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ON COMMUTATIVE TWISTED GROUP RINGS

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Abstract. Let G be an abelian group, R a commutative ring of prime characteristic p with identity and R_tG a commutative twisted group ring of G over R. Suppose p is a fixed prime, G_p and $S(R_tG)$ are the p-components of G and of the unit group $U(R_tG)$ of R_tG, respectively. Let R^* be the multiplicative group of R and let $f_\alpha(S)$ be the α-th Ulm-Kaplansky invariant of $S(R_tG)$ where α is any ordinal. In the paper the invariants $f_n(S)$, $n \in \mathbb{N} \cup \{0\}$, are calculated, provided $G_p = 1$.

Further, a commutative ring R with identity of prime characteristic p is said to be multiplicatively p-perfect if $(R^*)^p = R^*$. For these rings the invariants $f_\alpha(S)$ are calculated for any ordinal α and a description, up to an isomorphism, of the maximal divisible subgroup of $S(R_tG)$ is given.

Keywords: unit groups, isomorphism, Ulm-Kaplansky invariants, commutative twisted group rings

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1. Introduction

Hereafter, unless otherwise stated, R denotes a commutative ring with identity of prime characteristic p and G a multiplicatively written abelian group. Let RG be the group ring (group algebra) of G over R and let R_tG be a commutative twisted group ring of G over R. Denote by $U(R_tG)$ the multiplicative group of R_tG, i.e. the unit group of R_tG and by $V = V(RG)$ the group of normalized units (i.e. of augmentation 1) in RG. Let $V_p(RG)$ be the p-component of V. It is well known and quite elementary that if G is an abelian p-group, then $V_p(RG) = V(RG)$.

S. D. Berman [1] gives a description of $V(RG)$, provided G is a countable abelian p-group and R is a field of characteristic p such that if G is not a direct sum of cyclic groups then R is a perfect field, i.e. $R^p = R$, where $R^p = \{r^p \mid r \in R\}$. In [6] and [7]

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the Ulm-Kaplansky invariants $F_\alpha(V_p)$ of the group $V_p = V_p(RG)$ are calculated when G is an arbitrary abelian group and R is a field of positive characteristic p, and a description, up to an isomorphism, of the maximal divisible subgroup of V_p is given. A. A. Bovdi and Z. F. Pataj [3] calculate $f_\alpha(V_p)$ when R is a commutative ring of prime characteristic p without nilpotent elements and G is an abelian group, under the following restriction for R and G (which is valid in the case when R is a field, too): if the maximal divisible subgroup dG_p of G_p is not identity, then $R^p = R$, i.e. R is a perfect (p-divisible) ring. In [5] a description of V_p is given, when the ring R is finite and G is a finite abelian p-group, and in [11] when G is an arbitrary abelian p-group, R is a ring (of characteristic p) and $|G|\,|R| \geqslant \aleph_0$.

Let $S(R_tG) = U_p(R_tG)$, i.e. $S(R_tG)$ is the p-component of the multiplicative group $U(R_tG)$ of the commutative twisted group ring R_tG. The groups $U(R_tG)$ and $S(R_tG)$ play an important role in many investigations of the commutative twisted group rings but descriptions of $U(R_tG)$ and $S(R_tG)$, up to an isomorphism, are given only in some partial cases. In [13] a description of $U(K_tG)$, up to an isomorphism, is given when G is a cyclic p-group, p is odd and K is a field of characteristic different from p. In [9] $U(K_tG)$ is described when G is a cyclic p-group, K is a field of characteristic different from p and K is of the second kind with respect to p (for the definition of a field of the second kind with respect to p see [14, p. 683]). In [12] a description of $S(R_tG)$ is given when G is a finite abelian p-group and K is an infinite field of characteristic p.

When G is an infinite abelian group then one usually calculates the invariants $f_\alpha(S)$ of the group $S = S(R_tG)$. In many cases these invariants, together with the maximal divisible subgroup of S, give a complete description, up to an isomorphism, of the group S. Other invariants which could be calculated are the invariants used for the description of the $A_n(\mu)$-groups of W. Ullery [16] but we will note that they are constructed inductively on the basis of the Ulm-Kaplansky invariants.

Our basic aims are for a multiplicatively p-perfect ring R: 1) to calculate the invariants $f_\alpha(S)$ of $S = S(R_tG)$ and 2) to give a description, up to an isomorphism, of the maximal divisible subgroup of S.

Let p be a fixed prime number, $R(p) = \{r \in R \mid r^p = 0\}$ and let α be an arbitrary ordinal. We define inductive R^{p^α} as follows: $R^{p^0} = R$, $R^p = \{r^p \mid r \in R\}$,

$$R^{p^\alpha} = \begin{cases} (R^{p^{\alpha-1}})^p, & \text{if } \alpha - 1 \text{ exists;} \\ \bigcap\limits_{\beta<\alpha} R^{p^\beta}, & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Obviously R^{p^α} is a subring of R, $R^{p^\alpha} \supseteq R^{p^{\alpha+1}}$ and $1 \in R^{p^\alpha}$. It can be proved that where $\operatorname{rad} R$ is the radical of Baer of R then $(\operatorname{rad} R)^{p^\alpha} = \operatorname{rad} R^{p^\alpha}$. Denote by

$R^{p^\alpha}(p)/R^{p^{\alpha+1}}(p)$ an additive factor group, by $|M|$ the cardinality of the set M and by $\mathbb{N}_0$ the set $\mathbb{N}\cup\{0\}$. In the next theorem RG is a group ring of the group G over R and R is a commutative ring of an arbitrary characteristic. We recall that G is p-divisible, if $G^p = G$ [4, p. 118].

D. Passman [14, p. 20, Lemma 2.5] proves that if R is an algebraically closed field then an arbitrary commutative twisted group algebra R_tA is diagonally equivalent to the group algebra RA. It is also well known that if A is a free abelian group, then an arbitrary commutative twisted group algebra R_tA is isomorphic to the group algebra RA as R-algebras. In the following theorem we give two more cases in which an isomorphism $R_tA \cong RA$ as R-algebras holds.

Theorem 1. *Let R be a commutative ring with identity and G an abelian group. Suppose at least one of the following conditions holds:*

(a) *G is an abelian p-group and R^* is a p-divisible group;*

(b) *R^* is a divisible group.*

Then there exists a subgroup G' of $U(R_tG)$ such that $R_tG = RG'$ and $G' \cong G$, i.e. $R_tG \cong RG$ as R-algebras.

In the following theorem we calculate the invariants $f_n(S)$ of S for any $n \in \mathbb{N}_0$, when $G_p = 1$ (G_p is the p-component of G).

Theorem 2. *Let G be an abelian group, let R be a commutative ring with identity of prime characteristic p and $G_p = 1$, $n \in \mathbb{N}_0$. Then*

(i) *if $R^{p^n}(p) = 0$, then $f_n(S) = 0$;*

(ii) *if $R^{p^n}(p) = R^{p^{n+1}}(p) \neq 0$ and $G = G^p$, then $f_n(S) = 0$;*

(iii) *if $R^{p^n}(p) \neq 0$ and $G \neq G^p$, then*

$$f_n(S) = \max(|R^{p^n}(p)|, |G|);$$

(iv) *if $R^{p^n}(p) \neq R^{p^{n+1}}(p)$ and $|G|\,|R^{p^n}(p)/R^{p^{n+1}}(p)| < \aleph_0$, then*

$$f_n(S) = |G| \log_p |R^{p^n}(p)/R^{p^{n+1}}(p)|;$$

(v) *if $R^{p^n}(p) \neq R^{p^{n+1}}(p), G = G^p$ and $|G|\,|R^{p^n}(p)/R^{p^{n+1}}(p)| \geqslant \aleph_0$, then*

$$f_n(S) = \max(|R^{p^n}(p)/R^{p^{n+1}}(p)|, |G|).$$

For a multiplicatively p-perfect ring R the following assertions hold.

