{S_k}\Big(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \ldots, \sum_{i\in S} x^{ki}\Big) \tag{2}$$

is the number of partitions of $n$ into $S$ with $k$ parts.

(b) Summing over $k = 1, 2, \ldots$, we obtain

$$\sum_{k=1}^{\infty} Z_{S_k}\Big(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \ldots, \sum_{i\in S} x^{ki}\Big). \tag{3}$$

In (3), the coefficient of $x^n$ is the number of partitions of $n$ into $S$ with $k$ parts for $k = 1, 2, \ldots$, i.e., the coefficient of $x^n$ is the number of partitions of $n$ into $S$. □

**Example 2.** Let $D_3 = \{1, 2, 3\}$, let $S$ be the set of all positive integers $= \{1, 2, \ldots, n, \ldots\}$, $S^{D_3}$ the set of all functions from $D_3$ into $S$, let

$$S_3 = \{1, (123), (132), (12), (13), (23)\}$$

act on $D_3$, and let let $w\colon S \to Q[x]$ be defined as $w(i) = x^i$ for $i = 1, 2, 3, \ldots$ Then the cycle index is

$$Z_{S_3}(x_1, x_2, x_3) = \tfrac{1}{6}(x_1^3 + 2x_3 + 3x_1x_2),$$

and $\sum_{i\in S} w(i) = \sum_{i=1}^{\infty} x^i$ (a formal power series), $\sum_{i\in S}(w(i))^2 = \sum_{i=1}^{\infty} x^{2i}$, $\sum_{i\in S}(w(i))^3 = \sum_{i=1}^{\infty} x^{3i}$. By (2), we have

$$\begin{aligned}
Z_{S_3}\Big(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \sum_{i\in S} x^{3i}\Big) &= Z_{S_3}\Big(\sum_{i=1}^{\infty} x^i, \sum_{i=1}^{\infty} x^{2i}, \sum_{i=1}^{\infty} x^{3i}\Big)\\
&= \tfrac{1}{6}\big((x^1 + x^2 + x^3 + \ldots + x^m + \ldots)^3 + 2(x^3 + x^6 + x^9 + \ldots + x^{3m} + \ldots)\\
&\quad + 3(x^1 + x^2 + x^3 + \ldots + x^m + \ldots)(x^2 + x^4 + x^6 + \ldots + x^{2m} + \ldots)\big)\\
&= \tfrac{1}{6}\big((x^3 + 3x^4 + 6x^5 + 10x^6 + 15x^7 + 21x^8 + \ldots) + (2x^3 + 2x^6 + \ldots)\\
&\quad + (3x^3 + 3x^4 + 6x^5 + 6x^6 + 9x^7 + 9x^8 + \ldots)\big)\\
&= x^3 + x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + \ldots,
\end{aligned}$$

which means:

For $n = 1$ or 2, there is no partition of $n$ with 3 parts.

For $n = 3$, there is 1 partition of 3 with 3 parts. (Namely, $1 + 1 + 1$.)

For $n = 4$, there is 1 partition of 4 with 3 parts. (Namely, $1 + 1 + 2$.)

For $n = 5$, there are 2 partitions of 5 with 3 parts. (Namely, $1+1+3$ and $1+1+2$.)

For $n = 6$, there are 3 partitions of 6 with 3 parts. (Namely, $1 + 1 + 4, 1 + 2 + 3$ and $2 + 2 + 2$.)

For $n = 7$, there are 4 partitions of 7 with 3 parts. (Namely, $1 + 1 + 5, 1 + 2 + 4$, $1 + 3 + 3$ and $2 + 2 + 3$.)

For $n = 8$, there are 5 partitions of 8 with 3 parts. (Namely, $1 + 1 + 6, 1 + 2 + 5$, $1 + 3 + 4$, $2 + 2 + 4$ and $2 + 3 + 3$.)

