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SINGULAR POSITONE AND SEMIPOSITONE BOUNDARY VALUE PROBLEMS OF SECOND ORDER DELAY DIFFERENTIAL EQUATIONS

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Abstract. In this paper we present some new existence results for singular positone and semipositone boundary value problems of second order delay differential equations. Throughout our nonlinearity may be singular in its dependent variable.

Keywords: existence, positone problem, semipositone problem, singular delay differential equations

MSC 2000: 34B15

1. Introduction

This paper discusses the existence of nonnegative solutions for singular positone and semipositone boundary value problems of second order delay differential equations. In particular our nonlinear term $f(\cdot, y)$ may be singular at $y = 0$. In Section 2 we present some very general results for the existence of multiple solutions to positone problems (i.e. problems where f takes nonnegative values). In Section 3 we present a new result for the existence of one solution to semipositone problems (i.e. problems where f may take on negative values). Almost all papers in the literature [3], [6], [7], [8], [10] discuss the existence of one solution for singular and nonsingular positone problems of second order delay differential equations, and only recently (see for example [4], [9]) have papers appeared which discuss the semipositone nonsingular problems for ordinary differential equations. Very recently, R. P. Agarwal and D. O'Regan [1] discussed the semipositone singular problems for ordinary differential equations. For example in [1] they showed that the boundary value problem

$$\begin{cases} y'' + \mu(y^{-\alpha} + y^{\beta} - 1) = 0, & 0 < t < 1, \\ y(0) = y(1) = 0, & \alpha > 0,\ \beta > 1,\ \mu > 0 \text{ small}, \end{cases}$$

has a nonnegative solution $y \in C[0,1] \cap C^2(0,1)$ with $y(t) > 0$ for $t \in (0,1)$. (Existence is established in [1] by using a general cone fixed point theorem in [2], [5].) However no paper to date has discussed semipositone singular problems of delay differential equations. This paper attempts to fill this gap in the literature.

Some very general existence theorems (for positone problems) will be presented in Section 2 and there we will show, for example, that the boundary value problem

$$\begin{cases} y''(t) + \sigma(y^{-\alpha}(t-\tau) + y^{\beta}(t-\tau)) = 0, & t \in (0,1) \setminus \{\tau\}, \\ y(t) = (-t)^m, & -\tau \leqslant t \leqslant 0,\ 0 < m \leqslant 1, \\ y(1) = 0, & 0 < \alpha < 1 < \beta,\ 0 < \tau < 1,\ \sigma > 0 \text{ small}, \end{cases}$$

has two nonnegative solutions. Also a new existence theorem (for semipositone problems) will be presented in Section 3 and there we will show, for example, that the boundary value problem

$$\begin{cases} y''(t) + \mu(y^{-\alpha}(t-\tau) + y^{\beta}(t-\tau) - 1) = 0, & t \in (0,1) \setminus \{\tau\}, \\ y(t) = (-t)^m, & -\tau \leqslant t \leqslant 0,\ 0 < m \leqslant 1, \\ y(1) = 0, & 0 < \alpha < 1 < \beta,\ 0 < \tau < 1,\ \mu > 0 \text{ small}, \end{cases}$$

has one nonnegative solution.

Existence in this paper will be established using Krasnoselskii's fixed point theorem in a cone [5], which we state here for the convenience of the reader.

Theorem 1.1. *Let $E = (E, \|\cdot\|)$ be a Banach space and let $K \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$A\colon\ K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|Ay\| \leqslant \|y\|\ \forall\, y \in K \cap \partial\Omega_1$ *and* $\|Ay\| \geqslant \|y\|\ \forall\, y \in K \cap \partial\Omega_2$, *or*

(ii) $\|Ay\| \geqslant \|y\|\ \forall\, y \in K \cap \partial\Omega_1$ *and* $\|Ay\| \leqslant \|y\|\ \forall\, y \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. Singular positone problems

In this section we present some very general results for the singular problem

$$\begin{cases} y''(t) + q(t)f(t, y(t-\tau)) = 0, & t \in (0,1) \setminus \{\tau\}, \\ y(t) = \xi(t), & -\tau \leqslant t \leqslant 0, \\ y(1) = 0; \end{cases} \tag{2.1}$$

where $0 < \tau < 1$ is positive constant. Our nonlinearity $f(t, y)$ may be singular at $y = 0$.

Using Theorem 1.1 we establish the following main result.

Theorem 2.1. *Suppose the following conditions are satisfied:*

$$\xi \in C[-\tau, 0], \quad \xi(t) > 0 \text{ on } [-\tau, 0) \text{ and } \xi(0) = 0, \tag{2.2}$$

$$q \in C(0,1) \cap L^1[0,1] \text{ with } q > 0 \text{ on } (0,1), \tag{2.3}$$

$$f\colon\ [0,1] \times (0,\infty) \to (0,\infty) \text{ is continuous,} \tag{2.4}$$

$$\begin{cases} f(t,u) \leqslant g(u) + h(u) \text{ on } [0,1] \times (0,\infty) \text{ with } g > 0 \\ \text{continuous and nonincreasing on } (0,\infty),\ h \geqslant 0 \\ \text{continuous on } [0,\infty) \text{ and } h/g \text{ nondecreasing on } (0,\infty), \end{cases} \tag{2.5}$$

$$\exists\, K_0 \text{ with } g(ab) \leqslant K_0 g(a) g(b)\ \forall\, a > 0,\ b > 0, \tag{2.6}$$

$$a_0 = \int_\tau^1 s(1-s)q(s)g((s-\tau)(1+\tau-s))\,\mathrm{d}s < \infty, \tag{2.7}$$

