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IMPLICATIVE HYPER K-ALGEBRAS

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Abstract. In this note we first define the notions of (weak, strong) implicative hyper K-algebras. Then we show by examples that these notions are different. After that we state and prove some theorems which determine the relationship between these notions and (weak) hyper K-ideals. Also we obtain some relations between these notions and (weak) implicative hyper K-ideals. Finally, we study the implicative hyper K-algebras of order 3, in particular we obtain a relationship between the positive implicative hyper K-algebras and (weak, strong) implicative hyper K-algebras under a simple condition.

Keywords: hyper K-algebra, hyper K-ideal, (weak, strong) implicative hyper K-algebras, (weak) implicative hyper K-ideal

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1. Introduction

The hyperalgebraic structure theory was introduced by F. Marty [7] in 1934. Imai and Iseki [5] in 1966 introduced the notion of a BCK-algebra. Recently [3], [6], [11] Borzooei, Jun and Zahedi et al. applied the hyperstructure to BCK-algebras and introduced the concept of the hyper K-algebra which is a generalization of the BCK-algebra. It is well-known [9] that the category of bounded commutative BCK-algebras is equivalent to the category of MV-algebras. In particular, any bounded commutative BCK-algebra is an MV-algebra and vice-versa. On the other hand, an MV-algebra is an algebraic structure of the Lukasiewicz many-valued logic. Hence any bounded commutative BCK-algebra is somehow related to a many-valued logic. Since the concept of the hyper K-algebra is a generalization of the notion of the BCK-algebra, it is natural to search for a logic whose algebraic structure is a hyper K-algebra. To this end, we first need a deeper understanding of hyper K-algebras. Now, in this note we define the notions of (weak, strong) implicative hyper K-algebras, then we obtain some related results which have been mentioned in the abstract.

2. Preliminaries

Definition 2.1 ([3]). Let H be a nonempty set and "$\circ$" a *hyperoperation* on H, that is, "$\circ$" is a function from $H \times H$ to $\mathscr{P}^*(H) = \mathscr{P}(H) \backslash \{\emptyset\}$. Then H is called a *hyper K-algebra* if it contains a constant "0" and satisfies the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) < x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x < x$,
(HK4) $x < y,\ y < x \Rightarrow x = y$,
(HK5) $0 < x$

for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists\, a \in A$, $\exists\, b \in B$ such that $a < b$.

Note that if $A, B \subseteq H$, then by $A \circ B$ we mean the subset $\bigcup\limits_{\substack{a \in A \\ b \in B}} a \circ b$ of H.

Example 2.2 ([3]). Define the hyperoperation "$\circ$" on $H = [0, +\infty)$ as follows:

$$x \circ y = \begin{cases} [0, x] & \text{if } x \leqslant y, \\ (0, y] & \text{if } x > y \neq 0, \\ \{x\} & \text{if } y = 0 \end{cases}$$

for all $x, y \in H$. Then $(H, \circ, 0)$ is a hyper K-algebra.

Theorem 2.3 ([3]). *Let $(H, \circ, 0)$ be a hyper K-algebra. Then for all $x, y, z \in H$ and for all nonempty subsets A, B and C of H the following relations hold:*

(i) $x \circ y < z \Leftrightarrow x \circ z < y$,
(ii) $(x \circ z) \circ (x \circ y) < y \circ z$,
(iii) $x \circ (x \circ y) < y$,
(iv) $x \circ y < x$,
(v) $A \subseteq B \Rightarrow A < B$,
(vi) $x \in x \circ 0$,
(vii) $(A \circ C) \circ (A \circ B) < B \circ C$,
(viii) $(A \circ C) \circ (B \circ C) < A \circ B$,
(ix) $A \circ B < C \Leftrightarrow A \circ C < B$.

Definition 2.4 ([3]). Let I be a nonempty subset of a hyper K-algebra $(H, \circ, 0)$ and $0 \in I$. Then

(i) I is called a *weak hyper K-ideal* of H if $x \circ y \subseteq I$ and $y \in I$ imply that $x \in I$ for all $x, y \in H$;

(ii) I is called a *hyper K-ideal* of H if $x \circ y < I$ and $y \in I$ imply that $x \in I$ for all $x, y \in H$.

Theorem 2.5 ([3]). *Any hyper K-ideal of a hyper K-algebra H is a weak hyper K-ideal.*

Definition 2.6 ([4]). Let I be a nonempty subset of H. Then we say that I satisfies the *additive condition*, if for all $x, y \in H$, $x < y$ and $y \in I$ imply that $x \in I$.

Definition 2.7 ([2]). Let H be a hyper K-algebra. An element $a \in H$ is called a *left* (*right*) *scalar* if $|a \circ x| = 1$ ($|x \circ a| = 1$) for all $x \in H$. If $a \in H$ is both a left and a right scalar, we say that a is a *scalar element*.

Definition 2.8 ([2]). We say that a hyper K-algebra H satisfies the *transitive condition* if for all $x, y, z \in H$, $x < y$ and $y < z$ imply that $x < z$.