**Example 3.** Let $S = \{1, 2, 4\}$. Let $D_t = \{1, 2, \ldots, t\}$ for $t = 1, 2, 3, 4$, let $S^{D_t}$, $w$ and $S_t$ be defined similarly to Example 2. We know the following cycle indices:

$$\begin{aligned}
Z_{S_1}(x_1) &= x_1, \\
Z_{S_2}(x_1, x_2) &= \tfrac{1}{2}(x_1^2 + x_2), \\
Z_{S_3}(x_1, x_2, x_3) &= \tfrac{1}{6}(x_1^3 + 3x_1x_2 + 2x_3),
\end{aligned}$$

and

$$Z_{S_4}(x_1, x_2, x_3, x_4) = \tfrac{1}{24}(x_1^4 + 6x_1^2x_2 + 8x_1x_3 + 3x_2^2 + 6x_4).$$

Then

$$\begin{aligned}
&\sum_{k=1}^{4} Z_{S_k}\left(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \ldots\right) \\
&\quad = [(x + x^2 + x^4)] + \frac{1}{2}[(x + x^2 + x^4)^2 + (x^2 + x^4 + x^8)] \\
&\qquad + \frac{1}{6}[(x + x^2 + x^4)^3 + 3(x + x^2 + x^4)(x^2 + x^4 + x^8) + 2(x^3 + x^6 + x^{12})] \\
&\qquad + \frac{1}{24}[(x + x^2 + x^4)^4 + 6(x + x^2 + x^4)^2 + 8(x + x^2 + x^4)(x^3 + x^6 + x^{12}) \\
&\qquad + 3(x^2 + x^4 + x^8)^2 + 6(x^4 + x^8 + x^{16})] \\
&\quad = x + 2x^2 + 2x^3 + 4x^4 + 3x^5 + 4x^6 + 3x^7 + 4x^8 + 2x^9 + 3x^{10} + x^{11} \\
&\qquad + 2x^{12} + x^{13} + x^{14} + x^{16},
\end{aligned}$$

which means, for instance, for $n = 6$, there are 4 partitions of 6 with 1, 2, 3 or 4 parts. (Namely, $2 + 4$, $1 + 1 + 4$, $2 + 2 + 2$ and $1 + 1 + 2 + 2$.)

Using Theorem 1 we can obtain a recursive formula for $\mathrm{P}(n; S)$ with $n > 1$. Clearly, $\mathrm{P}(1; S) = 1$ if $1 \in S$, and $\mathrm{P}(1; S) = 0$ if $1 \notin S$.

**Corollary 1.1.**

$$\mathrm{P}(n;S)=\frac{1}{n}\bigg(\sum_{i|n,i\in S} i+\sum_{k=1}^{n-1}\bigg(\sum_{i|k,\, i\in S} i\bigg)\mathrm{P}(n-k;S)\bigg) \quad \textit{for } n>1. \tag{4}$$

**Remark.** If there exists no positive integer $i$ such that $i \mid k$ and $i \in S$, then $\sum_{i|k,\, i\in S} i = 0$.

In order to prove Corollary 1.1, we need two identities which can be found in [4]. First,

$$1+\sum_{k=1}^{\infty} Z_{S_k}(f(x), f(x^2), \ldots, f(x^k)) = \exp\bigg(\sum_{k=1}^{\infty}\frac{f(x^k)}{k}\bigg) \tag{5}$$

where $f(x)$ is a function of $x$ or a series of $x$.

Second, if

$$\sum_{m=0}^{\infty} A_m x^m = \exp\bigg(\sum_{m=1}^{\infty} a_m x^m\bigg), \tag{6}$$

then, for $m \geqslant 1$,

$$a_m = A_m - m^{-1}\bigg(\sum_{k=1}^{m-1} k a_k A_{m-k}\bigg).$$

P r o o f of Corollary 1.1. By Theorem 1, we have

$$1+\sum_{n=1}^{\infty}\mathrm{P}(n;S)x^n = 1+\sum_{k=1}^{\infty} Z_{S_k}\bigg(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \ldots, \sum_{i\in S} x^{ki}\bigg). \tag{7}$$