$$b_0 = \int_0^\tau s(1-s)q(s)f(s, \xi(s-\tau))\,\mathrm{d}s < \infty, \tag{2.8}$$

$$\exists\, r > b_0 \text{ with } \frac{r - b_0}{g(r) + h(r)} > K_0 a_0, \tag{2.9}$$

$$\begin{cases} \text{there exists } 0 < a < \tfrac{1}{2}(1-\tau) \text{ (choose and fix it) and a continuous,} \\ \text{nonincreasing function } g_1\colon\ (0,\infty) \to (0,\infty), \text{ and a continuous} \\ \text{function } h_1\colon\ [0,\infty) \to (0,\infty) \text{ with } h_1/g_1 \text{ nondecreasing on } (0,\infty) \\ \text{and with } f(t,u) \geqslant g_1(u) + h_1(u) \text{ for } (t,u) \in [\tau + a, 1-a] \times (0,\infty), \end{cases} \tag{2.10}$$

$$\begin{cases} \exists\, 0 < R_1 < r < R_2 \text{ with } (i = 1,2) \\ \dfrac{R_i g_1(a(a+\tau)R_i)}{g_1(R_i) g_1(a(a+\tau)R_i) + g_1(R_i) h_1(a(a+\tau)R_i)} < \displaystyle\int_{\tau+a}^{1-a} G(\sigma, s) q(s)\,\mathrm{d}s; \end{cases} \tag{2.11}$$

here $G(t,s)$ is the Green's function for

$$\begin{cases} y'' = 0 \quad \text{on } (0,1), \\ y(0) = y(1) = 0, \end{cases}$$

and $0 \leqslant \sigma \leqslant 1$ *is such that*

$$\int_{\tau+a}^{1-a} q(s)G(\sigma,s)\,\mathrm{d}s = \sup_{t\in[0,1]} \int_{\tau+a}^{1-a} q(s)G(t,s)\,\mathrm{d}s.$$

Then (2.1) *has two nonnegative solutions* $y_i \in C[-\tau,1]\cap C^2((0,1)\setminus\{\tau\})$ *with* $y_i(t) > 0$ *for* $t \in (0,1)$, $i = 1,2$.

P r o o f. To show (2.1) has two nonnegative solutions we will look at the boundary value problem

$$\begin{cases} y''(t) + q(t)f(t, y(t-\tau) + \eta(t-\tau)) = 0, & t \in (0,1)\setminus\{\tau\},\\ y(t) = 0, \quad -\tau \leqslant t \leqslant 0,\\ y(1) = 0 \end{cases} \tag{2.12}$$

where

$$\eta(t) = \begin{cases} 0, & 0 \leqslant t \leqslant 1,\\ \xi(t), & -\tau \leqslant t \leqslant 0. \end{cases}$$

We will show, using Theorem 1.1, that there exists two solutions y_i $(i = 1,2)$ to (2.12) with $y_i(t) > 0$ for $t \in (0,1)$ and $y_i(t) = 0$ for $t \in [-\tau, 0]$. If this is true then $u_i(t) = y_i(t) + \eta(t)$, $-\tau \leqslant t \leqslant 1$ are nonnegative solutions (positive on $(0,1)\cup[-\tau,0)$) of (2.1). As a result we will concentrate our study on (2.12).

Let

$$E = \{u \in C[-\tau,1]\colon\ u(t) = 0 \text{ as } t \in [-\tau,0],\ u(1) = 0\}$$

with the norm $\|u\| := \sup\{|u(t)|\colon\ -\tau \leqslant t \leqslant 1\}$ (note E is a Banach space). Now $\|u\| = \|u\|_{[0,1]}$ for $u \in E$, where $\|u\|_{[0,1]} = \sup\limits_{t\in[0,1]} |u(t)|$.

Let K be a cone in E defined by

$$K = \{u \in E;\ u(t) \geqslant t(1-t)\|u\|,\ t \in [0,1]\}.$$

First we will show that there exists a solution y_2 to (2.12) with $y_2(t) > 0$ for $t \in (0,1)$ and $r < \|y_2\| < R_2$. Let

$$\Omega_1 = \{u \in E;\ \|u\| < r\}, \quad \Omega_2 = \{u \in E;\ \|u\| < R_2\}.$$

Next let $A\colon K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to E$ be defined by

$$(Ay)(t) = \begin{cases} \displaystyle\int_0^1 G(t,s)q(s)f(s, y(s-\tau) + \eta(s-\tau))\,\mathrm{d}s, & 0 \leqslant t \leqslant 1,\\ 0, \quad -\tau \leqslant t \leqslant 0, \end{cases}$$

with the Green's function

$$G(t,s)=\begin{cases}(1-t)s, & 0\leqslant s\leqslant t\leqslant 1,\\ (1-s)t, & 0\leqslant t\leqslant s\leqslant 1.\end{cases}$$

One can see that

$$t(1-t)s(1-s)\leqslant G(t,s)\leqslant G(s,s)=s(1-s),\quad (t,s)\in[0,1]\times[0,1].$$

First we show A is well defined. To see this notice that if $y\in K\cap(\overline{\Omega}_2\setminus\Omega_1)$ then $r\leqslant\|y\|\leqslant R_2$ and $y(t)\geqslant t(1-t)\|y\|\geqslant t(1-t)r$, $0\leqslant t\leqslant 1$ and so $y(x-\tau)\geqslant(x-\tau)(1+\tau-x)r$, $x\in[\tau,1]$. Also notice that

$$f(x,y(x-\tau)+\eta(x-\tau))=f(x,\xi(x-\tau)),\quad \text{for } x\in(0,\tau)$$

and

$$\begin{aligned}f(x,y(x-\tau)+\eta(x-\tau))&=f(x,y(x-\tau))\\ &\leqslant g(y(x-\tau))+h(y(x-\tau))=g(y(x-\tau))\Big\{1+\frac{h(y(x-\tau))}{g(y(x-\tau))}\Big\}\\ &\leqslant g((x-\tau)(1+\tau-x)r)\Big\{1+\frac{h(R_2)}{g(R_2)}\Big\}\\ &\leqslant K_0g((x-\tau)(1+\tau-x))g(r)\Big\{1+\frac{h(R_2)}{g(R_2)}\Big\},\quad \text{for } x\in(\tau,1).\end{aligned}$$