Proposition 3. *If R is a multiplicatively p-perfect ring and $R(p) \neq 0$ then $|R(p)| = |\operatorname{rad} R| \geqslant \aleph_0$.*

Theorem 4. *Let p be a fixed prime number, let G be an abelian group and let R be a multiplicatively p-perfect ring. Denote by F the p-component G_p of G. Then*
(1) *if $F^{p^\alpha} = 1$ and $\operatorname{rad} R = 0$, then $f_\alpha(S) = 0$;*
(2) *if $F^{p^\alpha} = 1$, $G^{p^\alpha} = G^{p^{\alpha+1}}$ and $\operatorname{rad} R \neq 0$, then $f_\alpha(S) = 0$;*
(3) *if $F^{p^\alpha} = F^{p^{\alpha+1}} \neq 1$, $R^{p^\alpha} = R^{p^{\alpha+1}}$ and $G^{p^\alpha}/F^{p^\alpha} = (G^{p^\alpha}/F^{p^\alpha})^p$, then $f_\alpha(S) = 0$;*
(4) *if $F^{p^\alpha} = 1$, $\operatorname{rad} R \neq 0$ and $G^{p^\alpha} \neq G^{p^{\alpha+1}}$, then*

$$f_\alpha(S) = \max(|\operatorname{rad} R|, |G^{p^\alpha}|);$$

(5) *if $F^{p^\alpha} \neq 1$ and $G^{p^\alpha}/F^{p^\alpha} \neq (G^{p^\alpha}/F^{p^\alpha})^p$, then*

$$f_\alpha(S) = \max(|R^{p^\alpha}|, |G^{p^\alpha}|);$$

(6) *if $1 < |G^{p^\alpha}| < \aleph_0$ and $|R^{p^\alpha}| < \aleph_0$, then*

$$\begin{aligned} f_\alpha(S) &= |G^{p^\alpha}/F^{p^\alpha}| f_\alpha(V), \\ f_\alpha(V) &= (|F^{p^\alpha}| - 1)\log_p |R^{p^\alpha}| - 2(|F^{p^{\alpha+1}}| - 1)\log_p |R^{p^{\alpha+1}}| \\ &\quad + (|F^{p^{\alpha+2}}| - 1)\log_p |R^{p^{\alpha+2}}|; \end{aligned}$$

(7) *if $F^{p^\alpha} \neq F^{p^{\alpha+1}}$, $G^{p^\alpha}/F^{p^\alpha} = (G^{p^\alpha}/F^{p^\alpha})^p$ and $|G^{p^\alpha}|\,|R^{p^\alpha}| \geqslant \aleph_0$, then*

$$f_\alpha(S) = \max(|R^{p^\alpha}|, |G^{p^\alpha}|)$$

and
(8) *if $F^{p^\alpha} = F^{p^{\alpha+1}} \neq 1$, $G^{p^\alpha}/F^{p^\alpha} = (G^{p^\alpha}/F^{p^\alpha})^p$ and $R^{p^\alpha} \neq R^{p^{\alpha+1}}$, then*

$$f_\alpha(S) = \max(|R^{p^\alpha}/R^{p^{\alpha+1}}|, |G^{p^\alpha}|),$$

where $R^{p^\alpha}/R^{p^{\alpha+1}}$ is an additive factor group.

In the following theorem we denote by $\coprod_\lambda H$ the restricted direct product of λ copies of the group H where λ is a cardinal number, and by $Z(p^\infty)$ the quasicyclic group of type p^∞. If $\lambda = 0$ we set this direct product to be equal to identity.

Theorem 5. *Let p be a fixed prime number, let G be an abelian group and let R be a multiplicatively p-perfect ring. Denote by F the p-component of G and by α the first ordinal such that $R^{p^\alpha} = R^{p^{\alpha+1}}$ and $G^{p^\alpha} = G^{p^{\alpha+1}}$. Then the maximal divisible subgroup dS of $S(R_tG)$ is $S(R^{p^\alpha}{}_tG^{p^\alpha})$ and $dS \cong \coprod_\lambda Z(p^\infty)$, where*

$$\lambda = \begin{cases} \max(|G^{p^\alpha}|, |R^{p^\alpha}|), & \textit{if } F^{p^\alpha} \neq 1; \\ \max(|G^{p^\alpha}|, |\operatorname{rad} R|), & \textit{if } F^{p^\alpha} = 1 \ \textit{and} \ \operatorname{rad} R \neq 0; \\ 0, & \textit{if } F^{p^\alpha} = 1 \ \textit{and} \ \operatorname{rad} R = 0. \end{cases}$$

This paper is organized as follows. In Section 2 we give notation, definitions, some preliminary known and unknown results, and we prove Theorem 1. In Section 3 we introduce the class of multiplicatively p-perfect rings and prove Proposition 3 and some assertions which are connected with this class. In Section 4 we prove characteristic lemmas which are necessary for the proofs of Theorems 2, 4 and 5. In Section 5 we prove these theorems.

2. Definitions, preliminary results and proof of Theorem 1

Let A be a commutative semigroup with identity, let p be a fixed prime number and let A^* be the group of the invertible elements of A, i.e. the multiplicative group of A. We set

$$A_p = \{a \in A \mid a^{p^n} = 1,\ n \in \mathbb{N}_0\},\ A_p = S(A),\ \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$
$$A^p = \{a^p \mid a \in A\},\ A[p] = \{a \in A \mid a^p = 1\}.$$

Obviously $A[p]$ is a subgroup of A^*. Denote by R a commutative ring with idendity of prime characteristic p and by $R_p = S(R)$ the p-component of the multiplicative group R^* of R. If $B, C \subseteq R$ then we denote $B + C = \{b + c \mid b \in B,\ c \in C\}$. It is easy to see that for the ring R we have $R[p] = 1 + R(p)$, i.e. $S(R)[p] = 1 + R(p)$. Let $\operatorname{rad} R$ be the radical of Baer of R and α an arbitrary ordinal. We define inductive A^{p^α} as follows: $A^{p^0} = A$,

$$A^{p^\alpha} = \begin{cases} (A^{p^{\alpha-1}})^p, & \text{if } \alpha - 1 \text{ exists}; \\ \bigcap\limits_{\beta<\alpha} A^{p^\beta}, & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Obviously A^{p^α} is a subsemigroup of A and $A^{p^\alpha} \supseteq A^{p^{\alpha+1}}$. We can define $S^{p^\alpha}(A)$ as above. Let $Z(p^\infty)$ be the quasicyclic group of type p^∞ and $|M|$ the cardinality of the set M. If H is a subgroup of G then we denote by $\Pi(G/H)$ a transversal of H in G.

Let G be an abelian group. We denote by RG and by R_tG the group ring and a twisted group ring (twisted group algebra) of G over R, respectively. A twisted group ring R_tG of G over R [14, p. 13] is an associative R-algebra with basis $\{\bar{g} \mid g \in G\}$ and multiplication defined on the basis by

$$\bar{g}\bar{h} = (g,h)\overline{gh}, \quad (g,h) \in R^*.$$

The set $\varrho = \{(g,h) \mid g,h \in R^*\}$ is called a factor set of R_tG. The ring R_tG is commutative if and only if $(g,h) = (h,g)$ for any $g \in G$ and $h \in G$. We will use the notation $R_{t_1}G$ for a twisted group ring of G over R when $R_{t_1}G$ is different from the twisted group ring R_tG.

The algebra R_tG has an identity element $e = (1,1)^{-1}\bar{1}$ and every element $\bar{g}$ has an inverse element in R_tG. After replacing the basis element $\bar{1}$ by the identity element e, we can suppose that the factor set ϱ is normalized [15], i.e.

$$(g,1) = (1,g) = 1 \quad (g \in G)$$

and that the identity 1 of R coincides with $\bar{1}$, i.e. $\bar{1} = 1$.