Definition 2.9 ([2]). A hyper K-algebra H is called a *positive implicative hyper K-algebra*, if it satisfies $(x \circ z) \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in H$.

Definition 2.10 ([1]). We say that a hyper K-algebra H satisfies the *strong transitive condition* if for all $A, B, C \subseteq H$, $A < B$ and $B < C$ imply that $A < C$.

Definition 2.11 ([1]). Let H be a hyper K-algebra, then a nonempty subset I of H is called

(a) a *weak implicative hyper K-ideal* if it satisfies
   (i) $0 \in I$,
   (ii) $(x \circ z) \circ (y \circ x) \subseteq I$ and $z \in I$ imply $x \in I$ for all $x, y, z \in H$,

(b) an *implicative hyper K-ideal* if it satisfies
   (i) $0 \in I$,
   (ii) $(x \circ z) \circ (y \circ x) < I$ and $z \in I$ imply $x \in I$ for all $x, y, z \in H$.

Theorem 2.12 ([1]). *Let I be a weak hyper K-ideal of H. Then the following statements hold:*

(i) *If for all $x, y, z \in H$, $x \circ (y \circ x) \subseteq I$ implies $x \in I$, then I is a weak implicative hyper K-ideal.*

(ii) *Let $0 \in H$ be a right scalar element and I a weak implicative hyper K-ideal. Then for all $x, y \in H$, $x \circ (y \circ x) \subseteq I$ implies that $x \in I$.*

Theorem 2.13 ([1]). *Let I be a hyper K-ideal of H. Then I is an implicative hyper K-ideal if and only if*

$$x \circ (y \circ x) < I \quad \textit{implies that } x \in I \quad \textit{for any } x, y \in H.$$

Definition 2.14 ([10]). Let $H = \{0, 1, 2\}$ be a hyper K-algebra of order 3. We say that H satisfies the *simple condition* if $1 \not< 2$ and $2 \not< 1$.

Definition 2.15 ([10]). Let $H = \{0, 1, 2\}$ be a hyper K-algebra of order 3. We say that H satisfies the *normal condition* if $1 < 2$ or $2 < 1$.

3. Implicative hyper K-algebra

From now on H is a hyper K-algebra, unless stated otherwise.

Definition 3.1. H is said to be

(i) *weak implicative* if $x < x \circ (y \circ x)$ for all $x, y \in H$,
(ii) *implicative* if $x \in x \circ (y \circ x)$ for all $x, y \in H$,
(iii) *strong implicative* if $x \circ 0 \subseteq x \circ (y \circ x)$ for all $x, y \in H$.

Example 3.2. Let $H = \{0, 1, 2, 3\}$. Then the following table shows a hyper K-algebra structure on H:

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$
2	$\{2\}$	$\{2\}$	$\{0\}$	$\{2\}$
3	$\{2,3\}$	$\{1,2\}$	$\{0,2,3\}$	$\{0,1,2\}$

It can be checked that H is a weak implicative, implicative and strong implicative hyper K-algebra.

Theorem 3.3.

(i) *Any implicative hyper K-algebra is a weak implicative hyper K-algebra.*
(ii) *Any strong implicative hyper K-algebra is an implicative hyper K-algebra.*

Proof. The proof is trivial. □

The following example shows that the notions given in Definition 3.1 are not equivalent.

Example 3.4. (i) Let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1,2\}$	$\{0,2\}$	$\{0,2\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

We can see that H is a weak implicative hyper K-algebra. However it is not an implicative hyper K-algebra, because $1 \notin 1 \circ (2 \circ 1) = \{0, 2\}$.

(ii) Let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1\}$	$\{1,2\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

Now, H is an implicative hyper K-algebra. However it is not a strong implicative one because $0 \circ 0 = \{0, 1\} \nsubseteq 0 \circ (1 \circ 0) = \{0\}$.

Proposition 3.5. *Let $0 \in H$ be a right scalar element. Then the notions of implicative and strong implicative hyper K-algebras are equivalent.*

Proof. The proof follows from the fact that $x \circ 0 = x$. □

Proposition 3.6. *H is a weak implicative hyper K-algebra if and only if $x \circ 0 < x \circ (y \circ x)$ for all $x, y \in H$.*

Proof. Let $x \circ 0 < x \circ (y \circ x)$ for all $x, y \in H$. Then we have $x \circ (x \circ (y \circ x)) < 0$. Thus there exists $t \in x \circ (x \circ (y \circ x))$ such that $t < 0$. Hence $t = 0$, therefore $x < x \circ (y \circ x)$. The proof of the converse is trivial. □

Theorem 3.7. *Let H be a hyper K-algebra of order 3 that satisfies the simple condition. Then H is implicative if and only if it is weak implicative.*

Proof. Let H be a weak implicative hyper K-algebra. We show that $x \in x \circ (y \circ x)$ for all $x, y \in H$. If $x = 0$, then $0 \in 0 \circ (y \circ 0)$ for all $y \in H$. Let $x \neq 0$ and $x \notin x \circ (y \circ x)$. Since $x < x \circ (y \circ x)$, there exists $t \in x \circ (y \circ x)$ such that $x < t$. Clearly since $x \neq 0$, we must have $t \neq 0$ and $t \neq x$. Since H satisfies the simple condition, $x < t$ is impossible. Thus $x \in x \circ (y \circ x)$ for all $x, y \in H$. For the converse see Theorem 3.3 (i). □

Example 3.8. This example shows that in the above theorem, the simple condition can not be omitted. Indeed let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0, 1\}$	$\{0, 2\}$

Then H is weak implicative while it is not implicative, since $2 \notin 2 \circ (1 \circ 2)$.