These inequalities with (2.7) and (2.8) guarantee that $A\colon K\cap(\overline{\Omega}_2\setminus\Omega_1)\to E$ is well defined. Next we show that $A\colon K\cap(\overline{\Omega}_2\setminus\Omega_1)\to K$. If $y\in K\cap(\overline{\Omega}_2\setminus\Omega_1)$, then we have

$$\begin{cases}\|Ay\|_{[0,1]}\leqslant\displaystyle\int_0^1 s(1-s)q(s)f(s,y(s-\tau)+\eta(s-\tau))\,\mathrm{d}s,\\ (Ay)(t)\geqslant t(1-t)\displaystyle\int_0^1 s(1-s)q(s)f(s,y(s-\tau)+\eta(s-\tau))\,\mathrm{d}s\\ \qquad\qquad\geqslant t(1-t)\|Ay\|_{[0,1]}=t(1-t)\|Ay\|,\quad t\in[0,1],\end{cases}$$

i.e., $Ay\in K$ so $A\colon K\cap(\overline{\Omega}_2\setminus\Omega_1)\to K$. Now we show that $A\colon K\cap(\overline{\Omega}_2\setminus\Omega_1)\to K$ is continuous and compact. Let $y_n,y_0\in K\cap(\overline{\Omega}_2\setminus\Omega_1)$ with $\|y_n-y_0\|\to 0$ as $n\to\infty$. Of course $r\leqslant\|y_n\|=\|y_n\|_{[0,1]}\leqslant R_2$, $r\leqslant\|y_0\|=\|y_0\|_{[0,1]}\leqslant R_2$, $y_n(t)\geqslant t(1-t)r$, for $0 \leqslant t \leqslant 1$, and $y_n(x-\tau) \geqslant (x-\tau)(1+\tau-x)r$, for $x \in [\tau, 1]$. Notice also that

$$\begin{aligned}\varrho_n(x) &= |f(x, y_n(x-\tau)+\eta(x-\tau)) - f(x, y_0(x-\tau)+\eta(x-\tau))| \\ &= |f(x, \xi(x-\tau)) - f(x, \xi(x-\tau))| = 0, \quad \text{for } x \in (0, \tau),\end{aligned}$$

$$\begin{aligned}\varrho_n(x) &= |f(x, y_n(x-\tau)+\eta(x-\tau)) - f(x, y_0(x-\tau)+\eta(x-\tau))| \\ &= |f(x, y_n(x-\tau)) - f(x, y_0(x-\tau))| \to 0, \quad \text{as } n \to \infty,\ x \in (\tau, 1)\end{aligned}$$

and

$$\varrho_n(x) \leqslant 2K_0\Big\{1 + \frac{h(R_2)}{g(R_2)}\Big\} g(r) g((x-\tau)(1+\tau-x)) \quad \text{for } x \in (\tau, 1).$$

Now these together with the Lebesgue dominated convergence theorem guarantee that

$$\begin{aligned}\|Ay_n - Ay_0\| &= \|Ay_n - Ay_0\|_{[0,1]} \\ &\leqslant \sup_{t \in [0,1]} \int_0^1 G(t,s) q(s) \varrho_n(s)\,\mathrm{d}s \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

so $A\colon\ K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is continuous. Also for $y \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ we have

$$\begin{aligned}\|Ay\| &\leqslant b_0 + \int_\tau^1 s(1-s) q(s) K_0 g((s-\tau)(1+\tau-s)) g(r) \Big\{1 + \frac{h(R_2)}{g(R_2)}\Big\}\,\mathrm{d}s \\ &= b_0 + a_0 K_0 g(r) \Big\{1 + \frac{h(R_2)}{g(R_2)}\Big\},\end{aligned}$$

and for $t, t' \in [0,1]$ we have

$$\begin{aligned}&|(Ay)(t) - (Ay)(t')| \\ &\quad\leqslant \int_0^\tau |G(t,s) - G(t',s)| q(s) f(s, \xi(s-\tau))\,\mathrm{d}s \\ &\qquad + K_0 g(r) \Big\{1 + \frac{h(R_2)}{g(R_2)}\Big\} \int_\tau^1 |G(t,s) - G(t',s)| q(s) g((s-\tau)(1+\tau-s))\,\mathrm{d}s.\end{aligned}$$

Since $(Ay)(t) = 0$, for $t \in [-\tau, 0]$, the Arzela-Ascoli Theorem guarantees that $A\colon$ $K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is compact.

We now show that

$$\|Ay\| < \|y\| \quad \text{for } K \cap \partial\Omega_1. \tag{2.14}$$

To see this, let $y \in K \cap \partial\Omega_1$. Then $\|y\| = \|y\|_{[0,1]} = r$ and $y(t) \geqslant t(1-t)r$ for $t \in [0,1]$, $y(x-\tau) \geqslant (x-\tau)(1+\tau-x)r$ for $x \in [\tau,1]$. So for $t \in (0,1)$ we have

$$\begin{aligned}(Ay)(t) &\leqslant b_0 + \int_\tau^1 s(1-s)q(s)[g(y(s-\tau)) + h(y(s-\tau)]\,\mathrm{d}s \\ &\leqslant b_0 + K_0 g(r)\left\{1 + \frac{h(r)}{g(r)}\right\}\int_\tau^1 s(1-s)q(s)g((s-\tau)(1+\tau-s))\,\mathrm{d}s \\ &= b_0 + a_0 K_0[g(r) + h(r)].\end{aligned}$$

This together with (2.9) yields $\|Ay\| = \|Ay\|_{[0,1]} < r = \|y\|$, so (2.14) is satisfied.