The definition of R_tG yields that for every $g \in G$ and $n \in \mathbb{N}_0$ there exists $[g,n] \in R^*$ such that

$$\bar{g}^{p^n} = [g,n]\overline{g^{p^n}}.$$

In the sequel we will consider only commutative twisted group rings. Since R_tG is a commutative ring of characteristic p we have

$$\left(\sum_{g\in G} x_g\bar{g}\right)^{p^n} = \sum_{g\in G} x_g^{p^n}\bar{g}^{p^n}, \quad x_g \in \mathbb{R}, \ n \in \mathbb{N}_0. \tag{2.1}$$

Our abelian group terminology is in agreement with Fuchs [4]. Let $S = S(R_tG)$ be the p-component of the torsion subgroup of the unit group $U = U(R_tG)$ of R_tG, i.e. S is the Sylow p-subgroup of U. We use analogous notation $U(RG)$ and $S(RG)$ for an ordinary group ring RG. Moreover, we let $V(RG)$ denote the group of normalized units (i.e. of augmentation 1) in RG. If G is an abelian p-group then $V(RG)$ is obviously a subgroup of $S(RG)$ and

$$U(RG) = R^* \times V(RG), \;\; S(RG) = R_p \times V(RG), \;\; S(R_tG)[p] = 1 + (R_tG)(p).$$

We will give some results which are well known.

The following proposition is obtained as a particular case of a general result of A. A. Bovdi and S. V. Mihovski [2] for crossed products of arbitrary groups and rings.

Proposition 2.1. *Let H be a subgroup of an abelian group G and let R be a commutative ring with 1. The commutative twisted group algebra R_tG is R-isomorphic to a commutative twisted group algebra of the factor group G/H over the ring R_tH, i.e.*

$$R_tG \cong (R_tH)_{t_1}(G/H).$$

Theorem 2.2 [8]**.** *If R is a commutative ring with identity of prime characteristic p and $|\operatorname{rad} R| \geqslant \aleph_0$, then $|R(p)| = |\operatorname{rad} R|$.*

Theorem 2.3 [10]**.** *Let T be a subring of the commutative ring R with identity of prime characteristic p such that T contains the identity of R. Then the multiplicative factor group $R[p]/T[p]$ is isomorphic to the additive factor group $R(p)/T(p)$.*

Theorem 2.4 [11]**.** *Let F be an abelian p-group, let R be a commutative ring with identity of characteristic p and let α be an arbitrary ordinal. Then for $V = V(FG)$, the following holds:*

(a) *if $F^{p^\alpha} = 1$, then $f_\alpha(V) = 0$;*
(b) *if $F^{p^\alpha} = F^{p^{\alpha+1}} \neq 1$ and $R^{p^\alpha} = R^{p^{\alpha+1}}$, then $f_\alpha(V) = 0$;*
(c) *if $F^{p^\alpha} = F^{p^{\alpha+1}} \neq 1$ and $R^{p^\alpha} \neq R^{p^{\alpha+1}}$, then*

$$F_\alpha(V) = \max(|R^{p^\alpha}/R^{p^{\alpha+1}}|\,|F^{p^\alpha}|)$$

where $R^{p^\alpha}/R^{p^{\alpha+1}}$ is a factor group of the additive group R^{p^α} and
(d) *if $F^{p^\alpha} \neq F^{p^{\alpha+1}}$ and $|R^{p^\alpha}|\,|F^{p^\alpha}| \geqslant \aleph_0$, then*

$$f_\alpha(V) = \max(|R^{p^\alpha}|, |F^{p^\alpha}|).$$

Theorem 2.5 [5]**.** *Let α be an arbitrary ordinal, let F be an abelian p-group and let F^{p^α} and R^{p^α} be finite. Then*

$$\begin{aligned} f_\alpha(V) = {} & (|F^{p^\alpha}| - 1)\log_p |R^{p^\alpha}| - 2(|F^{p^{\alpha+1}}| - 1)\log_p |R^{p^{\alpha+1}}| \\ & + (|F^{p^{\alpha+2}}| - 1)\log_p |R^{p^{\alpha+2}}|. \end{aligned}$$

The following result was obtained by W. Ullery [16].

Theorem 2.6 [16]. *Let G be an abelian p-group and let R be a commutative ring with identity of characteristic p. If G is an $A_n(\mu)$-group for some positive n and a limit ordinal μ and $RG \cong RH$ as R-algebras for any group H, then $G \cong H$.*

The following two results are proved by standard induction with respect to ordinal α and we omit its proofs.

Lemma 2.7. *If A is a commutative semigroup with identity, p is a fixed prime number and α an arbitrary ordinal, then $(A^{p^\alpha})^* = (A^*)^{p^\alpha}$. If R is a commutative ring with identity of prime characteristic p, then $(R^{p^\alpha})^* = (R^*)^{p^\alpha}$.*

Lemma 2.8. *If A is a commutative semigroup with identity and p is a fixed prime number then for any ordinal α we have $(A^{p^\alpha})_p = (A_p)^{p^\alpha}$, i.e. $S^{p^\alpha}(A) = S(A^{p^\alpha})$. Moreover $(R^{p^\alpha})_p = (R_p)^{p^\alpha}$.*

In view of these two lemmas we will further use the notation $R^{*^{p^\alpha}} = (R^*)^{p^\alpha} = (R^{p^\alpha})^*$ and $R_p^{p^\alpha} = (R^{p^\alpha})_p = (R_p)^{p^\alpha}$ and analogously $G_p^{p^\alpha}$.

Proposition 2.9. *Let G be an abelian p-group and R a commutative ring with identity of characteristic p. If an R-isomorphism $R_tG \cong R_{t_1}H$ holds for any group H, then H is also an abelian p-group.*

P r o o f. Obviously H is an abelian group. For every $x \in R_tG$ there exists $n = n(x) \in \mathbb{N}_0$ and $\lambda = \lambda(x) \in R$ such that $x^{p^n} = \lambda$. The same property follows for $R_{t_1}H$. Therefore, if $h \in H$, then for $\bar{h} \in R_{t_1}H$ there exist $n \in \mathbb{N}$ and $\lambda \in R$ such that $\bar{h}^{p^n} = \lambda$, i.e. $[h,n]\overline{h^{p^n}} = \lambda$. Consequently, $h^{p^n} = 1$, i.e. H is an abelian p-group. □

P r o o f of Theorem 1. Let $\overline{G} = \{\bar{g} \mid g \in G\}$. Obviously, $R^*\overline{G} = \{r\bar{g} \mid r \in R^*, g \in G\}$ is a subgroup of $U(R_tG)$. We define a map $\varphi\colon R^*\overline{G} \to G$ by $\varphi(r\bar{g}) = g$, $r \in R^*$. It is evident that φ is a homomorphism. Since $\operatorname{Ker}\varphi = R^*$, we have $R^*\overline{G}/R^* \cong G$. Therefore the sequence

$$\mathfrak{a}\colon 1 \to R^* \to R^*\overline{G} \to G \to 1$$

is exact, i.e. $\mathfrak{a} \in \operatorname{Ext}(G, R^*)$. If either (a) or (b) is fulfilled, then by [4, p. 260] or [4, p. 258], respectively, $\operatorname{Ext}(G, R^*) = 0$ holds. Hence the sequence $\mathfrak{a}$ splits. Consequently, $R^*\overline{G} = R^* \times G'$, where G' is a subgroup of $R^*\overline{G} \subseteq U(R_tG)$ and $G' \cong G$. Hence G' can be chosen for a group basis of the twisted group ring R_tG, i.e. $R_tG = RG'$, $G' = G$. □

3. Twisted group rings over multiplicatively p-perfect rings. Proof of proposition 3

We recall that a commutative ring R with identity of prime characteristic p is called *perfect* (*p-perfect* or *p-divisible*) if $R^p = R$. We denote by $\mathcal{A}$ the class of perfect rings. We note that all finite fields and all algebraic closed fields of characteristic p and the group rings of a p-divisible abelian group over perfect rings belong to $\mathcal{A}$.