Example 3.9. (i) It is not necessary that a (weak, strong) implicative hyper K-algebra be a positive implicative hyper K-algebra. Example 3.2 shows a hyper K-algebra which is strong implicative while it is not a positive implicative hyper K-algebra. Indeed $(3 \circ 2) \circ (1 \circ 2) = \{0, 1, 2, 3\} \neq (3 \circ 1) \circ 2 = \{0, 1, 2\}$.

(ii) In general it is not needed that a positive implicative hyper K-algebra be a (weak, strong) implicative hyper K-algebra. Because let $H = \{0, 1, 2\}$. Then the following table shows a positive implicative hyper K-algebra structure on H:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0, 1\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0, 2\}$

but H is not a (weak, strong) implicative hyper K algebra. Indeed $1 \not< 1 \circ (2 \circ 1)$.

Theorem 3.10. *Let H be a positive implicative hyper K-algebra of order 3 that satisfies the simple condition. Then H is a (weak, strong) implicative hyper K-algebra.*

P r o o f. Since H satisfies the simple condition, we know that $1 \circ 0 = \{1\}$, $2 \circ 0 = \{2\}$, $1 \circ 2 \neq \{2\}$ and $2 \circ 1 \neq \{1\}$ by Theorem 3.17 of [10]. Now we show that H is a strong implicative hyper K-algebra, that is $x \circ 0 \subseteq x \circ (y \circ x)$ for all $x, y \in H$. To do this we consider three different cases:

(i) If $x = 0$, then we must show that $0 \circ 0 \subseteq 0 \circ (y \circ 0)$ for all $y \in H$. If $y = 0$, then we are done. We know that $0 \in 0 \circ 0$, so $0 \circ 0 = \{0\}$, $\{0, 1\}$, $\{0, 2\}$ or $\{0, 1, 2\}$. If $0 \circ 0 = \{0\}$, then clearly $0 \in 0 \circ 1$ and $0 \in 0 \circ 2$, and so we are done. Now let $0 \circ 0 = \{0, 1\}$. If $y = 1$, then we must show that $0 \circ 0 \subseteq 0 \circ (1 \circ 0) = 0 \circ 1$. We have $(0 \circ 0) \circ 0 = \{0, 1\} \circ 0 = (0 \circ 0) \cup (1 \circ 0) = \{0, 1\} \cup \{1\} = \{0, 1\}$. On the other hand, since H is positive implicative then $\{0, 1\} = (0 \circ 0) \circ 0 = (0 \circ 0) \circ (0 \circ 0) = \{0, 1\} \circ \{0, 1\} =$

$(0\circ 0)\cup(1\circ 0)\cup(0\circ 1)\cup(1\circ 1)$. Thus we conclude that $(0\circ 1)$ and $(1\circ 1)\subseteq\{0,1\}$. If $1\notin(0\circ 1)$, we get that $0\circ 1=\{0\}$. So $(0\circ 1)\circ 1=0\circ 1=\{0\}$ and on the other hand, since H is positive implicative we have $\{0\}=(0\circ 1)\circ 1=(0\circ 1)\circ(1\circ 1)\supseteq 0\circ 0=\{0,1\}$, which is a contradiction. Thus $0\circ 1=\{0,1\}$, and hence $0\circ 0=0\circ 1$. Now let $y=2$. Since $0\in 0\circ 2$ then $0\circ 2=\{0\}$, $\{0,1\}$, $\{0,2\}$ or $\{0,1,2\}$. If $0\circ 2=\{0\}$, then $(0\circ 2)\circ 2=0\circ 2=\{0\}$ and on the other hand, since H is positive implicative we have $\{0\}=(0\circ 2)\circ 2=(0\circ 2)\circ(2\circ 2)\supseteq 0\circ 0=\{0,1\}$, which is a contradiction. Hence $0\circ 2\neq\{0\}$. Let $0\circ 2=\{0,2\}$. Since $1\not<2$, then $0\notin 1\circ 2$. So $1\circ 2=\{1\}$ or $\{1,2\}$. If $1\circ 2=\{1\}$, then $\{0,2\}=0\circ 2\subseteq(1\circ 1)\circ 2=(1\circ 2)\circ 1=1\circ 1\subseteq\{0,1\}$, which is a contradiction. Hence $1\circ 2=\{1,2\}$. Now we have $0\in 2\circ 2\subseteq(1\circ 2)\circ(0\circ 2)=(1\circ 0)\circ 2=1\circ 2=\{1,2\}$, which is a contradiction. Hence $0\circ 2=\{0,1\}$ or $\{0,1,2\}$. Thus in the case $0\circ 0=\{0,1\}$, we conclude that $0\circ 0\subseteq 0\circ 2$. The proof for the case $0\circ 0=\{0,2\}$ is similar as above. If $0\circ 0=\{0,1,2\}$, then since H is a positive implicative we have $\{1\}=(1\circ 0)=(1\circ 0)\circ 0=(1\circ 0)\circ(0\circ 0)=1\circ\{0,1,2\}=(1\circ 0)\cup(1\circ 1)\cup(1\circ 2)$, thus we must have $1\circ 1=\{1\}$ and this is a contradiction with (HK3). Hence $0\circ 0\neq\{0,1,2\}$. Thus if $x=0$, then $0\circ 0\subseteq 0\circ(y\circ 0)$ for all $y\in H$.

