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REMARKS ON RESTRAINED DOMINATION AND TOTAL RESTRAINED DOMINATION IN GRAPHS

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Abstract. The restrained domination number $\gamma^r(G)$ and the total restrained domination number $\gamma_t^r(G)$ of a graph G were introduced recently by various authors as certain variants of the domination number $\gamma(G)$ of (G). A well-known numerical invariant of a graph is the domatic number $d(G)$ which is in a certain way related (and may be called dual) to $\gamma(G)$. The paper tries to define analogous concepts also for the restrained domination and the total restrained domination and discusses the sense of such new definitions.

Keywords: domination number, domatic number, total domination number, total domatic number, restrained domination number, restrained domatic number, total restrained domination number, total restrained domatic number

MSC 2000: 05C35, 05C69

The research of the domination in graphs has been an evergreen of the graph theory. Its basic concept is the dominating set and the domination number. A numerical invariant of a graph which is in a certain sense dual to it is the domatic number of a graph. And many variants of the dominating set were introduced and the corresponding numerical invariants were defined for them. Here we will study the restrained dominating set [4, 5] and the total restrained dominating set [1]. We consider finite undirected graphs without loops and multiple edges.

We start with definitions of various concepts concerning the domination in graphs. A subset $S \subseteq V(G)$ is called a dominating set (or a total dominating set) in G, if for each $x \in V(G) - S$ (or for each $x \in V(G)$, respectively) there exists a vertex $y \in S$ adjacent to x. A dominating set in G is called a restrained dominating set

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in G, if each vertex $x \in V(G) - S$ is adjacent both to a vertex $y \in S$ and to a vertex $z \in V(G) - S$. A set S which is simultaneously total dominating and restrained dominating in G is called a total restrained dominating set in G. The minimum number of vertices of a dominating set in a graph G is the domination number $\gamma(G)$ of G. Analogously the total domination number $\gamma_t(G)$, the restrained domination number $\gamma^r(G)$ and the total restrained domination number $\gamma_t^r(G)$ are defined.

The domatic number of a graph was introduced in [2] and the total domatic number in [3]. In an analogous way we will define the restrained domatic number and the total restrained domatic number and then we will discuss the purpose of defining them. Let $\mathscr{D}$ be a partition of the vertex set $V(G)$ of G. If all classes of $\mathscr{D}$ are dominating sets (or total dominating sets) in G, then $\mathscr{D}$ is called a domatic (or total domatic, respectively) partition of G. Quite analogously we may go on. If all classes of $\mathscr{D}$ are restrained dominating sets (or total restrained dominating sets) in G then $\mathscr{D}$ is called a restrained domatic (or total restrained domatic, respectively) partition of G.

The maximum number of classes of a domatic partition of G is the domatic number $d(G)$ of G. Analogously the total domatic number $d_t(G)$, the restrained domatic number $d^r(G)$ and the total restrained domatic number $d_t^r(G)$ are defined. Note that $d^r(G)$ is well-defined for all graphs, so as $d(G)$ is, while $d_t^r(G)$ is well-defined for all graphs without isolated vertices, so as $d_t(G)$ is. The sense of introducing $d_t^r(G)$ is brought into doubt by the following theorem.

Theorem 1. *Let G be a graph without isolated vertices. Then $d_t^r(G) = d_t(G)$.*

Proof. Each total restrained dominating set in G is a total dominating set in G; therefore each total restrained domatic partition of G is a total domatic partition of G and $d_t^r(G) \leqslant d_t(G)$. Now denote $d(G)$ by d and let $\mathscr{D}$ be a total domatic partition of G with d classes $D_1, \ldots, D_d$. Choose a class of $\mathscr{D}$, without loss of generality let it be D_1. Let $x \in V(G)$. As D_1 is a total dominating set in G, there exists $y \in D_1$ which is adjacent to x. Now suppose $x \in V(G) - D_1$. Then $x \in D_i$ for some $i \in \{2, \ldots, d\}$. The set D_i is also a total dominating set in G, therefore there exists $z \in D_i$ adjacent to x and evidently $z \in V(G) - D_1$, because $D_1 \cap D_i = \emptyset$. We have proved that D_1 is a total restrained dominating set in G. The set D_1 was chosen arbitrarily, therefore $\mathscr{D}$ is a total restrained domatic partition of G and $d_t(G) \leqslant d_t^r(G)$, which together with the former inequality gives the required result. $\square$

The following theorem is analogous, only a little more complicated.

