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ABELIAN GROUPS WHICH HAVE TRIVIAL ABSOLUTE COGALOIS GROUP

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Abstract. In this article we characterize those abelian groups for which the coGalois group (associated to a torsion free cover) is equal to the identity.

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1. Introduction

Given a field k and an algebraic closure $k \subset \Omega$ of k we have the associated absolute Galois group, i.e. the group of automorphisms of Ω that leave k fixed. This group is trivial if and only if k is separably algebraically closed. The notion of an absolute Galois group can be defined in any category where we have an enveloping class (the enveloping class in the above is the class of algebraically closed fields). The dual notion of an absolute coGalois group arises when we have a covering class. In this paper we will classify the abelian groups whose absolute coGalois group is trivial relative to the covering class of torsion free abelian groups (in the category of abelian groups).

2. Abelian groups which have trivial absolute coGalois group

Definition 2.1. If $\mathscr{C}$ is any category and $\mathscr{F}$ is a class of objects of $\mathscr{C}$, then by an $\mathscr{F}$-precover of an object X of $\mathscr{C}$ we mean a morphism $\varphi\colon F \to X$ where $F \in \mathscr{F}$ and where $\operatorname{Hom}(G, F) \to \operatorname{Hom}(G, X)$ is surjective for all $G \in \mathscr{F}$. Moreover, if every morphism $\varrho\colon F \to F$ such that $\varphi \circ \varrho = \varphi$ is an automorphism, then we say φ is an $\mathscr{F}$-cover of X. We say that $\mathscr{F}$ is a covering if every object admits an $\mathscr{F}$-cover.

If $\varphi\colon F \to X$ is an $\mathscr{F}$-cover, the group of $\varrho\colon F \to F$ such that $\varphi \circ \varrho = \varphi$ is called the (absolute) coGalois group of the cover and is denoted $G(\varphi)$ (this notion was introduced in [4, Definition 4.1]). The dual notions are $\mathscr{F}$-preenvelope, $\mathscr{F}$-envelope and Galois group of an $\mathscr{F}$-envelope $\varphi\colon X \to F$.

In [1] it was shown that every abelian group has a torsion free cover, which is unique up to isomorphism and so that the class of torsion free groups is covering in the category of abelian groups. So if $\varphi\colon T \to A$ is a torsion free cover of the abelian group A we will let $G(\varphi)$ denote the absolute or simply the coGalois group of the cover.

We recall that an abelian group C is cotorsion when $\mathrm{Ext}^1(F, C) = 0$ for all torsion free abelian groups F. If $\varphi\colon T \to A$ is a torsion free cover of A with kernel K, a special case of Wakamatsu's Lemma ([7] or [8, Lemma 2.1.1]) gives that K is a cotorsion group. Furthermore, it can be easily seen that no nonzero direct summand of T can be contained in K. In particular, K is reduced (i.e. K has no nonzero divisible subgroups). In this way, the kernel of a torsion free cover of an abelian group is a reduced, torsion free and cotorsion group.

Lemma 2.2. *Let $\varphi\colon T \to A$ be a torsion free cover of A. The following conditions are equivalent:*
(1) $G(\varphi) = \{1\}$ (i.e. $\{\mathrm{id}_T\}$);
(2) $\mathrm{Hom}(T, \ker \varphi) = 0$.

Proof. (1) $\Rightarrow$ (2) Let $\varrho\colon T \to T$ be an homomorphism such that $\mathrm{Im}\colon\ \varrho \subseteq \ker \varphi$. Then $\varrho + \mathrm{id}_T$ satisfies $\varphi(\varrho + \mathrm{id}_T) = \varphi\varrho + \varphi = \varphi$. Hence $\varrho + \mathrm{id}_T \in G(\varphi) = \{\mathrm{id}_T\}$ and so $\varrho = 0$.

(2) $\Rightarrow$ (1) Let $\sigma\colon T \to T$ be a homomorphism such that $\varphi\sigma = \varphi$. Since $\varphi(\sigma - \mathrm{id}_T) = \varphi\sigma - \varphi = 0$ then $\mathrm{Im}(\sigma - \mathrm{id}_T) \subseteq \ker \varphi$ and consequently, $\sigma = \mathrm{id}_T$. $\square$

By the uniqueness (up to isomorphism) of the torsion free covers, it is clear that if $\varphi\colon T \to A$ and $\varphi'\colon T' \to A$ are torsion free covers of A then $G(\varphi) \cong G(\varphi')$. So from now on, we refer to the coGalois group of any torsion free cover of A, as the coGalois group of A, denoted by $G(A)$. We can reformulate our main question as follows: for which abelian groups A is $G(A) = \{1\}$?