Definition. A commutative ring R with identity of prime characteristic p is said to be *multiplicatively p-perfect* or *multiplicatively p-divisible* if $R^{*^p} = R^*$, i.e. if the group of units R^* of R is p-divisible.

We observe that if R is a multiplicatively p-perfect ring, then $R^{*^{p^\alpha}} = R^*$ for every ordinal α.

Let $\mathcal{B}$ be the class of all multiplicatively p-perfect rings. We give the following examples.

1) $\mathcal{A} \subseteq \mathcal{B}$
2) If a ring K of prime characteristic p is perfect, then the ring $R = K[X_i]_{i\in I}$, where $\{X_i\}_{i\in I}$ is an arbitrary set of variables X_i, $i \in I$, is multiplicatively p-perfect but R is not perfect. In fact $R^p \neq R$, since

$$R^p = K[X_i^p]_{i\in I} \neq K[X_i]_{i\in I} = R,$$

   but $R^* = K^* = K^{*^p} = R^{*^p}$, where the second equality follows from $K = K^p$ and Lemma 2.7. Therefore R is a multiplicatively p-perfect ring.
3) Class $\mathcal{B}$ is closed under the operation of taking direct products.
4) If K is a perfect field of characteristic p, then the field $F = K(X_i)_{i\in I}$ is a multiplicatively p-perfect ring, what is not perfect. The proof is analogously to 2).
5) Every commutative ring R with identity of prime characteristic p without nilpotent elements, whose group R^* of units is torsion, is multiplicatively p-perfect. Really, let q be prime and let R_q be the q-component of R^*. Since R^* is a restricted direct product of the groups R_q, $q \neq p$ and $R_q = R_q^p$, then $R^{*^p} = R^*$.

P r o o f of Proposition 3. The equality $R^{*^p} = R^*$ implies $R_p^p = R_p$, i.e. R_p is a divisible abelian p-group. Since $R_p[p] = 1 + R(p) \neq 1$, then $|R_p| \geqslant \aleph_0$ and $|\operatorname{rad} R| = |R_p|$ yields $|\operatorname{rad} R| \geqslant \aleph_0$. Therefore, by Theorem 2.2, $|R(p)| = |\operatorname{rad} R| \geqslant \aleph_0$. The proposition is proved. □

Lemma 3.1. *If R is a multiplicatively p-perfect ring and α is an ordinal, then R^{p^α} is a multiplicatively p-perfect ring, i.e. $R^{*^p} = R^*$ implies $(R^{p^\alpha})^{*^p} = (R^{p^\alpha})^*$.*

Proof. We have

$$(R^{p^\alpha})^* = (R^*)^{p^\alpha} = R^* = (R^*)^{p^{\alpha+1}} = ((R^*)^{p^\alpha})^p = (R^{p^\alpha})^{*^p},$$

where the first and the last equalities follow from Lemma 2.7. □

Corollary 3.2. *If R is a multiplicatively p-perfect ring, then $R^{p^\alpha}(p) = R(p)$ holds for any ordinal α.*

Proof. The equality $R^{*^{p^\alpha}} = R^*$ implies $R_p^{p^\alpha} = R_p$. Therefore $R_p^{p^\alpha}[p] = R_p[p]$, which yields $1 + R^{p^\alpha}(p) = 1 + R(p)$. □

Lemma 3.3. *Let R be a multiplicatively p-perfect ring and let G be an abelian group. Then for any ordinal α*

$$(R_tG)^{p^\alpha} = R^{p^\alpha}{}_tG^{p^\alpha}$$

holds.

Proof. For $\alpha = 0$ the lemma is evident. We will prove it for $\alpha = 1$. For every $g \in G$ we have $[g,1] \in R^* = R^{*^p}$. Therefore $[g,1] = \lambda_g^p$, $\lambda_g \in R^*$. If $x \in (R_tG)^p$, then

$$x = \left(\sum_{g\in G} \alpha_g \bar{g}\right)^p = \sum_{g\in G} \alpha_g^p [g,1]\overline{g^p} = \sum_{g\in G} \alpha_g^p \lambda_g^p \overline{g^p} \in R^p{}_tG^p,$$

where $\alpha_g \in R$.

Conversely, if $x \in R^p{}_tG^p$, then

$$x = \sum_{g\in G} \alpha_g^p \overline{g^p} = \left(\sum_{g\in G} \alpha_g \lambda_g^{-1} \bar{g}\right)^p \in (R_tG)^p,$$

where $\alpha_g \in R$.

Let us assume that the lemma holds for every ordinal $\beta < \alpha$. If $\alpha - 1$ exists, then applying the lemma for $\beta = 1$ we obtain

$$(R_tG)^{p^\alpha} = (R^{p^{\alpha-1}}{}_tG^{p^{\alpha-1}})^p = (R^{p^{\alpha-1}})^p{}_t(G^{p^{\alpha-1}})^p = R^{p^\alpha}{}_tG^{p^\alpha}.$$

Now if $\alpha - 1$ does not exist, then

$$(R_tG)^{p^\alpha} = \bigcap_{\beta<\alpha} (R_tG)^{p^\beta} = \bigcap_{\beta<\alpha} R^{p^\beta}{}_tG^{p^\beta} = R^{p^\alpha}{}_tG^{p^\alpha},$$

where the second equality follows from the inductive assumption and we prove the last equality in the following way. Obviously, the inclusion $\supseteq$ holds. Conversely, if

$$x \in \bigcap_{\beta<\alpha} R^{p^\beta}{}_t G^{p^\beta},$$

then $x = \sum_{g\in G} x_g \bar{g} \in R_t G$ with $x_g \in R$ such that $x_g \in R^{p^\beta}$ and $g \in G^{p^\beta}$ for every $\beta < \alpha$. Consequently $x_g \in R^{p^\alpha}$ and $g \in G^{p^\alpha}$, i.e. $x \in R^{p^\alpha}{}_t G^{p^\alpha}$. $\square$

Lemma 3.4. *If R is a multiplicatively p-perfect ring, then for every ordinal α we have*

$$S^{p^\alpha}(R_t G) = S(R^{p^\alpha}{}_t G^{p^\alpha}).$$

P r o o f. We have

$$S^{p^\alpha}(R_t G) = S((R_t G)^{p^\alpha}) = S(R^{p^\alpha}{}_t G^{p^\alpha}),$$

where the first equality follows from Lemma 2.8 and the second equality from Lemma 3.3. $\square$

4. Characteristic lemmas

Throughout this section R denotes a commutative ring with identity of prime characteristic p and G is an abelian group. We will use the fact that $(R_t G)^{p^n}$, $n \in \mathbb{N}$, is an R^{p^n}-algebra.

Lemma 4.1. *Let $G_p = 1$. Then the set $\overline{G}^{p^n} = \{\bar{g}^{p^n} \mid g \in G\}$ is an R^{p^n}-basis of the algebra $(R_t G)^{p^n}$. If $G = G^p$, then $\overline{G}^{p^{n+k}}$, $k \in \mathbb{N}$, is an R^{p^n}-basis of the algebra $(R_t G)^{p^n}$, too.*

P r o o f. Obviously, any element $x \in (R_t G)^{p^n} = A$ is expressed linearly over R^{p^n} by elements of $\overline{G}^{p^n}$. Besides, the set $\overline{G}^{p^n}$ is linearly independent over R^{p^n}. Really, let $\sum_{g\in G} \alpha_g^{p^n} \bar{g}^{p^n} = 0$, $\alpha_g \in R$. Then $\sum_{g\in G} \alpha^{p^n}[g,n]\overline{g^{p^n}} = 0$. Since $\overline{g^{p^n}}$, in view of $G_p = 1$, are different elements, we have $\alpha_g^{p^n}[g,n] = 0$. Therefore $\alpha_g^{p^n} = 0$.