Next we show that

$$\|Ay\| > \|y\| \quad \text{for } K \cap \partial\Omega_2. \tag{2.15}$$

To see this let $y \in K \cap \partial\Omega_2$ so $\|y\| = \|y\|_{[0,1]} = R_2$ and $y(t) \geqslant t(1-t)R_2$ for $t \in [0,1]$, $y(x-\tau) \geqslant (x-\tau)(1+\tau-x)R_2$ for $x \in [\tau,1]$. Moreover, $y(x-\tau) \geqslant a(a+\tau)R_2$ for $x \in [\tau+a, 1-a]$, since $a \in \left(0, \frac{1-\tau}{2}\right)$.

Now with σ as in the statement of Theorem 2.1, we have

$$\begin{aligned}(Ay)(\sigma) &\geqslant \int_{\tau+a}^{1-a} G(\sigma,s)q(s)[g_1(y(s-\tau)) + h_1(y(s-\tau)]\,\mathrm{d}s \\ &\geqslant g_1(R_2)\int_{\tau+a}^{1-a} G(\sigma,s)q(s)\left\{1 + \frac{h_1(a(a+\tau)R_2)}{g_1(a(a+\tau)R_2)}\right\}\mathrm{d}s.\end{aligned}$$

This together with (2.11) yields that $\|Ay\| > R_2 = \|y\|$, so (2.15) holds.

Now Theorem 1.1 implies A has a fixed point $y_2 \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, i.e. $r \leqslant \|y_2\| = \|y_2\|_{[0,1]} \leqslant R$ and $y_2(t) \geqslant t(1-t)r$ for $t \in [0,1]$. It follows from (2.14) and (2.15) that $\|y_2\| \neq r$, $\|y_2\| \neq R_2$, so we have $r < \|y_2\| < R_2$.

Similarly, if we put

$$\Omega_1 = \{u \in E;\ \|u\| < R_1\}, \quad \Omega_2 = \{u \in E;\ \|u\| < r\},$$

we can show that there exists a solution y_1 to (2.12) with $y_1(t) > 0$ for $t \in (0,1)$ and $R_1 < \|y_1\| < r$.

This completes the proof of Theorem 2.1. □

The following result can be extracted from the proof of Theorem 2.1.

Theorem 2.2. *Suppose* (2.2)–(2.10) *hold. In addition suppose that*

$$(2.16)\qquad \begin{cases} \exists\, 0 < R_1 < r \text{ with} \\ \dfrac{R_1 g_1(a(a+\tau)R_1)}{g_1(R_1)g_1(a(a+\tau)R_1) + g_1(R_1)h_1(a(a+\tau)R_1)} < \displaystyle\int_{\tau+a}^{1-a} G(\sigma,s)q(s)\,\mathrm{d}s; \end{cases}$$

here σ is as in Theorem 2.1. Then (2.1) *has a nonnegative solution $y_1 \in C[-\tau,1] \cap C^2((0,1)\setminus\{\tau\})$ with $y_1(t) > 0$ for $t \in (0,1)$.*

Remark 2.1. If in (2.16) we have $R_1 > r$ then (2.1) has a nonnegative solution $y_2 \in C[-\tau,1] \cap C^2((0,1)\setminus\{\tau\})$ with $y_2(t) > 0$ for $t \in (0,1)$.

It is easy to use Theorem 2.2 and Remark 2.1 to write theorems which guarantee the existence of more than two solutions to (2.1). We state one such result.

Theorem 2.3. *Suppose* (2.2)–(2.8) *and* (2.10) *hold. Assume that $\exists\, m \in \{1,2,\ldots\}$ and constants R_i, r_i $(i = 1,\ldots,m)$, with $r_1 > b_0$, and*

$$0 < R_1 < r_1 < R_2 < r_2 < \ldots < R_m < r_m.$$

In addition suppose for each $i = 1,\ldots,m$ that

$$(2.17)\qquad \frac{r_i - b_0}{g(r_i) + h(r_i)} > K_0 a_0$$

and

$$(2.18)\qquad \frac{R_i g_1(a(a+\tau)R_i)}{g_1(R_i)g_1(a(a+\tau)R_i) + g_1(R_i)h_1(a(a+\tau)R_i)} < \int_{\tau+a}^{1-a} G(\sigma,s)q(s)\,\mathrm{d}s$$

hold. Then (2.1) *has nonnegative solutions $y_1,\ldots,y_m \in C[-\tau,1] \cap C^2((0,1)\setminus\{\tau\})$ with $y_i(t) > 0$ for $t \in (0,1)$.*

Example. Consider the boundary value problem

$$(2.19)\qquad \begin{cases} y''(t) + \sigma(y^{-\alpha}(t-\tau) + y^{\beta}(t-\tau)) = 0, \quad t \in (0,1)\setminus\{\tau\}, \\ y(t) = (-t)^m, \quad -\tau \leqslant t \leqslant 0,\ 0 < m \leqslant 1, \\ y(1) = 0, \quad 0 < \alpha < 1 < \beta,\ 0 < \tau < 1 \end{cases}$$

where $\sigma \in (0,\sigma_0)$ is such that

$$\sigma_0 \leqslant \frac{1}{2a_1 + b_1};$$

here

$$a_1 = \int_\tau^1 s(1-s)(s-\tau)^{-\alpha}(1+\tau-s)^{-\alpha}\,\mathrm{d}s < \infty,$$

$$b_1 = \int_0^\tau s(1-s)[(\tau-s)^{-m\alpha} + (\tau-s)^{m\beta}]\,\mathrm{d}s < \infty.$$

Then (2.19) has two solutions y_1, y_2 with $y_1(t) > 0$, $y_2(t) > 0$ for $t \in (0,1)$, $i = 1,2$.