Proposition 2.3. *If D is a divisible group then $G(D) = \{1\}$.*

Proof. Let $\varphi\colon T \to D$ be a torsion free cover of D. By [2, Corollary 1], T is divisible. But then $\mathrm{Hom}(T, \ker \varphi) = \{0\}$ since $\ker \varphi$ is reduced and epimorphic image of a divisible group is divisible. The result follows by Lemma 2.2. $\square$

It turns out that the converse of our previous proposition holds for torsion abelian groups, as we shall see in the following results.

Lemma 2.4. *Let A be an abelian group such that $G(A) = \{1\}$. Then $G(B) = \{1\}$ for every direct summand B of A.*

P r o o f. Assume that $A = B \oplus C$ and let $\varphi\colon U \to B$ and $\psi\colon V \to C$ be torsion free covers of B and C, respectively. Then by [3, Proposition 4.1], $U \oplus V \overset{\varphi\oplus\psi}{\longrightarrow} B \oplus C = A$ is a torsion free cover of A. Since $G(A) = \{1\}$ then $\operatorname{Hom}(U \oplus V, \ker\varphi \oplus \ker\psi) = 0$ and so $\operatorname{Hom}(U, \ker\varphi) = \operatorname{Hom}(V, \ker\psi) = 0$. Consequently, $G(B) = G(C) = \{1\}$. $\square$

For a prime number p, we denote by $\hat{\mathbb{Z}}_p$ the group of p-adic numbers. It is well known that for every integer $n \geqslant 1$, the canonical epimorphism $\hat{\mathbb{Z}}_p \to \mathbb{Z}/(p^n)$ is a torsion free cover of $\mathbb{Z}/(p^n)$ ([8, Proposition 4.1.6]).

Proposition 2.5. *Let A be a torsion abelian group such that $G(A) = \{1\}$. Then A is divisible.*

P r o o f. Let $A = \bigoplus\limits_p A(p)$ (over all primes p), where $A(p)$ is the p-primary part of A (i.e. all $x \in A$ such that $p^k x = 0$ for some integer $k \geqslant 0$). Choose a prime p and let $A' = \bigoplus\limits_{q\neq p} A(q)$. Then $A = A(p) \oplus A'$. If $A(p)$ is not divisible then $A(p)$ has a direct summand isomorphic to $\mathbb{Z}/(p^n)$, for some $n \geqslant 1$ ([6]). Since $G(A) = \{1\}$, it follows from Lemma 2.4, that $G(\mathbb{Z}/(p^n)) = \{1\}$. But this is a contradiction since $\hat{\mathbb{Z}}_p \to \mathbb{Z}/(p^n)$ is a torsion free cover of $\mathbb{Z}/(p^n)$ with kernel $p^n\hat{\mathbb{Z}}_p$ and $\operatorname{Hom}(\hat{\mathbb{Z}}_p, p^n\hat{\mathbb{Z}}_p) \neq 0$. Hence $A(p)$ is divisible for each prime p and so A is divisible. $\square$

Proposition 2.6. *Let $t(A)$ be the torsion subgroup of the abelian group A. If $G(A) = \{1\}$ then $G(t(A)) = \{1\}$.*

P r o o f. Let $\varphi\colon T \to A$ be a torsion free cover of A with kernel K. By [5, Proposition 3.1] we know that $\varphi^{-1}(t(A)) \to t(A)$ is a torsion free cover of $t(A)$ with kernel K. From the short exact sequence

$$0 \to \varphi^{-1}(t(A)) \to T \to T/\varphi^{-1}(t(A)) \to 0$$

we get the exact sequence

$$\operatorname{Hom}(T, K) \to \operatorname{Hom}(\varphi^{-1}(t(A)), K) \to \operatorname{Ext}^1(T/\varphi^{-1}(t(A)), K).$$

But $T/\varphi^{-1}(t(A)) \cong A/t(A)$ is a torsion free abelian group and K is a cotorsion group. Consequently, $\operatorname{Ext}^1(T/\varphi^{-1}(t(A)), K) = 0$. On the other hand, since $G(A) = \{1\}$ it follows that $\operatorname{Hom}(T, K) = 0$. Hence $\operatorname{Hom}(\varphi^{-1}(t(A)), K) = 0$ and so $G(t(A)) = \{1\}$. $\square$

Let A be an abelian group with torsion subgroup $t(A)$. We can decompose $t(A)$ into its p-primary parts $t(A) = \bigoplus_p t(A)(p)$. For each p let $U(p) \to t(A)(p)$ be a torsion free cover of $t(A)(p)$. It was shown in [5, Section 3] that $\ker(U(p) \to t(A)(p)) = T_p$, where T_p is a direct summand of a product of copies of $\hat{\mathbb{Z}}_p$. Note that $T_p \to T_p/pT_p$ is a torsion free cover of T_p/pT_p ([8, Proposition 4.1.6]).