By (5), the right-hand side of (7) is $\exp\Big(\sum\limits_{k=1}^{\infty}\Big(\sum\limits_{i\in S} x^{ki}/k\Big)\Big)$. So,

$$\begin{aligned}1+\sum_{n=1}^{\infty}\mathrm{P}(n;S)x^n &= \exp\bigg(\sum_{k=1}^{\infty}\bigg(\sum_{i\in S}\frac{x^{ki}}{k}\bigg)\bigg) = \exp\bigg(\sum_{k=1}^{\infty}\bigg(\sum_{i\in S}\frac{ix^{ki}}{ki}\bigg)\bigg)\\ &= \exp\bigg(\sum_{n=1}^{\infty}\frac{1}{n}\bigg(\sum_{i|n,\, i\in S} i\bigg)x^n\bigg).\end{aligned}$$

By (6), we have

$$\frac{1}{n}\sum_{i|n,\, i\in S} i = \mathrm{P}(n;S) - \frac{1}{n}\bigg(\sum_{k=1}^{n-1} k\cdot\frac{1}{k}\bigg(\sum_{i|k,\, i\in S} i\bigg)\mathrm{P}(n-k;S)\bigg).$$

Hence,

$$P(n;S) = \frac{1}{n}\Bigl(\sum_{i|n,\, i\in S} i + \sum_{k=1}^{n-1}\Bigl(\sum_{i|k,\, i\in S} i\Bigr)P(n-k,S)\Bigr).$$

Now we consider the set of positive integers $I = \{1,2,\ldots,n,\ldots\}$ and the set of positive odd integers $O = \{1,3,\ldots,2n-1,\ldots\}$. By using Corollary 1.1, we obtain recursive formulas $P(n;I)$ and $P(n;O)$ where $n$ is a positive integer.

**Corollary 1.2.** (a) $P(1;I) = 1$ *and for* $n > 1$,

(8) $$P(n;I) = \frac{1}{n}\Bigl(\sum_{i|n,\, i\in S} i + \sum_{k=1}^{n-1}\Bigl(\sum_{i|k,\, i\in S} i\Bigr)P(n-k;I)\Bigr).$$

(b) $P(I;O) = 1$ *and for* $n > 1$,

(9) $$P(n;O) = \frac{1}{n}\Bigl(\sum_{i|n,\, i\in O} I + \sum_{k=1}^{n-1}\Bigl(\sum_{i|k,\, i\in O} i\Bigr)P(n-k;O)\Bigr).$$

Proof. (a) (8) is obtained by substituting $I$ for $S$ in (4) in Corollary 1.1 and (9) is obtained by substituting $O$ for $S$ in (4) in Corollary 1.1. □

**Example 4.** We use Corollary 1.2 to compute $P(n;I)$ and $P(n;O)$ for $n = 1,2,3,4,5$.

$$\begin{aligned}
P(1;I) &= 1,\\
P(2;I) &= \tfrac{1}{2}(3 + P(1;I)) = \tfrac{1}{2}(3+1) = 2,\\
P(3;I) &= \tfrac{1}{3}(4 + P(2;I) + 3P(1;I) = \tfrac{1}{3}(4+2+3) = 3,\\
P(4;I) &= \tfrac{1}{4}(7 + P(3;I) + 3P(2;I) + 4P(1;I)) = \tfrac{1}{4}(7+3+6+4) = 5,\\
P(5;I) &= \tfrac{1}{5}(6 + P(4;I) + 3P(3;I) + 4P(2;I) + 7P(1;I))\\
&= \tfrac{1}{5}(6+5+9+8+7) = 7,\\
P(1;O) &= 1,\\
P(2;O) &= \tfrac{1}{2}(1 + P(1;O)) = \tfrac{1}{2}(1+1) = 1,\\
P(3;O) &= \tfrac{1}{3}(4 + P(2;O) + P(1;O) = \tfrac{1}{3}(4+1+1) = 2,\\
P(4;O) &= \tfrac{1}{4}(1 + P(3;O) + P(2;O) + 4P(1;O)) = \tfrac{1}{4}(1+2+1+4) = 2,\\
P(5;O) &= \tfrac{1}{5}(6 + P(4;O) + P(3;O) + 4P(2;O) + P(1;O))\\
&= \tfrac{1}{5}(6+2+2+4+1) = 3.
\end{aligned}$$