To see this we will apply Theorem 2.1 with (here $0 < R_1 < 1 < R_2$ will be chosen below)

$$g(y) = g_1(y) = y^{-\alpha}, \quad h(y) = h_1(y) = y^{\beta}, \quad q(t) = \sigma,$$
$$\xi(t) = (-t)^m, \quad K_0 = 1, \quad a = \frac{1-\tau}{4}.$$

Clearly (2.2)–(2.8) and (2.10) hold, and $a_0 = \sigma a_1$, $b_0 = \sigma b_1$. Now (2.9) holds with $r = 1$ since

$$\frac{r - b_0}{g(r) + h(r)} = \frac{1 - b_1\sigma}{2} > \frac{1 - b_1\sigma_0}{2} \geqslant a_1\sigma_0 > K_0 a_0.$$

Finally notice that (2.11) is satisfied for R_1 small and R_2 large since

$$\frac{R_i}{g_1(R_i)\{1 + \frac{h_1(a(a+\tau)R_i)}{g_1(a(a+\tau)R_i)}\}} = \frac{R_i^{1+\alpha}}{1 + a^{\alpha+\beta}(a+\tau)^{\alpha+\beta}R_i^{\alpha+\beta}} \to 0,$$

as $R_1 \to 0$, $R_2 \to \infty$, since $\beta > 1$. Thus all the conditions of Theorem 2.1 are satisfied so the existence is guaranteed.

3. Singular semipositone problems

In this section we present a new result for the semipositone singular problem

$$\text{(3.1)} \qquad \begin{cases} y''(t) + \mu q(t) f(t, y(t-\tau)) = 0, & t \in (0,1) \setminus \{\tau\}, \\ y(t) = \xi(t), & -\tau \leqslant t \leqslant 0, \\ y(1) = 0; \end{cases}$$

here $\mu > 0$ and $0 < \tau < 1$ are positive constants. Our nonlinearity $f(t, y)$ may be singular at $y = 0$.

Before we prove our main result we first state a result from [1].

Lemma 3.1 ([1]). *Suppose $q \in L^1[0,1]$ with $q > 0$ on $(0,1)$. Then the boundary value problem*

$$\begin{cases} y'' + q(t) = 0, & 0 < t < 1, \\ y(0) = 0, & y(1) = 0; \end{cases}$$

has a solution w with

$$w(t) \leqslant t(1-t)C_0 \quad \text{for } t \in [0,1];$$

here

$$C_0 = \max_{t\in[0,1]} \left\{ \frac{1}{1-t} \int_t^1 (1-x)q(x)\,\mathrm{d}x + \frac{1}{t}\int_0^t xq(x)\,\mathrm{d}x \right\}.$$

The above Lemma together with Theorem 1.1 establish our main result.

Theorem 3.1. *Suppose the following conditions are satisfied:*

$$\xi \in C[-\tau,0], \quad \xi(t)>0 \text{ on } [-\tau,0) \text{ and } \xi(0)=0, \tag{3.2}$$

$$q \in C(0,1)\cap L^1[0,1] \quad \text{with } q>0 \text{ on } (0,1), \tag{3.3}$$

$$\begin{cases} f\colon\ [0,1]\times(0,\infty)\to\mathbb{R} \text{ is continuous and there exists} \\ \text{a constant } M>0 \text{ with } f(u)+M\geqslant 0 \\ \text{for } (t,u)\in[0,1]\times(0,\infty), \end{cases} \tag{3.4}$$

$$\begin{cases} f^*(t,u)=f(t,u)+M\leqslant g(u)+h(u) \text{ on } [0,1]\times(0,\infty) \text{ with } g>0 \\ \text{continuous and nonincreasing on } (0,\infty),\ h\geqslant 0 \\ \text{continuous on } [0,\infty) \text{ and } h/g \text{ nondecreasing on } (0,\infty), \end{cases} \tag{3.5}$$

$$\exists\, K_0 \text{ with } g(ab)\leqslant K_0 g(a)g(b)\ \forall\, a>0,\ b>0, \tag{3.6}$$

$$a_0=\int_\tau^1 s(1-s)q(s)g((s-\tau)(1+\tau-s))\,\mathrm{d}s<\infty, \tag{3.7}$$

$$b_0=\int_0^\tau s(1-s)q(s)f^*(s,\xi(s-\tau))\,\mathrm{d}s<\infty, \tag{3.8}$$

$$\exists\, r>\max\{\mu MC_0,\mu b_0\} \text{ with } \frac{r-\mu b_0}{g(r-\mu MC_0)\{1+h(r)/g(r)\}}\geqslant \mu K_0 a_0, \tag{3.9}$$

$$\begin{cases} \text{there exists } 0<a<\frac{1}{2}(1-\tau) \text{ (choose and fix it) and a continuous,} \\ \text{nonincreasing function } g_1\colon\ (0,\infty)\to(0,\infty), \text{ and a continuous} \\ \text{function } h_1\colon\ [0,\infty)\to(0,\infty) \text{ with } h_1/g_1 \text{ nondecreasing on } (0,\infty) \\ \text{and with} f(t,u)+M\geqslant g_1(u)+h_1(u) \text{ for} \\ (t,u)\in[\tau+a,1-a]\times(0,\infty) \end{cases} \tag{3.10}$$