We will prove the second part of the lemma. Since $G = G^p$ and $G_p = 1$, g can be uniquely expressed in the form $g = h^{p^k}$. Therefore, we can represent any element $x \in A$ in the form

$$x = \left(\sum_{g\in G} \alpha_g \bar{g}\right)^{p^n} = \left(\sum_{h\in G} \alpha_g [h,k]^{-1} \bar{h}^{p^k}\right)^{p^n} = \sum_{h\in G} \beta_h^{p^n} \bar{h}^{p^{n+k}},$$

where $\alpha_g \in R$, $\beta_h \in R$, i.e. x is expressed linearly over R^{p^n} by $\overline{G}^{p^{n+k}}$. Analogously to the first part we prove that $\overline{G}^{p^{n+k}}$ is linearly independent over R^{p^n}. $\square$

Lemma 4.2. *Let $G_p = 1$ and $x \in R_tG$. Then*

(i) *$x \in (R_tG)^{p^n}(p)$ if and only if x can be uniquely represented in the form*

$$x = \sum_{g \in G} x_g \bar{g}^{p^n}, \tag{4.1}$$

where $x_g \in R^{p^n}(p)$;

(ii) *if $|G|\,|R^{p^n}(p)| \geqslant \aleph_0$, then*

$$|(R_tG)^{p^n}(p)| = \max(|R^{p^n}(p)|, |G|).$$

P r o o f. (i) If x is represented in the form (4.1) and $x_g \in R^{p^n}(p)$, then obviously $x \in (R_tG)^{p^n}(p)$.

Conversely, let $x \in (R_tG)^{p^n}(p)$. Since, by Lemma 4.1, $\overline{G}^{p^n}$ is an R^{p^n}-basis of $A = (R_tG)^{p^n}$, x can be uniquely represented in the form (4.1) with $x_g \in R^{p^n}$. Besides, $x_g \in R^{p^n}(p)$. Really, we have $x^p = 0$ and

$$x^p = \sum_{g \in G} x_g^p \bar{g}^{p^{n+1}} \in A^p.$$

Since, by Lemma 4.1, $\overline{G}^{p^{n+1}}$ is an $R^{p^{n+1}}$-basis of algebra A^p, we have $x_g^p = 0$, i.e. $x_g \in R^{p^n}(p)$. This proves (i).

(ii) Since $G_p = 1$, we have $|\overline{G}^{p^n}| = |G^{p^n}| = |G|$ and (i) directly implies (ii). $\square$

Corollary 4.3. *If $G_p = 1$ and $R^{p^n}(p) = 0$, then $(R_tG)^{p^n}(p) = 0$ and $S^{p^n}(R_tG) = 1$.*

P r o o f. The first part directly follows from Lemma 4.2 (i). Besides,

$$S^{p^n}(R_tG)[p] = S((R_tG)^{p^n})[p] = 1 + (R_tG)^{p^n}(p) = 1.$$

Therefore $S^{p^n}(R_tG) = 1$. $\square$

Lemma 4.4. *Let $G_p = 1$, $G = G^p$, $k \in \mathbb{N}_0$ and $x \in R_tG$. Then $x \in (R_tG)^{p^n}(p)$ if and only if x can be represented uniquely in the form*

$$x = \sum_{g \in G} x_g \bar{g}^{p^{n+k}}, \tag{4.2}$$

where $x_g \in R^{p^n}(p)$.

P r o o f. Let $x \in (R_tG)^{p^n}(p)$. Since, by Lemma 4.1, $\overline{G}^{p^{n+k}}$ is an R^{p^n}-basis of $(R_tG)^{p^n}$, we conclude that for x there exists a unique representation (4.2) with $x_g \in R^{p^n}$. Further, as in Lemma 4.2 it can be proved that $x_g \in R^{p^n}(p)$ and also the converse part of the lemma. $\square$

In the following lemma $(R_tG)^{p^n}(p)/(R_tG)^{p^{n+1}}(p)$ is an additive factor group.

Lemma 4.5. *Let $G_p = 1$, $G = G^p$ and $A = (R_tG)^{p^n}(p)/(R_tG)^{p^{n+1}}(p)$. Suppose $\Pi = \Pi(R^{p^n}(p)/R^{p^{n+1}}(p))$, $0 \in \Pi$ and*

$$A(\Pi, G) = \Big\{\sum_{g \in G} \alpha_g \bar{g}^{p^{n+1}} \mid \alpha_g \in \Pi\Big\}.$$

Then $|A| = |A(\Pi, G)|$.

P r o o f. We define a map $\varphi\colon A(\Pi, G) \to A$ by

$$\varphi\Big(\sum_{g \in G} \alpha \bar{g}^{p^{n+1}}\Big) = \sum_{g \in G} \alpha_g \bar{g}^{p^{n+1}} + B, \quad B = (R_tG)^{p^{n+1}}(p).$$

We will prove that φ is a surjective map. By Lemma 4.4, an arbitrary element $x \in A$ can be uniquely represented in the from

$$x = \sum_{g \in G} x_g \bar{g}^{p^{n+1}} + B, \quad x_g \in R^{p^n}(p).$$

Since x can be represented in the form $x_g = \alpha_g + \beta_g$, $\alpha_g \in \Pi$, $\beta_g \in R^{p^{n+1}}(p)$, we have

$$x = \sum_{g \in G} \alpha_g \bar{g}^{p^{n+1}} + B, \quad \alpha_g \in \Pi.$$

Then the element $y = \sum\limits_{g \in G} \alpha_g \bar{g}^{p^{n+1}}$, $\alpha_g \in \Pi$, is a preimage of x under the map φ.

We will prove that φ is an injection. Let $x, y \in A(\Pi, G)$,

$$x = \sum_{g \in G} \alpha_g \bar{g}^{p^{n+1}}, \quad y = \sum_{g \in G} \beta_g \bar{g}^{p^{n+1}} \qquad (\alpha_g, \beta_g \in \Pi)$$

and $\varphi(x) = \varphi(y)$. Then $\sum\limits_{g\in G}(\alpha_g - \beta_g)\bar{g}^{p^{n+1}} \in B$. Therefore Lemma 4.2, applied for $n+1$, implies $\alpha_g - \beta_g \in R^{p^{n+1}}(p)$. Since $\alpha_g, \beta_g \in \Pi$, we obtain $\alpha_g = \beta_g$ for any g, i.e. $x = y$. Consequently, φ is a bijection. □

In the proof of the following corollary due to L. Fuchs [4], $r(A)$ denotes the rank of an abelian group A.

Corollary 4.6. *Let $G_p = 1$, $G = G^p$ and $f_n(S) = f_n(S(R_tG))$. Then*
(a) *if $R^{p^n}(p) \neq R^{p^{n+1}}(p)$ and $|G|\,|R^{p^n}(p)/R^{p^{n+1}}(p)| < \aleph_0$, then*

$$f_n(S) {=} |G| \log_p |R^{p^n}(p)/R^{p^{n+1}}(p)|;$$

(b) *if $R^{p^n}(p) \neq R^{p^{n+1}}(p)$ and $|G|\,|R^{p^n}(p)/R^{p^{n+1}}(p)| \geqslant \aleph_0$, then*

$$f_n(S) {=} \max(|G|, |R^{p^n}(p)/R^{p^{n+1}}(p)|)$$

and
(c) *if $R^{p^n}(p) = R^{p^{n+1}}(p)$, then $f_n(S) = 0$.*

P r o o f. It is known that $f_n(S) = r((R_tG)^{p^n}[p]/(R_tG)^{p^{n+1}}[p])$. Then Theorem 2.3, applied to the ring $(R_tG)^{p^n}$ and its subring $(R_tG)^{p^{n+1}}$, implies $f_n(S) = r(A)$, where $A = (R_tG)^{p^n}(p)/(R_tG)^{p^{n+1}}(p)$ is an additive group. Therefore

$$f_n(S) = \begin{cases} \log_p |A|, & \text{if } |A| < \aleph_0; \\ |A|, & \text{if } |A| \geqslant \aleph_0. \end{cases} \tag{4.3}$$

Since, by Lemma 4.1, $\overline{G}^{p^{n+1}}$ is an R^{p^n}-basis of the algebra $(R_tG)^{p^n}$ and $|\overline{G}^{p^{n+1}}| = |G|$, Lemma 4.5 yields
(i) $|A| = |R^{p^n}(p)/R^{p^{n+1}}|^{|G|}$, if condition (a) is fulfilled;
(ii) $A = \max(|G|, |R^{p^n}(p)/R^{p^{n+1}}(p)|)$, if condition (b) holds and
(iii) $|A| = 1$, if condition (c) is valid.