## 4. Counting partitions of a positive integer into distinct parts

**Theorem 2.** (a) *Let $k$, $D_k$, $S$, $S_k$, $S^{D_k}$ and $w$ be the same as in Theorem 1. Also, let $F$ be a subset of all one-to-one functions in $S^{D_k}$, and for $f, g \in F, f \sim g$ if and only if there exists a $\sigma \in S_k$ such that $f(\sigma d) = g(d)$ for every $d$ in $D_k$. Then the number of partitions of a positive integer $n$ with $k$ distinct parts into $S$ is the coefficient of $x^n$ in the counting series*

$$Z_{A_k}\Big(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \dots, \sum_{i\in S} x^{ki}\Big) - Z_{S_k}\Big(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \dots, \sum_{i\in S} x^{ki}\Big) \tag{10}$$

*where $A_k$ is the alternating subgroup of $S_k$, and $Z_{A_1}(x_1) - Z_{S_1}(x_1)$ is defined to be $x_1$.*

(b) *The number $\mathrm{DP}(n; S)$ of partitions of $n$ into $S$ with distinct parts is the coefficient of $x^n$ in the counting series*

$$\sum_{k=1}^{\infty}\Big(Z_{A_k}\Big(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \dots, \sum_{i\in S} x^{ki}\Big) - Z_{S_k}\Big(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \dots, \sum_{x\in S} x^{ki}\Big)\Big). \tag{11}$$

The proofs are similar to those of Theorem 1 using the cycle indices of $A_k$ and $S_k$ for one-to-one functions. (See p. 48 in [4].)

Similarly to (5), we have

$$\begin{aligned} 1 + \sum_{k=1}^{\infty} Z_{A_k}(f(x), f(x^2), \dots, f(x^k)) - Z_{S_k}(f(x), f(x^2), \dots, f(x^k)) \\ = \exp\Big(\sum_{k=1}^{\infty}(-1)^{k+1}\frac{f(x^k)}{k}\Big) \end{aligned} \tag{12}$$

where $f(x)$ is a function of $x$ or a series of $x$. By using Theorem 2, (12) and (6), we obtain a recursive formula for $\mathrm{DP}(n; S)$ with $n > 1$. Clearly, $\mathrm{DP}(1; S) = 1$ if $1 \in S$, and $\mathrm{DP}(1; S) = 0$ if $1 \notin S$.

**Corollary 2.1.**

$$\mathrm{DP}(n; S) = \frac{1}{n}\Big(\sum_{i\mid n,\, i\in S}(-1)^{n/i+1}\, i + \sum_{k=1}^{n-1}\Big(\sum_{i\mid k,\, i\in S}((-1)^{k/i+1}\, i)\mathrm{DP}(n-k; S)\Big)\Big) \tag{13}$$

*for $n > 1$.*

By using Corollary 1.2 and Corollary 2.1, we can prove a well-known result which can be found in [1].

**Corollary 2.2** (Euler). $\mathrm{DP}(n;I)=\mathrm{P}(n;O)$, *i.e., the number of partitions of $n$ into distinct parts is equal to the number of partitions of $n$ into odd parts.*

P r o o f. For $n=1$, $\mathrm{DP}(1;I)=\mathrm{P}(I;O)=1$. For $n>1$, comparing the formulas for $\mathrm{P}(n;O)$ in Corollary 1.2 with the formula for $\mathrm{DP}(n;S)$ with $S=I$ in Corollary 2.1, we need only to prove that

$$\sum_{i|n,\, i\in I}(-1)^{(n/i)+1}\, i+\sum_{i|n,\, i\in O} i. \tag{14}$$

There are two cases to be considered.

*Case 1.* $n$ is odd. $i\mid n$ implies $i$ and $n/i$ are odd, so $i\in O$ and $(n/i)+1$ is even. Thus, (14) holds.