and $\exists\, R>r$ *with*

$$\frac{Rg_1(\varepsilon a(a+\tau)R)}{g_1(R)g_1(\varepsilon a(a+\tau)R)+g_1(R)h_1(\varepsilon a(a+\tau)R)}\leqslant \mu\int_{\tau+a}^{1-a} G(\sigma,s)q(s)\,\mathrm{d}s; \tag{3.11}$$

here $\varepsilon>0$ *is any constant (choose and fix it) so that* $1-\mu MC_0/R\geqslant\varepsilon$ *(note* ε *exists since* $R>r>\mu MC_0$*) and* $G(t,s)$ *is the Green's function for*

$$\begin{cases} y''=0 \quad \text{on } (0,1) \\ y(0)=y(1)=0, \end{cases}$$

and $0\leqslant\sigma\leqslant 1$ *is such that*

$$\int_{\tau+a}^{1-a} q(s)G(\sigma,s)\,\mathrm{d}s=\sup_{t\in[0,1]}\int_{\tau+a}^{1-a} q(s)G(t,s)\,\mathrm{d}s.$$

Then (3.1) *has a solution* $y\in C[-\tau,1]\cap C^2((0,1)\setminus\{\tau\})$ *with* $y(t)>0$ *for* $t\in(0,1)$.

P r o o f. To show that (3.1) has a nonnegative solution we will look at the boundary value problem

$$\text{(3.12)}\qquad \begin{cases} y''(t)+\mu q(t)f^*(t,y(t-\tau)-\varphi(t-\tau))=0, & t\in(0,1)\setminus\{\tau\},\\ y(t)=0, \quad -\tau\leqslant t\leqslant 0,\\ y(1)=0 \end{cases}$$

where

$$\text{(3.13)}\qquad \varphi(t)=\begin{cases} \mu M w(t), & 0\leqslant t\leqslant 1,\\ -\xi(t), & -\tau\leqslant t\leqslant 0 \end{cases}$$

(w is as in Lemma 3.1).

We will show, using Theorem 1.1, that there exists a solution y_1 to (3.12) with $y_1(t)>\varphi(t)$ for $t\in(0,1)$ and $y_1(t)=0$ for $t\in[-\tau,0]$. If this is true then $u(t)=y_1(t)-\varphi(t)$, $-\tau\leqslant t\leqslant 1$ is a nonnegative solution (positive on (0,1)) of (3.1), since $u(t)=\xi(t)$ for $-\tau\leqslant t\leqslant 0$ and

$$\begin{aligned} u''(t)=y_1''(t)-\varphi''(t)&=-\mu q(t)f^*(t,y_1(t-\tau)-\varphi(t-\tau))+\mu Mq(t)\\ &=-\mu q(t)[f(t,y_1(t-\tau)-\varphi(t-\tau))+M]+\mu Mq(t)\\ &=-\mu q(t)f(t,y_1(t-\tau)-\varphi(t-\tau))\\ &=-\mu q(t)f(t,u(t-\tau)), \quad 0<t<1. \end{aligned}$$

As a result, we will concentrate our study on (3.12). Let E, K be as in Section 2, and let

$$\Omega_1=\{u\in E;\ \|u\|<r\},\quad \Omega_2=\{u\in E;\ \|u\|<R\}.$$

Next let $A\colon K\cap(\overline{\Omega}_2\setminus\Omega_1)\to E$ be defined by

$$(Ay)(t)=\begin{cases} \mu\displaystyle\int_0^1 G(t,s)q(s)f^*(s,y(s-\tau)-\varphi(s-\tau))\,\mathrm{d}s, & 0\leqslant t\leqslant 1,\\ 0, \quad -\tau\leqslant t\leqslant 0. \end{cases}$$

First we show that A is well defined. To see this notice if $y\in K\cap(\overline{\Omega}_2\setminus\Omega_1)$ then $r\leqslant\|y\|\leqslant R$ and $y(t)\geqslant t(1-t)\|y\|\leqslant t(1-t)r$, $0\leqslant t\leqslant 1$. Also notice for $t\in(0,1)$ that Lemma 3.1 implies

$$\begin{aligned} y(t)-\varphi(t)=y(t)-\mu Mw(t)&\geqslant t(1-t)r-\mu Mt(1-t)C_0\\ &=t(1-t)(r-\mu MC_0), \quad t\in[0,1], \end{aligned}$$

since $y(t) \geqslant t(1-t)r$, $w(t) \leqslant t(1-t)C_0$ and $r > \mu M C_0$. So for $t \in (0,\tau)$ we have

$$f^*(t, y(t-\tau) - \varphi(t-\tau)) = f^*(t, \xi(t-\tau)),$$

and for $t \in (\tau, 1)$ we have

$$\begin{aligned} f^*(t, y(t-\tau) - \varphi(t-\tau)) &= f(t, y(t-\tau) - \varphi(t-\tau)) + M \\ &\leqslant g(y(t-\tau) - \varphi(t-\tau)) + h(y(t-\tau) - \varphi(t-\tau)) \\ &= g(y(t-\tau) - \varphi(t-\tau)) \left\{ 1 + \frac{h(y(t-\tau) - \varphi(t-\tau))}{g(y(t-\tau) - \varphi(t-\tau))} \right\} \\ &\leqslant g((t-\tau)(1+\tau-t)(r-\mu M C_0)) \left\{ 1 + \frac{h(R)}{g(R)} \right\} \\ &\leqslant K_0 g((t-\tau)(1+\tau-t)) g(r-\mu M C_0) \left\{ 1 + \frac{h(R)}{g(R)} \right\}. \end{aligned}$$