Hence we obtain cases (a), (b) and (c) of the lemma, respectively. □

Let F be an abelian p-group. Obviously, the equality $S(RF) = R_p \times V(RF)$ implies

$$S(RF)[p] = R[p] \times V(RF)[p]. \tag{4.4}$$

At the end of this section we will suppose that F is an abelian p-group and $A = (RF)(p)/(R^pF^p)(p)$ is an additive abelian group.

Lemma 4.7. *The additive group $A = (RF)(p)/(R^pF^p)(p)$ is infinite if and only if at least one of the following conditions holds:*
(i) $f_0(R_p) \geqslant \aleph_0$;
(ii) $F \neq F^p$ *and* $|R|\,|F| \geqslant \aleph_0$;
(iii) $F = F^p \neq 1$ *and* $R \neq R^p$.

Proof. By Theorem 2.3, we obtain $A \cong S(RF)[p]/S(R^pF^p)[p]$, which in view of (4.4) yields

$$A \cong R[p]/R^p[p] \times V(RF)[p]/V(R^pF^p[p]. \tag{4.5}$$

Consequently, $|A| \geqslant \aleph_0$ if and only if either $f_0(R_p) \geqslant \aleph_0$, i.e. condition (i) of the lemma holds, or $f_0(V(RF)) \geqslant \aleph_0$. The last inequality, by Theorem 2.4, is fulfilled if and only if at least one of conditions (ii) or (iii) is valid. □

Corollary 4.8. *The inequality $|A| < \aleph_0$ holds if and only if at least one of the following conditions is fulfilled:*
(i) $f_0(R_p) < \aleph_0$ *and* $F = 1$;
(ii) $F = F^p \neq 1$ *and* $R = R^p$;
(iii) $F \neq 1$ *and* $|F|\,|R| < \aleph_0$.
Moreover, if at least one of conditions (i) *or* (ii) *is valid, then* $f_0(S) = f_0(R_p)$ *and if condition* (iii) *holds then*

$$f_0(S) = f_0(R_p) + (|F| - 1)\log_p |R| - 2(|F^p| - 1)\log_p |R^p| + (|F^{p^2}| - 1)\log_p |R^{p^2}|.$$

Proof. The corollary is obtained from (4.5) using Theorems 2.4 and 2.5. □

Lemma 4.9. *The equality $(RF)(p) = 0$ is fulfilled if and only if $R(p) = 0$ and $F = 1$.*

The lemma follows from (4.4).

Lemma 4.10. *The equality $(RF)(p) = (R^pF^p)(p)$ is fulfilled if and only if at least one of the following conditions holds:*
(i) $F = 1$ *and* $R(p) = R^p(p)$ *or*
(ii) $F = F^p \neq 1$ *and* $R = R^p$.

Proof. The equality $(RF)(p) = (R^pF^p)(p)$ is valid if and only if $S(RF)[p] = S(R^pF^p)[p]$. The last equality, by (4.4), is equivalent to $R(p) = R^p(p)$ and

$$V(RF)[p] = V(R^pF^p)[p]. \tag{4.6}$$

We will prove that (4.6) is fulfilled if and only if at least one of the following conditions holds: (a) $F = 1$ or (b) $F = F^p \neq 1$ and $R = R^p$. Let (4.6) be valid. If $F^p = 1$, then (4.6) implies $F = 1$, i.e. (a) holds. Let $F^p \neq 1$. If $F \neq F^p$, then there exists $f \in F \setminus F^p$ and therefore $f \in V(RF) \setminus V(R^p F^p) = B$, which contradicts (4.6). If $R \neq R^p$ then there exist $\alpha \in R \setminus R^p$ and $g \in F^p[p] \neq 1$ such that $1 + \alpha(g - 1) \in B$, which contradicts (4.6). Consequently (b) is fulfilled.

Conversely, conditions (a) or (b) trivially yield (4.6). The proof is completed. $\square$

Corollary 4.11. *The inequality $(RF)(p) \neq (R^p F^p)(p)$ holds if and only if at least one of the following conditions holds:*

(i) $F \neq F^p$;

(ii) $F = F^p \neq 1$ *and* $R \neq R^p$;

(iii) $F = 1$ *and* $R(p) \neq R^p(p)$.

P r o o f. Using the negation of Lemma 4.10 we conclude that $(RF)(p) \neq (R^p F^p)(p)$ if and only if at least one of the following conditions is valid:

(a1) $F \neq F^p$;

(a2) $F \neq F^p = 1$;

(a3) $F \neq 1$ and $R \neq R^p$;

(a4) $R(p) \neq R^p(p)$ and $F \neq F^p$;

(a5) $R(p) \neq R^p(p)$ and $F = 1$;

(a6) $R(p) \neq R^p(p)$.

Case (a2) is contained in (a1). Cases (a4) and (a5) are contained in (a6). It is not difficult to prove that at least one of cases (a1), (a3) and (a6) is fulfilled if and only if at least one of cases (i), (ii) or (iii) holds. To this end we divide case (a3) into two subcases: (a3.1) $F = F^p$, $F^p \neq 1$ and $R \neq R^p$ and (a3.2) $F \neq F^p$ and $R \neq R^p$. We divide also case (a6) into two subcases: (a6.1) $F = 1$ and $R(p) \neq R^p(p)$ and (a6.2) $F \neq 1$ and $R(p) \neq R^p(p)$. Further, in case (a6.2) we consider two possibilities: $F \neq F^p$ and $F = F^p$. Thus the proof is completed. $\square$

The following lemma is proved for group algebras in [7] provided R is a field of characteristic p.

Lemma 4.12. *Let R be a multiplicatively p-perfect ring, G an abelian group, $G_p \neq 1$ and $|R|\,|G| \geqslant \aleph_0$. Then $|S(R_t G)[p]| = \max(|R|, |G|)$.*

P r o o f. We choose an element $g \in G[p] \setminus \{1\}$. Since $R^* = R^{*^p}$, there exists $\beta \in R^*$ such that $[g, 1] = \beta^p$. We consider the elements

$$z(\alpha, f) = 1 + \alpha f(\bar{g} - \beta), \quad \alpha \in R, \quad f \in \Pi(G/\langle g \rangle) = \Pi, \quad 1 \in \Pi.$$

Obviously, $z(\alpha, f) \in S(R_tG)[p] = S[p]$. Further, we consider two subcases (i) and (ii).

(i) Let $|G| \geqslant |R|$. Then we set $\alpha = 1$. If $h \neq f$, $h \in \Pi$, then $z(1, f) \neq z(1, h)$ holds. Therefore $|S[p]| \geqslant |\Pi| = |G| = \max(|R|, |G|)$.

(ii) Let $|R| > |G|$. Then we set $f = 1$. If $\gamma \neq \alpha$ and $\gamma \in R$, we have $z(\alpha, 1) \neq z(\gamma, 1)$. Consequently, $|S[p]| \geqslant |R| = \max(|R|, |G|)$. The lemma is proved. □

Lemma 4.13. *Let R be a commutative ring with identity of prime characteristic p and $G_p = 1$. Then*

(a) $\operatorname{rad}(R_tG) = (\operatorname{rad} R)_tG$ *and* $S(R_tG) = 1 + (\operatorname{rad} R)_tG$*;*

(b) $(R_tG)(p^n) = R(p^n)_tG$*;*

(c) *if* $\operatorname{rad} R = 0$*, then* $S(R_tG) = 1$.