(ii) If $x=1$, then we must show that $1\in 1\circ(y\circ 1)$ for all $y\in H$. If $y=0$ or 1 it is trivial, so let $y=2$. Since $2\not<1$, then $0\notin 2\circ 1$ and $2\circ 1\neq\{1\}$. Thus we conclude that $2\circ 1=\{2\}$ or $\{1,2\}$. Since $1\not<2$, then $0\notin 1\circ 2$ and $1\circ 2\neq\{2\}$. Therefore $1\circ 2=\{1\}$ or $\{1,2\}$. Hence in all cases by some manipulations we can get that $1\in 1\circ(2\circ 1)$.

(iii) If $x=2$, then by the same argument as in (ii) we can show that $2\in 2\circ(y\circ 2)$ for all $y\in H$. □

Remark 3.11. The following example shows that in the above theorem the simple condition can not be omitted. Let $H=\{0,1,2\}$. Then the following table shows a positive implicative hyper K-algebra structure on H where H does not satisfy the simple condition:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0,2\}$

and H is not an implicative hyper K-algebra, either, because $2\notin 2\circ(1\circ 2)=\{0\}$.

Theorem 3.12. *Let H be a weak implicative hyper K-algebra. Then each hyper K-ideal of H is a weak implicative hyper K-ideal.*

Proof. Let I be a hyper K-ideal and $(x\circ z)\circ(y\circ x)\subseteq I$, $z\in I$. Then for all $t\in x\circ(y\circ x)$ we have $t\circ z\subseteq(x\circ(y\circ x))\circ z=(x\circ z)\circ(y\circ x)\subseteq I$ and $z\in I$. Thus

$t \in I$ and hence $x \circ (y \circ x) \subseteq I$. Since H is weak implicative, then $x < x \circ (y \circ x) \subseteq I$. So there exists $r \in I$ such that $x < r$. Thus $0 \in x \circ r$, hence $x \circ r < I$ and $r \in I$ which implies that $x \in I$. $\square$

Remark 3.13. (i) The following example shows that in the above theorem we can not use "weak hyper K-ideal" instead of "hyper K-ideal". Let $H = \{0, 1, 2\}$. Then the following table shows a weak implicative hyper K-algebra structure on H:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1,2\}$	$\{2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,1\}$

Now $I = \{0, 1\}$ is a weak hyper K-ideal and $(2 \circ 0) \circ (1 \circ 2) = \{0, 1\} \subseteq I$ and $0 \in I$, but $2 \notin I$. Hence I is not a weak implicative hyper K-ideal.

(ii) The following example shows that in the above theorem, if we use "weak hyper K-ideal" instead of "hyper K-ideal", we can not conclude that "any weak hyper K-ideal is implicative". Let $H = \{0, 1, 2\}$. Then the following table shows a weak implicative hyper K-algebra structure on H:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1\}$	$\{1,2\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

Then $I = \{0\}$ is a weak hyper K-ideal and $1 \circ (0 \circ 1) = \{0, 1, 2\} < I$, but $1 \notin I$. Hence I is not an implicative hyper K-ideal.

(iii) The following example shows that the conditions of the above theorem do not imply that any hyper K-ideal is implicative. Let $H = \{0, 1, 2\}$. Then the following table shows a weak implicative hyper K-algebra structure on H:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

We see that $I = \{0\}$ is a hyper K-ideal and $2 \circ (2 \circ 2) = \{0, 1, 2\} < I$, but $2 \notin I$. Hence I is not an implicative hyper K-ideal.

Theorem 3.14. *Let H be an implicative hyper K-algebra. Then each weak hyper K-ideal of H is a weak implicative hyper K-ideal.*

Proof. Let I be a weak hyper K ideal and $(x \circ z) \circ (y \circ x) \subseteq I$, $z \in I$. Then $(x \circ (y \circ x)) \circ z \subseteq I$. Since H is implicative, we have $x \in (x \circ (y \circ x))$. Therefore $x \circ z \subseteq (x \circ (y \circ x)) \circ z \subseteq I$ and since $z \in I$, we conclude that $x \in I$. □

Corollary 3.15. *Let H be an implicative hyper K-algebra. Then each hyper K ideal of H is a weak implicative hyper K-ideal.*

Remark 3.16. The following example shows that the conditions of the above corollary do not imply that any hyper K-ideal is implicative. Let $H = \{0, 1, 2\}$. Then the following table shows an implicative hyper K-algebra structure on H:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0, 2\}$

Now $I = \{0\}$ is a hyper K-ideal while it is not an implicative hyper K-ideal, since $(2 \circ 0) \circ (2 \circ 2) = \{0, 2\} < I$ and $0 \in I$, but $2 \notin I$.