*Case 2.* $n$ is even. $n$ can be written as $n=2^t d$ where $t\geqslant 1$ and $d$ is odd. Thus, a factor of $n$ must have the form $2^j h$ where $0\leqslant j\leqslant t$ and $h\mid d$, and

$$\sum_{i|n,\, i\in I}(-1)^{(n/i)+1}\, i=\sum_{h|d,\, 0\leqslant j\leqslant t}(-1)^{(2^t d/(2^j h))+1}\, 2^j h=\sum_{h|d} h\Bigg(\sum_{j=0}^{t}(-1)^{2^{t-j}(d/h)+1}\, 2^j\Bigg).$$

Since

$$\begin{aligned}\sum_{j=0}^{t}(-1)^{2^{t-j}(d/h)+1}\, 2^j &= -1-2-2^2-2^3-\ldots-2^{t-1}+2^t\\ &= -(2^t-1)+2^t=1,\end{aligned}$$

we have

$$\sum_{i|n,\, i\in I}(-1)^{(n/i)+1}\, i=\sum_{h|d} h=\sum_{h|n,\, h\in O} h=\sum_{i|n,\, i\in O} i.$$

Thus, (14) again holds. □

**Example 5.** Let $D_3$, $S$, $S^{D_3}$, $S_3$ and $w$ be the same as in Example 2. We know that $A_3=\{1,(123),(132)\}$ and $Z_{A_3}(x_1,x_2,x_3)=\frac{1}{3}(x_1^3+2x_3)$.

From Example 2 we have

$$\begin{aligned}Z_{S_3}\Bigg(\sum_{i=1}^{\infty}x^i,\sum_{i=1}^{\infty}x^{2i},\sum_{i=1}^{\infty}x^{3i}\Bigg)&=x^3+x^4+2x^5+3x^6+4x^7+5x^8+\ldots\\ Z_{A_3}\Bigg(\sum_{i=1}^{\infty}x^i,\sum_{i=1}^{\infty}x^{2i},\sum_{i=1}^{\infty}x^{3i}\Bigg)&=\frac{1}{3}\big((x^3+3x^4+6x^5+10x^6+15x^7+21x^8+\ldots)\\ &\qquad+2(x^3+x^6+x^9+\ldots)\big)\\ &=x^3+x^4+2x^5+4x^6+5x^7+7x^8+\ldots\end{aligned}$$

By Theorem 2(a), the number of partitions of $n$ with 3 distinct parts into the set of positive integers is the coefficient of $x^n$ in the counting series

$$Z_{A_3}\left(\sum_{i=1}^{\infty} x^i, \sum_{i=1}^{\infty} x^{2i}, \sum_{i=1}^{\infty} x^{3i}\right) = Z_{S_3}\left(\sum_{i=1}^{\infty} x^i, \sum_{i=1}^{\infty} x^{2i}, \sum_{i=1}^{\infty} x^{3i}\right) = x^6 + x^7 + 2x^8 + \dots$$

For $n = 6$, $1 + 2 + 3$ is the only partition of 6 with 3 distinct parts.

For $n = 7$, $1 + 2 + 4$ is the only partition of 7 with 3 distinct parts.

For $n = 8$, $1+2+5$ and $1+3+4$ are the only partitions of 8 with 3 distinct parts.

**Example 6.** Let $D_k = \{1, 2, \dots, k\}$ and $S = \{1, 2, 4\}$. We want to compute $\mathrm{DP}(n; S)$. Since $S$ contains only 3 positive integers, none of the functions from $D_k$, $k \geqslant 4$, into $S$ can be one-to-one. Hence, to compute $\mathrm{DP}(n; S)$, we only have to compute the following with $Z_{A_1}(x_1) - Z_{S_1}(x_1)$ being defined to be $x_1$:

$$\begin{aligned}
&\sum_{k=1}^{3}\left(Z_{A_k}\left(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \dots, \sum_{i\in S} x^{ki}\right) - Z_{S_k}\left(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \dots, \sum_{i\in S} x^{ki}\right)\right) \\
&\quad = (x + x^2 + x^4) + (x + x^2 + x^4)^2 - \tfrac{1}{2}\big((x + x^2 + x^4)^2 + (x^2 + x^4 + x^8)\big) \\
&\qquad + \tfrac{1}{3}\big((x + x^2 + x^4)^3 + 2(x^3 + x^6 + x^{12})\big) \\
&\qquad - \tfrac{1}{6}\big((x + x^2 + x^4)^3 + 3(x + x^2 + x^4)(x^2 + x^4 + x^8) + 2(x^3 + x^6 + x^{12})\big) \\
&\quad = (x + x^2 + x^4) + (x^3 + x^5 + x^6) + x^7 = x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7.
\end{aligned}$$

Thus, the distinct partitions of $n$ into $S = \{1, 2, 4\}$ are 1, 2, 4, $1+2$, $1+4$, $2+4$ and $1 + 2 + 4$.