Theorem 2. *Let G be a graph, let $d(G) \geqslant 3$. Then $d^r(G) = d(G)$.*

P r o o f. Each restrained dominating set in G is a dominating set in G; therefore each restrained domatic partition of G is a domatic partition of G and $d^r(G) \leqslant d(G)$. Now denote $d(G)$ by d and let $= \{D_1, \ldots, D_d\}$ be a domatic partition of G with d classes. Choose a class of $\mathscr{D}$; without loss of generality let it be D_1. Let $x \in V(G) - D_1 = \bigcup_{i=2}^{d} D_i$. Without loss of generality let $x \in D_2$. As D_1 is a dominating set in G, there exists $y \in D_1$ adjacent to x. Also D_3 is a dominating set in G and therefore there exists $z \in D_3$ adjacent to x. We have $z \in V(G) - D_1$, because $D_1 \cap D_3 = \emptyset$. We have proved that D_1 is a restrained dominating set in G. The set D_1 was chosen arbitrarily, therefore $\mathscr{D}$ is a restrained dominating set in G and $d^r(G) \geqslant d(G)_\gamma$, which together with the former inequality gives the required result. □

The case $d(G) \leqslant 2$ will be treated separately.

Theorem 3. *Let G be a graph, let $d(G) \leqslant 2$. If G has no isolated vertex, then $d^r(G) = d_t(G)$, otherwise $d^r(G) = 1$.*

P r o o f. If G has no isolated vertex, then $d_t^r(G)$ is well-defined and obviously $d^r(G) \leqslant d(G) \leqslant 2$. As any restrained dominating set in G is a dominating set in G, we have also $d^r(G) \leqslant d(G) \leqslant 2$. Suppose $d(G) = 2$ and let $\{D_1, D_2\}$ be a total domatic partition of G with two classes. Let $x \in D_1$. There exists $y \in V(G) - D_1 = D_2$ adjacent to x. As D_2 is a total dominating set in G, there exists $z \in D_2$ adjacent to y. Therefore D_1 is a restrained dominating set in G; analogously we prove that so is D_2 and thus $\{D_1, D_2\}$ is a restrained domatic partition of G and $d^r(G) = 2 = d_t(G)$. Now suppose $d^r(G) = 2$ and let $\{D_1', D_2'\}$ be a restrained domatic partition of G with two classes. Each vertex of D is adjacent to a vertex of D_1' and to a vertex of D_2', because D_2' is a restrained dominating set in G. Analogously also each vertex of D_2' is adjacent to a vertex of $V(G) - D_2' \equiv D_1'$ and to a vertex of D_2'. Both sets D_1', D_2' are total dominating sets in G and $\{D_1', D_2'\}$ is a total domatic partition of G and $d_t(G) = 2 = d^r(G)$. We have proved that $d^r(G) = 2$ if and only if $d_t(G) = 2$. If $d(G) \leqslant 2$, then there is only one other possibility $d^r(G) = 1$ and $d_t(G) = 1$, therefore $d^r(G) = d_t(G)$ again. If G contains an isolated vertex r, then all dominating sets in G contain r and therefore no two of them are disjoint. We have $d(G) = 1$ and thus also $d^r(G) = 1$. □

The numbers $\gamma^r(G)$ and $\gamma_t^r(G)$ where studied in [1], [5], [6]. An interesting motivation for the research of $\gamma_t^r(G)$ is in [1] in applications in guarding prisons. But the concept of our paper shows that probably there is no reason to introduce $d^r(G)$ and $d_t^r(G)$ as new numerical invariants of graphs.

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