Note that the following theorem says that if we restrict ourselves to the hyper K-algebras of order 3, then the above corollary holds even if H is not implicative.

Theorem 3.17. *If H is a hyper K-algebra of order 3, then each nonzero hyper K-ideal is a weak implicative hyper K-ideal.*

Proof. Let $H = \{0, 1, 2\}$. Without loss of generality let $I = \{0, 1\}$ be a hyper K-ideal of H. By Theorem 2.11 it is enough to show that $x \circ (y \circ x) \subseteq I$ implies that $x \in I$. If $x = 0, 1$ then we are done. Now let $x = 2$, then $2 \circ (y \circ 2) \subseteq I$ for all $y \in H$ and we will get a contradiction. To obtain it, consider three different cases:

(i) Let $y = 0$. Then $2 \in 2 \circ (0 \circ 2) \subseteq I$, and this is a contradiction.

(ii) Let $y = 1$. If $1 < 2$, then $0 \in 1 \circ 2$. Therefore $2 \in 2 \circ 0 \subseteq 2 \circ (1 \circ 2) \subseteq I$, and this is a contradiction. If $1 \not< 2$, then $0 \notin 1 \circ 2$, so we must have $1 \circ 2 = \{1\}$, $\{2\}$ or $\{1, 2\}$. If $1 \circ 2 = \{1\}$, then $2 \circ 1 = 2 \circ (1 \circ 2) \subseteq I$ and $1 \in I$ imply that $2 \in I$, which is a contradiction. If $1 \circ 2 = \{2\}$, then $0 \in 0 \circ 2 \subseteq (1 \circ 1) \circ 2 = (1 \circ 2) \circ 1 = 2 \circ 1$. Hence $2 \circ 1 < I$ and $1 \in I$ imply that $2 \in I$, which is a contradiction. If $1 \circ 2 = \{1, 2\}$, consider $(2 \circ 1) \cup (2 \circ 2) = 2 \circ \{1, 2\} = 2 \circ (1 \circ 2) \subseteq I$, therefore $2 \circ 1 \subseteq I$ and $1 \in I$ imply that $2 \in I$, which is a contradiction.

(iii) If $y = 2$ then $2 \in 2 \circ (2 \circ 2) \subseteq I$, which is a contradiction. □

Remark 3.18. (i) The converse of the above theorem is not correct. Indeed let $H = \{0, 1, 2, 3\}$. Then the following table shows a hyper K-algebra structure on H:

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$
2	$\{2\}$	$\{2\}$	$\{0,2\}$	$\{2\}$
3	$\{2,3\}$	$\{1,2,3\}$	$\{0,1,3\}$	$\{0,1,2,3\}$

Then $I = \{0, 1\}$ is a weak implicative hyper K-ideal, which is not a hyper K-ideal, since $3 \circ 1 = \{1, 2, 3\} < I$ and $1 \in I$, but $3 \notin I$.

(ii) The following example shows that the condition "nonzero hyper K-ideal" in the above theorem can not be omitted. Let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{1\}$	$\{0,1\}$

Now it is easy to see that $I = \{0\}$ is a hyper K-ideal while it is not a weak implicative one since $(1 \circ 0) \circ (2 \circ 1) \subseteq I$ and $0 \in I$, but $1 \notin I$.

Lemma 3.19. *Let H be a positive implicative hyper K-algebra of order 3 that satisfies the normal condition. Then the following statements hold:*

(i) $1 \circ 0 = \{1\}$,

(ii) $2 \circ 0 = \{2\}$.

Proof. (i) We know that $1 \in 1 \circ 0$ and $0 \notin 1 \circ 0$, thus $1 \circ 0 = \{1\}$ or $\{1, 2\}$. Let $1 \circ 0 = \{1, 2\}$. Since H satisfies the normal condition, then $1 < 2$ or $2 < 1$. Now we consider the following two cases.

Case 1: Let $1 < 2$. Then $0 \notin 2 \circ 1$. Since $0 \in 2 \circ 2 \subseteq (2 \circ 0) \circ \{1, 2\} = (2 \circ 0) \circ (1 \circ 0) = (2 \circ 1) \circ 0$, thus $2 \circ 1 < 0$. So there is $x \in 2 \circ 1$ such that $x < 0$, therefore $x = 0$. Hence $0 \in 2 \circ 1$, which is a contradiction.