## 5. Algorithms

(a) An algorithm for computing $\mathrm{P}(n; S)$ where $n$ is a positive integer and $S$ is a non-empty subset of positive integers.

Based on the recursive formula for $\mathrm{P}(n; S)$ from Corollary 1.1, an algorithm for computing $\mathrm{P}(n; S)$ can be given as follows:

- **Step 1:** Determine $\mathrm{P}(1; S)$, $\mathrm{P}(1; S) = 1$ if $1 \in S$; otherwise, $\mathrm{P}(1; S) = 0$. This is the base case of the algorithm.
- **Step 2:** Compute the sum of factors of a positive integer $k \leqslant n$. The factors should be in $S$. Use SumOfFactors $(k, S)$ to denote the sum.
- **Step 3:** Recursively compute $\mathrm{P}(n; S)$ using the formula. An implementation of the algorithm is described as follows:

Input $n$ and $S$;
If ($n == 1$ and $1 \in S$) then
  $\mathrm{P}(n; S) := 1$;
If ($n == 1$ and $1 \notin S$) then
  $\mathrm{P}(n; S) := 0$;
Sum := 0;
For $k := 1$ to $n - 1$ do
  Begin
  Sum := Sum + SumOfFactors($k, S$) $*$ $\mathrm{P}(n - k; S)$;
  End
Sum := Sum + SumOfFactors($n; S$);
$\mathrm{P}(n; S) :=$ Sum$/n$;

Using the algorithm we can obtain $\mathrm{P}(n; I)$ and $\mathrm{P}(n; O)$ for all positive integers $n$ where $I$ is the set of all positive integers and $O$ is the set of all positive odd integers. The following is a table of $\mathrm{P}(n; I)$ and $\mathrm{P}(n; O)$ for $n = 1, 2, \ldots, 20$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathrm{P}(n; I)$ | 1 | 2 | 3 | 5 | 7 | 11 | 15 | 22 | 30 | 42 | 56 | 77 | 101 | 135 | 176 | 231 | 297 | 385 | 490 | 627 |
| $\mathrm{P}(n; O)$ | 1 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 8 | 10 | 12 | 15 | 18 | 22 | 27 | 32 | 38 | 46 | 54 | 64 |

(b) An algorithm for computing $\mathrm{DP}(n; S)$ where $n$ is a positive integer and $S$ is a non-empty subset of positive integers.

Based on the recursive formula for $\mathrm{DP}(n; S)$ from Corollary 2.1, an algorithm for computing $\mathrm{DP}(n; S)$ can be given as follows:

**Step 1:** Determine $\mathrm{DP}(1; S)$. $\mathrm{DP}(1; S) = 1$ if $1 \in S$; otherwise, $\mathrm{DP}(1; S) = 0$. This is the base case of the algorithm.

**Step 2:** Compute the sum of signed factors of a positive integer $k \leqslant n$. The factors should be in $S$. The sign of a factor $i$ is $+$ (or $-$) if $k/i + 1$ is even (or odd). Use SumOfSignedFactors($k, S$) to denote the sum.

**Step 3:** Recursively compute $\mathrm{DP}(n; S)$ using the formula. An implementation of the algorithm is described as follows:

Input $n$ and $S$;
If ($n == 1$ and $1 \in S$) then
  $\mathrm{DP}(n; S) := 1$;
If ($n == 1$ and $\notin S$) then
  $\mathrm{DP}(n; S) := 0$;
Sum:= 0;
For $k := 1$ to $n - 1$ do

Begin

Sum := Sum + SumOfSignedFactors($k, S$) to denote the sum.