These inequalities with (3.7)–(3.8) guarantee that $A\colon K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to E$ is well defined. If $y \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, then we have

$$\begin{cases} \|Ay\|_{[0,1]} \leqslant \mu \displaystyle\int_0^1 s(1-s)q(s) f^*(s, y(s-\tau) - \varphi(s-\tau))\,\mathrm{d}s, \\ (Ay)(t) \geqslant t(1-t)\mu \displaystyle\int_0^1 s(1-s)q(s) f^*(s, y(s-\tau) - \varphi(s-\tau))\,\mathrm{d}s \\ \qquad\qquad \geqslant t(1-t)\|Ay\|_{[0,1]} = t(1-t)\|Ay\|, \quad t \in [0,1], \end{cases}$$

i.e., $Ay \in K$ so $A\colon K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$. Next we show that $A\colon K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is continuous and compact. Let $y_n, y \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $\|y_n - y\| \to 0$ as $n \to \infty$. Of course $r \leqslant \|y_n\| = \|y_n\|_{[0,1]} \leqslant R$, $r \leqslant \|y\| = \|y\|_{[0,1]} \leqslant R$, $y_n(t) \geqslant t(1-t)r$ and $y(t) \geqslant t(1-t)r$, for $0 \leqslant t \leqslant 1$. Notice also that $y_n(s) - \varphi(s) \geqslant s(1-s)(r - \mu M C_0)$ and $y(s) - \varphi(s) \geqslant s(1-s)(r-\mu M C_0)$ for $s \in [0,1]$, so

$$\begin{aligned} \varrho_n(s) &= |f^*(s, y_n(s-\tau) - \varphi(s-\tau)) - f^*(s, y(s-\tau) - \varphi(s-\tau))| = 0, \\ &\qquad\qquad s \in (0,\tau), \\ \varrho_n(s) &= |f^*(s, y_n(s-\tau) - \varphi(s-\tau)) - f^*(s, y(s-\tau) - \varphi(s-\tau))| \to 0, \\ &\qquad\qquad \text{as } n \to \infty,\ s \in (\tau, 1) \end{aligned}$$

and

$$\varrho_n(s) \leqslant 2K_0 \left\{ 1 + \frac{h(R)}{g(R)} \right\} g(r - \mu M C_0) g((s-\tau)(1+\tau-s))$$

for $s \in (\tau, 1)$. Now these together with the Lebesgue dominated convergence theorem guarantee that

$$\begin{aligned}\|Ay_n - Ay\| &= \|Ay_n - Ay\|_{[0,1]} \\ &\leqslant \sup_{t\in[0,1]} \mu \int_0^1 G(t,s)q(s)\varrho_n(s)\,\mathrm{d}s \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

so $A\colon K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is continuous. Also for $y \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ we have

$$\begin{aligned}\|Ay\| &\leqslant \mu b_0 \\ &\quad + \mu \int_\tau^1 s(1-s)q(s)g(y(s-\tau) - \varphi(s-\tau))\left\{1 + \frac{h(y(s-\tau)-\varphi(s-\tau))}{g(y(s-\tau)-\varphi(s-\tau))}\right\}\mathrm{d}s \\ &\leqslant \mu b_0 + \mu \int_\tau^1 s(1-s)q(s)K_0 g((s-\tau)(1+\tau-s))g(r-\mu MC_0)\left\{1+\frac{h(R)}{g(R)}\right\}\mathrm{d}s \\ &= \mu b_0 + \mu a_0 K_0 g(r-\mu MC_0)\left\{1+\frac{h(R)}{g(R)}\right\},\end{aligned}$$

and for $t, t' \in [0,1]$ we have

$$\begin{aligned}|(Ay)(t) - (Ay)(t')| \leqslant \mu &\int_0^\tau |G(t,s) - G(t',s)|q(s)f^*(s,\xi(s-\tau))\,\mathrm{d}s \\ &+ \mu K_0 g(r-\mu MC_0)\left\{1+\frac{h(R)}{g(R)}\right\} \\ &\times \int_\tau^1 |G(t,s)-G(t',s)|q(s)g((s-\tau)(1+\tau-s))\,\mathrm{d}s.\end{aligned}$$

Since $(Ay)(t) = 0$, for $t \in [-\tau, 0]$, the Arzela-Ascoli Theorem guarantees that $A\colon K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is compact.

We now show that

$$\|Ay\| \leqslant \|y\| \quad \text{for } K \cap \partial\Omega_1. \tag{3.14}$$

To see this, let $y \in K \cap \partial\Omega_1$. Then $\|y\| = \|y\|_{[0,1]} = r$ and $y(t) \geqslant t(1-t)r$ for $t \in [0,1]$. Now for $t \in (0,1)$ (as above)

$$y(t) - \varphi(t) \geqslant t(1-t)r - \mu Mt(1-t)C_0 \geqslant t(1-t)(r-\mu MC_0),$$

so for $t \in [0,1]$ we have

$$\begin{aligned}
(Ay)(t) &\leqslant \mu b_0 + \mu \int_\tau^1 s(1-s)q(s)g(y(s-\tau)-\varphi(s-\tau)) \\
&\qquad\qquad\qquad\qquad \times \left\{1 + \frac{h(y(s-\tau)-\varphi(s-\tau))}{g(y(s-\tau)-\varphi(s-\tau))}\right\} \mathrm{d}s \\
&\leqslant \mu b_0 + \mu \int_\tau^1 s(1-s)q(s)g((s-\tau)(1+\tau-s)(r-\mu M C_0))\left\{1 + \frac{h(r)}{g(r)}\right\} \mathrm{d}s \\
&\leqslant \mu b_0 + \mu a_0 K_0 g(r-\mu M C_0)\left\{1 + \frac{h(r)}{g(r)}\right\}.
\end{aligned}$$

This together with (3.9) yields $\|Ay\| = \|Ay\|_{[0,1]} \leqslant r = \|y\|$, so (3.14) is satisfied.