Case 2: Let $2 < 1$. Then $0 \notin 1 \circ 2$. Since $0 \in 2 \circ 2 \subseteq \{1, 2\} \circ (2 \circ 0) = (1 \circ 0) \circ (2 \circ 0) = (1 \circ 2) \circ 0$, thus there is $x \in 1 \circ 2$ such that $x < 0$, so $x = 0$. Hence $0 \in 2 \circ 1$, which is a contradiction. Thus we must have $1 \circ 0 = \{1\}$.

(ii) The proof is similar to the proof of (i). □

Theorem 3.20. *Let H be a hyper K-algebra of order 3 and $I \subset H$. Then*

(i) *If H satisfies the simple condition, then I is a weak implicative hyper K-ideal if and only if I is a weak hyper K-ideal;*

(ii) *if H is positive implicative and satisfies the normal condition then $I \neq \{0\}$ is a weak implicative hyper K-ideal if and only if I is a weak hyper K-ideal.*

Proof. (i) Let $I = \{0\}$ be a weak hyper K-ideal and $(x \circ z) \circ (y \circ x) \subseteq I$ and $z \in I$. Then $x \circ (y \circ x) \subseteq (x \circ 0) \circ (y \circ x) = \{0\}$. We must show that $x = 0$. On the contrary, let $x = 1$. Then $1 \circ (y \circ 1) = \{0\}$. If $y = 0$ or 1, we get the contradiction $1 \in \{0\}$. If $y = 2$, since H satisfies the simple condition, then $1 \circ (2 \circ 1) \neq \{0\}$, which is a contradiction, hence $x = 1$ is impossible. By a similar argument we show that $x = 2$ is also impossible. Thereby $x = 0 \in I$. Note that since $I = \{0\}$ is always a weak hyper K-ideal the converse is trivial. For the case $I \neq \{0\}$ see Theorem 4.11 of [1].

(ii) Without loss of generality let $I = \{0, 1\}$ be a weak hyper K-ideal. By Theorem 2.11 (i), it is enough to show that if $x \circ (y \circ x) \subseteq I$ for $x, y \in H$, then $x \in I$. If $x = 0$ or 1 we are done. Now let $x = 2$. So $2 \circ (y \circ 2) \subseteq I$ for arbitrary $y \in H$. We consider three cases for y and show that none of these cases holds.

(a) Let $y = 0$. Then $2 \in 2 \circ 0 \subseteq 2 \circ (0 \circ 2) \subseteq I$, which is a contradiction.

(b) Let $y = 1$. If $1 < 2$, then $0 \in 1 \circ 2$, hence $2 \in 2 \circ 0 \subseteq 2 \circ (1 \circ 2) \subseteq I$, which is a contradiction. If $1 \nless 2$, then $1 \circ 2 = \{1\}$, $\{2\}$ or $\{1, 2\}$. If $1 \circ 2 = \{1\}$, then $2 \circ 1 = 2 \circ (1 \circ 2) \subseteq I$. Since $1 \in I$, we get that $2 \in I$, which is a contradiction. If $1 \circ 2 = \{2\}$, then we have $2 \circ 2 = 2 \circ (1 \circ 2) \subseteq I = \{0, 1\}$. Since H is positive implicative we have $2 \in 2 \circ 0 \subseteq 2 \circ (2 \circ 2) = (1 \circ 2) \circ (2 \circ 2) = (1 \circ 2) \circ 2 = 2 \circ 2 \subseteq I = \{0, 1\}$ which is a contradiction. If $1 \circ 2 = \{1, 2\}$, then $(2 \circ 1) \cup (2 \circ 2) = 2 \circ \{1, 2\} = 2 \circ (1 \circ 2) \subseteq I$. Hence $2 \circ 1 \subseteq I$. Since I is a weak hyper K-ideal and $1 \in I$, so $2 \in I$, which is a contradiction.

(c) Let $y = 2$. Then $2 \in 2 \circ 0 \subseteq 2 \circ (2 \circ 2) \subseteq I$, which is a contradiction. Hence $x = 2$ is impossible and therefore I is a weak implicative hyper K-ideal.

Conversely, let $I = \{0, 1\}$ be a weak implicative hyper K-ideal and $x \circ y \subseteq I$ and $y \in I$. We must show that $x \in I$. If $x = 0$ or 1 then $x \in I$ and we are done. Now we show that $x = 2$ is impossible. If $x = 2$ then we have $2 \circ y \subseteq I$ and $y \in I$. We consider three different cases for y:

(a′) If $y = 0$, then $2 \in 2 \circ 0 \subseteq I$, which is a contradiction.

(b′) The case $y = 2$ never occurs since we must have $y \in I$.

(c′) If $y = 1$, since $2 \circ 1 \subseteq I$ we conclude that $2 \circ 1 = \{0\}$, $\{1\}$ or $\{0, 1\}$. Now consider the following cases:

(c$'$1) If $2 \circ 1 = \{0\}$, then $0 \notin 1 \circ 2$, therefore $1 \circ 2 = \{1\}$, $\{2\}$ or $\{1,2\}$. Thus we have to consider the following three subcases:

(c$'$1.1) If $1 \circ 2 = \{1\}$, since by Lemma 3.19, $2 \circ 0 = \{2\}$, hence $(2 \circ 0) \circ (1 \circ 2) = 2 \circ 1 = \{0\} \subseteq I$. Since I is a weak implicative hyper$K$-ideal and $0 \in I$, we have $2 \in I$, which is a contradiction.