P r o o f. The proof follows from $\bar{g}^{p^n} = [g, n]\overline{g^{p^n}}$ and from $G_p = 1$. □

5. Proofs of Theorems 2, 4 and 5

P r o o f of Theorem 2. Let $A = (R_tG)^{p^n}(p)/(R_tG)^{p^{n+1}}(p)$ be an additive factor group. It is seen, as in the proof of Corollary 4.6, that for $f_n(S)$ formula (4.3) holds. We will consider successively the separate cases of the theorem.

(i) Let $R^{p^n}(p) = 0$. Then Corollary 4.3 implies $f_n(S) = 0$.

(ii) Let $R^{p^n}(p) = R^{p^{n+1}}(p) \neq 0$ and $G = G^p$. Then Corollary 4.6 yields $f_n(S) = 0$.

(iii) Let $R^{p^n}(p) \neq 0$ and $G \neq G^p$. Then $G^{p^n} \neq G^{p^{n+1}}$ and $|G^{p^n}| = |G| = |G \setminus G^p| \geqslant \aleph_0$. Let $g \in G \setminus G^p$. Then $g^{p^n} \in G^{p^n} \setminus G^{p^{n+1}}$. We consider the elements

$$z(\alpha, g) = \alpha\bar{g}^{p^n}, \quad \alpha \in R^{p^n}(p) \setminus \{0\}.$$

There are two subcases: (a) and (b).

(a) Let $|G| \geqslant |R^{p^n}(p)|$. We fix $\alpha \in R^{p^n}(p) \setminus \{0\}$. Let $g_1 \neq g$, $g_1 \in G \setminus G^p$, $B = (R_tG)^{p^{n+1}}(p)$. Then the elements $z(\alpha, g) + B$ and $z(\alpha, g_1) + B$ belong to A and are different. Otherwise, by Lemma 4.1, we receive

$$\alpha\bar{g}^{p^n} - \alpha\bar{g}_1^{p^n} = \sum_{h \in G} x_h \bar{h}^{p^{n+1}}, \quad x_h \in R^{p^{n+1}},$$

which implies the contradiction $g^{p^n} = h^{p^{n+1}} \in G^{p^{n+1}}$. Therefore, $|A| \geqslant |G \setminus G^p| = |G| = \max(|R^{p^n}(p)|, |G|)$. Since, by Lemma 4.2 (ii), the inequality $|A| \leqslant \max(|R^{p^n}(p)|, |G|)$ holds, we have $|A| = \max(|R^{p^n}(p)|, |G|)$.

(b) Let $|R^{p^n}(p)| > |G|$. Let $g \in G \setminus G^p$ be a fixed element and let $\beta \in R^{p^n}(p) \setminus \{0\}$, $\beta \neq \alpha$. Then $z(\alpha, g) + B \neq z(\beta, g) + B$. Otherwise, by Lemma 4.1, we obtain

$$\alpha \bar{g}^{p^n} - \beta \bar{g}^{p^n} = \sum_{h \in G} x_h \bar{h}^{p^{n+1}},$$

which yields the contradiction $g^{p^n} = h^{p^{n+1}} \in G^{p^{n+1}}$. Consequently, $|A| \geqslant |R^{p^n}(p)|\,|G|$ and as in case (a) we obtain $|A| = \max(|R^{p^n}(p)|, |G|)$.

(iv) Let $R^{p^n}(p) \neq R^{p^{n+1}}(p)$ and $|G|\,|R^{p^n}(p)/R^{p^{n+1}}(p)| < \aleph_0$. Then the conclusion of the case is obtained by Corollary 4.6 (a).

(v) Let $R^{p^n}(p) \neq R^{p^{n+1}}(p)$, $G = G^p$ and $|G|\,|R^{p^n}(p)/R^{p^{n+1}}(p)| \geqslant \aleph_0$. Then, by Corollary 4.6 (b), the conclusion of the case is obtained. $\square$

In the following proof of Theorem 4 we will use that, by Corollary 3.2, $R^{p^\alpha}(p) = R(p)$ for every ordinal α. Therefore $R^{p^\alpha}(p) \neq R^{p^{\alpha+1}}(p)$ will be impossible and the equalities $R^{p^\alpha}[p] = R^{p^{\alpha+1}}[p]$ and $f_\alpha(R_p) = 0$ will be fulfilled.

P r o o f of Theorem 4. In the proof of this theorem

$$A = (R^{p^\alpha} F^{p^\alpha})(p)/(R^{p^{\alpha+1}} F^{p^{\alpha+1}})(p)$$

denotes an additive abelian p-group. The equalities

$$f_\alpha(S(R_tG)) = f_0(S^{p^\alpha}(R_tG)) = f_0(S(R^{p^\alpha}{}_tG^{p^\alpha}))$$

hold, where the last equality follows from Lemma 3.4. Then

$$R^{p^\alpha}{}_tG^{p^\alpha} \cong (R^{p^\alpha}{}_tF^{p^\alpha})_{t_1}(G^{p^\alpha}/F^{p^\alpha}) \cong (R^{p^\alpha} F^{p^\alpha})_{t_1}(G^{p^\alpha}/F^{p^\alpha})$$

where the first isomorphism holds by Proposition 2.1 and the second by Theorem 1. Hence

$$f_\alpha(S) = f_0(S((R^{p^\alpha} F^{p^\alpha})_{t_1}(G^{p^\alpha}/F^{p^\alpha}))).$$

Since the p-component of $G^{p^\alpha}/F^{p^\alpha}$ is identity, $f_\alpha(S)$ will be equal to the zero Ulm-Kaplansky invariant in Theorem 2, applied for the group $G^{p^\alpha}/F^{p^\alpha}$ and the ring $R^{p^\alpha} F^{p^\alpha}$. In this connection we will use cases (i)–(v) of Theorem 2 under $n = 0$. We start with case (i).

(i) Now condition (i) of Theorem 2 is replaced equivalently by $(RF)^{p^\alpha}(p) = 0$ and by the conclusion $f_\alpha(S) = 0$, i.e., by Lemmas 3.3 and 4.9 the conditions are $R^{p^\alpha}(p) = 0$ and $F^{p^\alpha} = 1$. The first condition is equivalent to $\operatorname{rad} R = 0$. Hence (i) is equivalent to the condition (1) of the theorem.

(ii) Condition (ii) of Theorem 2, by Lemma 3.3, is now replaced equivalently by

$$(R^{p^\alpha}F^{p^\alpha})(p) = (R^{p^{\alpha+1}}F^{p^{\alpha+1}})(p) \neq 0 \quad \text{and} \quad G^{p^\alpha}/F^{p^\alpha} = (G^{p^\alpha}/F^{p^\alpha})^p$$

and by the conclusion $F_\alpha(S) = 0$. The first equality, by Lemma 4.10, holds if and only if at least one of the following conditions is fulfilled:

(a1): $F^{p^\alpha} = 1$ and $R^{p^\alpha}(p) = R^{p^{\alpha+1}}(p)$ or

(a2): $F^{p^\alpha} = F^{p^{\alpha+1}} \neq 1$ and $R^{p^\alpha} = R^{p^{\alpha+1}}$.

The condition $(R^{p^\alpha}F^{p^\alpha})(p) \neq 0$, by Lemma 4.9, is fulfilled if and only if at least one of the following conditions holds:

(b1) $R^{p^\alpha}(p) \neq 0$ or (b2) $F^{p^\alpha} \neq 1$. We replace condition (b1) equivalently by $\operatorname{rad} R \neq 0$. Since case (a1) cannot combine with case (b2) we finally obtain the following conditions:

(c1) $G^{p^\alpha} = G^{p^{\alpha+1}}$, $F^{p^\alpha} = 1$ and $\operatorname{rad} R \neq 0$;

(c2) $G^{p^\alpha}/F^{p^\alpha} = (G^{p^\alpha}/F^{p^\alpha})^p$, $F^{p^\alpha} = F^{p^{\alpha+1}} \neq 1$, $\operatorname{rad} R \neq 0$ and $R^{p^\alpha} = R^{P^{\alpha+1}}$;

(c3) $G^{p^\alpha}/F^{p^\alpha} = (G^{p^\alpha}/F^{p^\alpha})^p$, $F^{p^\alpha} = F^{p^{\alpha+1}} \neq 1$ and $R^{p^\alpha} = R^{p^{\alpha+1}}$.