Lemma 2.7. *Let A be an abelian group, $t(A)$ its torsion subgroup and $\varphi\colon U \to t(A)$ a torsion free cover of $t(A)$. The following conditions are equivalent:*
(1) $\operatorname{Hom}(A/t(A), \ker\varphi) = 0$;
(2) $p(A/t(A)) = A/t(A)$ *for every prime p such that $t(A)(p) \neq 0$.*

P r o o f. (1) $\Rightarrow$ (2) Let p be a prime such that $t(A)(p) \neq 0$ and $U(p) \to t(A)(p)$ be a torsion free cover of $t(A)(p)$ with kernel T_p. The fact that $\operatorname{Hom}(A/t(A), \ker\varphi) = 0$ implies that $\operatorname{Hom}(A/t(A), T_p) = 0$. On the other hand, $t(A)(p) \neq 0$ if and only if $T_p \neq 0$ if and only if $pT_p \neq T_p$. So assume that $p(A/t(A)) \neq A/t(A)$. Then there exists a nonzero homomorphism $(A/t(A))/(p(A/t(A))) \longrightarrow T_p/pT_p$ since $(A/t(A))/(p(A/t(A)))$ and T_p/pT_p are nonzero vector spaces over $\mathbb{Z}/(p)$ (and so, direct sum of $\mathbb{Z}/(p)$'s). Since $T_p \to T_p/pT_p$ is a torsion free cover of T_p/pT_p, we can complete the diagram

$$\begin{tikzcd} A/t(A) \arrow[r, dashed] \arrow[d] & T_p \arrow[d] \\ (A/t(A))/(p(A/t(A))) \arrow[r] & T_p/pT_p \end{tikzcd}$$

with a (nonzero) homomorphism $A/t(A) \to T_p$. This yields a contradiction. Consequently, $p(A/t(A)) = A/t(A)$.

(2) $\Rightarrow$ (1) It is clearly sufficient to prove that $\operatorname{Hom}(A/t(A), T_p) = 0$ for all prime p such that $T_p \neq 0$. If p is such prime then $t(A)(p) \neq 0$ and so $p(A/t(A)) = A/t(A)$. Since T_p is a direct summand of a product of copies of $\hat{\mathbb{Z}}_p$, for every nonzero homomorphism $A/t(A) \to T_p$ we can construct a nonzero homomorphism $A/t(A) \to T_p \to \hat{\mathbb{Z}}_p$, which implies according to [5, Lemma 3.4] that $p(A/t(A)) \neq A/t(A)$. In consequence, $\operatorname{Hom}(A/t(A), T_p) = 0$. $\square$

Now we can prove our main result.

Theorem 2.8. *Let A be an abelian group with torsion subgroup $t(A)$. The following conditions are equivalent:*
(1) $G(A) = \{1\}$;
(2) $t(A)$ *is divisible and* $p(A/t(A)) = A/t(A)$ *for every prime p such that $t(A)(p) \neq 0$.*

Proof. Let $\varphi\colon U \to t(A)$ be a torsion free cover of $t(A)$.

(1) $\Rightarrow$ (2) If $G(A) = \{1\}$ then by Proposition 2.6, $G(t(A)) = \{1\}$. It follows from Proposition 2.5 that $t(A)$ is divisible. In consequence, $A = t(A) \oplus L$ for a torsion free abelian group L and $U \oplus L \to t(A) \oplus L = A$ is a torsion free cover of A such that $\ker(U \oplus L \to t(A) \oplus L) \cong \ker\varphi$. Since $G(A) = \{1\}$ we know that $\operatorname{Hom}(U \oplus L, \ker\varphi) = 0$ which implies $0 = \operatorname{Hom}(L, \ker\varphi) \cong \operatorname{Hom}(A/t(A), \ker\varphi)$. The result follows from Lemma 2.7.

(2) $\Rightarrow$ (1) $t(A)$ divisible implies $A = t(A) \oplus L$ for a torsion free abelian group L. By Proposition 2.3, $\operatorname{Hom}(U, \ker\varphi) = 0$ and by our previous lemma, $\operatorname{Hom}(L, \ker\varphi) = 0$. It follows that $\operatorname{Hom}(U \oplus L, \ker(U \oplus L \to t(A) \oplus L)) \cong \operatorname{Hom}(U \oplus L, \ker\varphi) = 0$. Hence $G(A) = \{1\}$ since $U \oplus L \to t(A) \oplus L$ is a torsion free cover of A. $\square$

Example 2.9. Let X be any set of primes and let $A = \Big(\bigoplus_{p \in X} \mathbb{Z}(p^\infty)\Big) \oplus U$ where $U \subset \mathbb{Q}$ is generated by all $1/p^n$ for $p \in X$, $n \geqslant 1$. Then $G(A) = \{1\}$.

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