(c$'$1.2) If $1 \circ 2 = \{2\}$, since H is positive implicative we have $\{0\} = 2 \circ 1 = (1 \circ 2) \circ 1 = (1 \circ 1) \circ 2 = (1 \circ 2) \circ (1 \circ 2) = 2 \circ 2$. Since by Lemma 3.19, $2 \circ 0 = \{2\}$, $(2 \circ 0) \circ (1 \circ 2) = 2 \circ (1 \circ 2) = 2 \circ 2 = \{0\} \subseteq I$ and $0 \in I$, we conclude that $2 \in I$, which is a contradiction.

(c$'$1.3) If $1 \circ 2 = \{1,2\}$, since $1 \circ 0 = \{1\}$ and $2 \circ 0 = \{2\}$ hence $(1 \circ 0) \circ 2 = 1 \circ 2 = \{1,2\}$ and $(1 \circ 0) \circ 2 = (1 \circ 2) \circ (0 \circ 2) = \{1,2\} \circ (0 \circ 2)$. If 1 or 2 belongs to $0 \circ 2$, then $0 \in (1 \circ 0) \circ 2 = \{1,2\}$ which is a contradiction, thus $0 \circ 2 = \{0\}$. We know that $0 \in 2 \circ 2$; if $2 \circ 2 = \{0\}$ then $(2 \circ 0) \circ (1 \circ 2) = 2 \circ (1 \circ 2) = 2 \circ \{1,2\} = (2 \circ 1) \cup (2 \circ 2) = \{0\} \subseteq I$. Since $0 \in I$ and I is weak implicative, we get that $2 \in I$, which is a contradiction. Thus $2 \circ 2 \neq \{0\}$. Since H is positive implicative we have $\{0\} = 0 \circ 2 = (2 \circ 1) \circ 2 = (2 \circ 2) \circ (1 \circ 2) = (2 \circ 2) \circ \{1,2\} \neq \{0\}$, hence this case is impossible.

(c$'$2) If $2 \circ 1 = \{1\}$, we know that $1 \circ 0 = \{1\}$. Then $\{1\} = 1 \circ 0 = (2 \circ 1) \circ 0 = (2 \circ 0) \circ 1 = (2 \circ 1) \circ (0 \circ 1) = 1 \circ (0 \circ 1)$, therefore $0 \circ 1 = \{0\}$. We have $0 \in 0 \circ 2$. If $0 \circ 2 = \{0\}$, then by considering $(2 \circ 1) \circ (0 \circ 2) = 1 \circ 0 = \{1\} \subseteq I$, $1 \in I$ and I is weak implicative, we conclude that $2 \in I$, which is a contradiction. Hence $0 \circ 2 \neq \{0\}$. Consider $(0 \circ 2) \circ 1 = (0 \circ 1) \circ (2 \circ 1) = 0 \circ \{1\} = \{0\}$. But if $0 \circ 2 \neq \{0\}$ then $(0 \circ 2) \circ 1 \neq \{0\}$, hence this case is impossible.

(c$'$3) If $2 \circ 1 = \{0,1\}$, since $0 \in 2 \circ 1$, hence $0 \notin 1 \circ 2$. Therefore $1 \circ 2 = \{1\}$, $\{2\}$ or $\{1,2\}$. Now we discuss the following three different subcases.

(c$'$3.1) If $1 \circ 2 = \{1\}$, since $(2 \circ 0) \circ (1 \circ 2) = 2 \circ 1 \subseteq I$, $0 \in I$ and I is weak implicative, we get that $2 \in I$, which is impossible.

(c$'$3.2) Suppose $1 \circ 2 = \{2\}$. Consider $(2 \circ 0) \circ (1 \circ 2) = 2 \circ (1 \circ 2) = 2 \circ 2 = (1 \circ 2) \circ (1 \circ 2) = (1 \circ 1) \circ 2 = (1 \circ 2) \circ 1 = 2 \circ 1 = \{0,1\} \subseteq I$. Since $0 \in I$ and I is weak implicative, we get that $2 \in I$, which is a contradiction.

(c$'$3.3) If $1 \circ 2 = \{1,2\}$, then since H is positive implicative we have $\{1,2\} = 1 \circ 2 = (1 \circ 0) \circ 2 = (1 \circ 2) \circ (0 \circ 2) = \{1,2\} \circ (0 \circ 2)$. If 1 or $2 \in 0 \circ 2$, then $0 \in (1 \circ 0) \circ 2 = \{1,2\}$ which is a contradiction, hence $0 \circ 2 = \{0\}$. Consider $\{1\} = 1 \circ 0 = (1 \circ 0) \circ 0 = (1 \circ 0) \circ (0 \circ 0) = 1 \circ (0 \circ 0)$. Then we conclude that $0 \circ 0 = \{0\}$. Then $(2 \circ 1) \circ (0 \circ 2) = \{0,1\} \circ 0 = (0 \circ 0) \cup (1 \circ 0) = \{0,1\} \subseteq I$. Since $1 \in I$ we get that $2 \in I$, which is a contradiction.