Next we show that

$$\|Ay\| \geqslant \|y\| \quad \text{for } K \cap \partial\Omega_2. \tag{3.15}$$

To see this let $y \in K \cap \partial\Omega_2$ so $\|y\| = \|y\|_{[0,1]} = R$ and $y(t) \geqslant t(1-t)R$ for $t \in [0,1]$. Also for $t \in [0,1]$ we have

$$\begin{aligned}
y(t) - \varphi(t) = y(t) - \mu M w(t) &\geqslant t(1-t)R - \mu M C_0 t(1-t) \\
&\geqslant t(1-t)R\Big(1 - \frac{\mu M C_0}{R}\Big) \geqslant \varepsilon t(1-t)R.
\end{aligned}$$

As a result

$$y(t-\tau) - \varphi(t-\tau) \geqslant \varepsilon a(a+\tau)R \quad \text{for } t \in [a+\tau, 1-a].$$

Now with σ as in the statement of Theorem 3.1, we have

$$\begin{aligned}
(Ay)(\sigma) &\geqslant \mu \int_{\tau+a}^{1-a} G(\sigma,s)q(s)[g_1(y(s-\tau)-\varphi(s-\tau)) + h_1(y(s-\tau)-\varphi(s-\tau)]\,\mathrm{d}s \\
&\geqslant \mu g_1(R) \int_{\tau+a}^{1-a} G(\sigma,s)q(s)\left\{1 + \frac{h_1(\varepsilon a(a+\tau)R)}{g_1(\varepsilon a(a+\tau)R)}\right\} \mathrm{d}s.
\end{aligned}$$

This together with (3.11) yields $\|Ay\| \geqslant R = \|y\|$, so (3.15) holds.

Now Theorem 1.1 implies that A has a fixed point $y_1 \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, i.e. $r \leqslant \|y\| = \|y\|_{[0,1]} \leqslant R$ and $y_1(t) \geqslant t(1-t)r$ for $t \in [0,1]$. Thus $y_1(t)$ is a solution of (3.12) with $y_1(t) > \varphi(t)$ for $t \in (0,1)$ and $y_1(t) = 0$ for $t \in [-\tau, 0]$.

Example. Consider the boundary value problem

$$\begin{cases}
y''(t) + \mu(y^{-\alpha}(t-\tau) + y^{\beta}(t-\tau) - 1) = 0, \quad t \in (0,1) \setminus \{\tau\}, \\
y(t) = (-t)^m, \quad -\tau \leqslant t \leqslant 0,\ 0 < m \leqslant 1, \\
y(1) = 0, \quad 0 < \alpha < 1 < \beta,\ 0 < \tau < 1
\end{cases} \tag{3.16}$$

where $\mu \in (0, \mu_0)$ is such that

$$\frac{\mu_0}{2} + \Big(\frac{2\mu_0 a_0}{1-\mu_0 b_0}\Big)^{1/\alpha} \leqslant 1, \quad \mu_0 < \frac{1}{b_0}; \tag{3.17}$$

here

$$\begin{aligned} a_0 &= \int_\tau^1 s(1-s)(s-\tau)^{-\alpha}(1+\tau-s)^{-\alpha}\,\mathrm{d}s < \infty, \\ b_0 &= \int_0^\tau s(1-s)[(\tau-s)^{-m\alpha} + (\tau-s)^{m\beta}]\,\mathrm{d}s < \infty. \end{aligned}$$

Then (3.16) has a solution y with $y(t) > 0$ for $t \in (0,1)$.

To see this we will apply Theorem 3.1 with (here $R > 1$ will be chosen later, in fact here we choose $R > 1$ so that $\varepsilon = \frac{1}{2}$ works, i.e. we choose R so that $1 - \frac{1}{2}\mu/R \geqslant \frac{1}{2}$),

$$g(y) = g_1(y) = y^{-\alpha}, \quad h(y) = h_1(y) = y^\beta, \quad q(t) = 1, \quad M = 1, \quad \xi(t) = (-t)^m,$$
$$K_0 = 1, \quad C_0 = \frac{1}{2}, \quad \varepsilon = \frac{1}{2}, \quad a = \frac{1-\tau}{4}.$$

Clearly (3.2)–(3.8) and (3.10) hold. Now (3.9) holds with $r = 1$ since

$$\begin{aligned} \mu K_0 a_0 = \mu a_0 < \mu_0 a_0 &\leqslant \frac{1}{2}(1-\mu_0 b_0)\Big(1 - \frac{\mu_0}{2}\Big)^\alpha \\ &\leqslant \frac{1}{2}(1-\mu b_0)\Big(1-\frac{\mu}{2}\Big)^\alpha \\ &= \frac{r - \mu b_0}{\{1 + \frac{h(r)}{g(r)}\} g(r - \mu M C_0)} \end{aligned}$$

from (3.17). Finally notice that (3.11) is satisfied for R large since

$$\begin{aligned} &\frac{R g_1(\varepsilon a(a+\tau)R)}{g_1(R) g_1(\varepsilon a(a+\tau)R) + g_1(R) h_1(\varepsilon a(a+\tau)R)} \\ &\quad = \frac{[\varepsilon a(a+\tau)]^{-\alpha} R^{1+\alpha}}{[\varepsilon a(a+\tau)]^{-\alpha} + [\varepsilon a(a+\tau)]^\beta R^{\alpha+\beta}} \to 0, \end{aligned}$$

as $R \to \infty$, since $\beta > 1$. Thus all the conditions of Theorem 3.1 are satisfied so existence is guaranteed. □

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