End

Sum := Sum + SumOfSignedFactors($n; S$);

DP($n; S$) := Sum/$n$;

Using the algorithm we can obtain DP($n; I$) and DP($n; O$) for all positive integers $n$ where $I$ is the set of all positive integers and $O$ is the set of all positive odd integers. The following table shows DP($n; I$) and DP($n; O$) for $n = 1, 2, \ldots, 20$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| DP($n; I$) | 1 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 8 | 10 | 12 | 15 | 18 | 22 | 27 | 32 | 38 | 46 | 54 | 64 |
| DP($n; O$) | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 | 5 | 6 | 7 |

(c) Let $E_i$ be the set of all positive integers each of which is not a multiple of the positive integer $i$ for $i = 3, 4, 5, 6$. Using the first algorithm, we can obtain P($n; E_i$) for all positive integers $n$ and for $i = 3, 4, 5, 6$. The following table gives (P($n; i$) for $n = 1, 2, \ldots, 20$ and $i = 3, 4, 5, 6$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| P($n; E_3$) | 1 | 2 | 2 | 4 | 5 | 7 | 9 | 13 | 16 | 22 | 27 | 36 | 44 | 57 | 70 | 89 | 108 | 135 | 163 | 202 |
| P($n; E_4$) | 1 | 2 | 3 | 4 | 6 | 9 | 12 | 16 | 22 | 29 | 38 | 50 | 64 | 82 | 105 | 132 | 166 | 208 | 258 | 320 |
| P($n; E_5$) | 1 | 2 | 3 | 5 | 6 | 10 | 13 | 19 | 25 | 34 | 44 | 60 | 76 | 100 | 127 | 164 | 205 | 262 | 325 | 409 |
| P($n; E_6$) | 1 | 2 | 3 | 5 | 7 | 10 | 14 | 20 | 27 | 39 | 49 | 65 | 85 | 111 | 143 | 184 | 234 | 297 | 374 | 470 |

Using the second algorithm, we have the following table.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| DP($n; E_3$) | 1 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 14 | 16 | 18 |
| DP($n; E_4$) | 1 | 1 | 2 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 11 | 13 | 16 | 18 | 21 | 24 | 27 |
| DP($n; E_5$) | 1 | 1 | 2 | 2 | 2 | 3 | 4 | 4 | 6 | 7 | 8 | 10 | 12 | 14 | 16 | 19 | 22 | 26 | 30 | 35 |
| DP($n; E_6$) | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 6 | 8 | 9 | 11 | 13 | 16 | 19 | 22 | 26 | 30 | 35 | 41 |

(d) Using both algorithms, we have the following table for $S = \{1, 2, 4\}$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| P($n; S$) | 1 | 2 | 2 | 4 | 4 | 6 | 6 | 9 | 9 | 12 | 12 | 16 | 16 | 20 | 20 | 25 | 25 | 30 | 30 | 36 |
| DP($n; S$) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

(e) Let $F$ be the set of all positive integers each of which is not a multiple of 3 or 4. Let $G$ be the set of all positive integers each of which is not a multiple of 3, 4 or 5. Let $H$ be the set of all positive integers each of which is not a multiple of 3, 4, 5 or 6. Using both algorithms, we have the following table:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathrm{P}(n;F)$ | 1 | 2 | 2 | 3 | 4 | 5 | 7 | 8 | 10 | 13 | 16 | 20 | 24 | 30 | 36 | 43 | 52 | 61 | 73 | 86 |
| $\mathrm{DP}(n;F)$ | 1 | 1 | 1 | 0 | 1 | 1 | 2 | 2 | 1 | 2 | 2 | 3 | 4 | 4 | 4 | 4 | 5 | 6 | 7 | 7 |
| $\mathrm{P}(n;G)$ | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 | 10 | 11 | 14 | 17 | 20 | 23 | 27 | 31 | 36 | 41 |
| $\mathrm{DP}(n;G)$ | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 3 | 2 | 2 | 2 | 2 | 3 | 4 |
| $\mathrm{P}(n;H)$ | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 | 10 | 11 | 14 | 17 | 20 | 23 | 27 | 31 | 36 | 41 |
| $\mathrm{DP}(n;H)$ | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 3 | 2 | 2 | 2 | 2 | 3 | 4 |


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