Thus the above arguments show that $x = 2$ is impossible, hence I is a weak hyper K-ideal of H.

Remark 3.21. (i) In part (ii) of the above theorem the condition "positive implicative" can not be omitted. Let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H which satisfies the normal condition:

$\circ$	0	1	2
0	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{1,2\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

We can check that $I = \{0, 1\}$ is a weak implicative hyper K-ideal, but it is not a weak hyper K-ideal, since $2 \circ 1 \subseteq I$ and $1 \in I$ but $2 \notin I$. Note that H is not a positive implicative hyper K-algebra, since $\{1, 2\} = (1 \circ 2) \circ 0 \neq (1 \circ 0) \circ (2 \circ 0) = \{0, 1, 2\}$.

(ii) In part (ii) of above theorem the condition "$I \neq \{0\}$" can not be omitted, since hyper K-algebra H of Example 3.9 (ii) is positive implicative and normal and $I = \{0\}$ is weak hyper K-ideal but is not weak implicative hyper K-ideal, since $2 \circ (1 \circ 2) = \{0\} \subseteq I$ but $2 \notin I$.

Theorem 3.22. *Let H be an implicative hyper K-algebra that satisfies the strong transitive condition and let I be a hyper K-ideal of H. Then I is an implicative hyper K-ideal.*

Proof. Let $x \circ (y \circ x) < I$. Since H is implicative, then $x \in x \circ (y \circ x)$. Hence $x < I$ and I is a hyper K-ideal, so $x \in I$. Thus by Theorem 2.13, I is an implicative hyper K-ideal. □

Note that the example given in Remark 3.16 shows that the "strong transitive condition" is necessary in the above proposition.

Theorem 3.23. *Let H be a hyper K-algebra of order 3 and $0 \in H$ a right scalar element. If $I = \{0\}$ is an implicative hyper K-ideal, then H is a strong implicative hyper K-algebra.*

Proof. Since 0 is a right scalar element, it is enough to show that $x \in x \circ (y \circ x)$ for all $x, y \in H$. To do this consider the following cases:

(i) If $x = 0$, then it is clear that $0 \in 0 \circ (y \circ 0)$ for all $y \in H$.

(ii) If $x = 1$, we consider three cases: (a) if $y = 0$, then $1 \in 1 \circ 0 \subseteq 1 \circ (0 \circ 1)$. (b) if $y = 1$, then $1 \in 1 \circ 0 \subseteq 1 \circ (1 \circ 1)$. (c) Let $y = 2$, consider two cases $2 < 1$ and $2 \not< 1$. If $2 < 1$, then $0 \in 2 \circ 1$. Therefore $1 \in 1 \circ 0 \subseteq 1 \circ (2 \circ 1)$. If $2 \not< 1$, then $0 \notin 2 \circ 1$. Thus $2 \circ 1 = \{1\}$, $\{2\}$ or $\{1, 2\}$. If $2 \circ 1 = \{1\}$, then $0 \in 1 \circ (2 \circ 1)$ and therefore $1 \circ (2 \circ 1) < I$. Since I is implicative, by Theorem 2.13 we conclude that $1 \in I$, which is a contradiction. If $2 \circ 1 = \{2\}$, we show that $1 \circ 2 = \{1\}$.

To do this, we show that $0 \notin 1 \circ 2$ and $2 \notin 1 \circ 2$. If $0 \in 1 \circ 2$, then we have $0 \in 1 \circ 2 = 1 \circ (2 \circ 1)$, so $1 \circ (2 \circ 1) < I$. Since I is implicative, by Theorem 2.13 we conclude that $1 \in I$, which is a contradiction. If $2 \in 1 \circ 2$, then $0 \in 2 \circ (1 \circ 2)$, therefore $2 \circ (1 \circ 2) < I$. Since I is implicative, by Theorem 2.12 we conclude that $2 \in I$, which is a contradiction. Therefore $1 \circ 2 = \{1\}$, so $1 \in 1 \circ (2 \circ 1)$. Now, let $2 \circ 1 = \{1, 2\}$. Hence $0 \in (1 \circ 1) \cup (1 \circ 2) = 1 \circ \{1, 2\} = 1 \circ (2 \circ 1)$, thus $1 \circ (2 \circ 1) < I$. Since I is implicative, by Theorem 2.13 we conclude that $1 \in I$, which is a contradiction.

(iii) If $x = 2$ then by the same argument as in the case (ii) we can obtain that $2 \in 2 \circ (y \circ 2)$ for all $y \in H$. □

Remark 3.24. If in the above theorem we replace $I = \{0\}$ by $I = \{0, 1\}$, then the theorem does not hold. Let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

Here $0 \in H$ is a scalar element and $I = \{0, 1\}$ is an implicative hyper K-ideal, but H is not an implicative hyper$K$-algebra since $1 \notin 1 \circ (2 \circ 1)$.

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