Case (c2) is contained in (c3). Obviously conditions (c1) and (c3) coincide with conditions (2) and (3) of the theorem, respectively, and the conclusion is $F_\alpha(S) = 0$.

(iii) We replace case (iii) of Theorem 2 equivalently by the conditions

$$(R^{p^\alpha}F^{p^\alpha})(p) \neq 0 \quad \text{and} \quad G^{p^\alpha}/F^{p^\alpha} \neq (G^{p^\alpha}/F^{p^\alpha})^p$$

and by the conclusion

$$f_\alpha(S) = \max(|(R^{p^\alpha}F^{p^\alpha})(p)|, |G^{p^\alpha}/F^{p^\alpha}|). \tag{5.1}$$

These conditions, by Lemma 4.9, are fulfilled if and only if at least one of the following conditions holds:

(a) $\operatorname{rad} R \neq 0$ and $G^{p^\alpha}/F^{p^\alpha} \neq (G^{p^\alpha}/F^{p^\alpha})^p$ or

(b) $F^{p^\alpha} \neq 1$ and $G^{p^\alpha}/F^{p^\alpha} \neq (G^{p^\alpha}/F^{p^\alpha})^p$.

Conditions (b) coincide with conditions (5) of the theorem. We divide the system of conditions (a) into two subcases: conditions (4) of the theorem and the conditions

$$F^{p^\alpha} \neq 1, \quad \operatorname{rad} R \neq 0, \quad G^{p^\alpha}/F^{p^\alpha} \neq (G^{p^\alpha}/F^{p^\alpha})^p.$$

The last system of conditions is contained in (b), i.e. in conditions (5) of the theorem. Now we see that in case (4) of the theorem the conclusion (5.1) coincides with the conclusion of (4) of the theorem. In case (5) of the theorem the conclusion (5.1), by Lemma 4.12 applied to $S(R^{p^\alpha}F^{p^\alpha})[p]$, gives conclusion (5) of the theorem, since $|G^{p^\alpha}/F^{p^\alpha}| \geqslant \aleph_0$.

(iv) We replace case (iv) of Theorem 2 equivalently by

$$(R^{p^\alpha}F^{p^\alpha})(p) \neq (R^{p^{\alpha+1}}F^{p^{\alpha+1}})(p) \quad \text{i.e. } A \neq 0,\ |G^{p^\alpha}/F^{p^\alpha}| < \aleph_0,\ |A| < \aleph_0$$

and by the conclusion $f_\alpha(S) = |G^{p^\alpha}/F^{p^\alpha}| \log_p |A|$. The condition $A \neq 0$ holds, by Corollary 4.11, if and only if at least one of the following conditions is fulfilled:

(a) $F^{p^\alpha} \neq F^{p^{\alpha+1}}$;

(b) $F^{p^\alpha} = F^{p^{\alpha+1}} \neq 1$ and $R^{p^\alpha} \neq R^{p^{\alpha+1}}$ or

(c) $F^{p^\alpha} = 1$ and $R^{p^\alpha}(p) \neq R^{p^{\alpha+1}}(p)$.

The condition $|A| < \aleph_0$, by Corollary 4.8, is valid if and only if at least one of the following conditions holds:

(a1) $F^{p^\alpha} = 1$ and $f_\alpha(R_p) < \aleph_0$;

(a2) $F^{p^\alpha} = F^{p^{\alpha+1}} \neq 1$ and $R^{p^\alpha} = R^{p^{\alpha+1}}$ or

(a3) $F^{p^\alpha} \neq 1$, $|F^{p^\alpha}| < \aleph_0$ and $|R^{p^\alpha}| < \aleph_0$.

We note the following: by Corollary 3.2, case (c) is impossible and $f_\alpha(R_p) = 0$, (a) cannot combine with (a1) and (a2), and (b) with (a1), (a2) and (a3). Thus conditions (iv) coincide with conditions (6) of the theorem with the exception of the case $F^{p^\alpha} = 1$. This case is considered separately and the formula for $f_\alpha(S)$ is obtained trivialy.

For the conclusion of case (iv), i.e. for case (6) of the theorem, it is sufficient to note that, by Theorem 2.3, $A \cong R^{p^\alpha}[p]/R^{p^{\alpha+1}}[p] \times A'$, where

$$A' \cong V(R^{p^\alpha}F^{p^\alpha})[p]/V(R^{p^{\alpha+1}}F^{p^{\alpha+1}})[p].$$

Since $R^{p^\alpha}[p] = R^{p^{\alpha+1}}[p]$, we have $\log_p |A| = f_\alpha(V(RF))$. If we take $f_\alpha(V)$ from Theorem 2.5, then we obtain the indicated value of $f_\alpha(V)$ in case (6) of the theorem.

(v) We replace case (v) of Theorem 2 equivalently by

$$A \neq 0, \quad G^{p^\alpha}/F^{p^\alpha} = (G^{p^\alpha}/F^{p^\alpha})^p \quad \text{and } |A|\,|G^{p^\alpha}/F^{p^\alpha}| \geqslant \aleph_0,$$

and by the conclusion $f_\alpha(S) = \max(|A|, |G^{p^\alpha}/F^{p^\alpha}|)$. By Corollary 4.11, $A \neq 0$ holds if and only if at least one of the following conditions is fulfilled:

(a) $F^{p^\alpha} \neq F^{p^{\alpha+1}}$;

(b) $F^{p^\alpha} = F^{p^{\alpha+1}} \neq 1$ and $R^{p^\alpha} \neq R^{p^{\alpha+1}}$ or

(c) $F^{p^\alpha} = 1$ and $R^{p^\alpha}(p) \neq R^{p^{\alpha+1}}(p)$. Since $f_\alpha(R_p) = 0$, we have by Lemma 4.7 that $|A|\,|G^{p^\alpha}/F^{p^\alpha}| \geqslant \aleph_0$ if and only if at least one of the following conditions is valid:

(a1) $F^{p^\alpha} \neq F^{p^{\alpha+1}}$ and $|R^{p^\alpha}|\,|F^{p^\alpha}| \geqslant \aleph_0$;

(a2) $F^{p^\alpha} = F^{p^{\alpha+1}} \neq 1$ and $R^{p^\alpha} \neq R^{p^{\alpha+1}}$ or

(a3) $|G^{p^\alpha}/F^{p^\alpha}| \geqslant \aleph_0$.

We note the following: case (c) is impossible, case (a) cannot combine with (a2) and (b) with (a1). Thus we obtain the following four conditions:

(b1) $G^{p^\alpha}/F^{p^\alpha} = (G^{p^\alpha}/F^{p^\alpha})^p$, $F^{p^\alpha} \neq F^{p^{\alpha+1}}$ and $|R^{p^\alpha}|\,|F^{p^\alpha}| \geqslant \aleph_0$;

(b2) $G^{p^\alpha}/F^{p^\alpha} = (G^{p^\alpha}/F^{p^\alpha})^p$, $F^{p^\alpha} \neq F^{p^{\alpha+1}}$ and $|G^{p^\alpha}/F^{p^\alpha}| \geqslant \aleph_0$;

(b3) $G^{p^\alpha}/F^{p^\alpha} = (G^{p^\alpha}/F^{p^\alpha})^p$, $F^{p^\alpha} = F^{p^{\alpha+1}} \neq 1$ and $R^{p^\alpha} \neq R^{p^{\alpha+1}}$;

(b4) $G^{p^\alpha}/F^{p^\alpha} = (G^{p^\alpha}/F^{p^\alpha})^p$, $F^{p^\alpha} = F^{p^{\alpha+1}} \neq 1$, $R^{p^\alpha} \neq R^{p^{\alpha+1}}$ and $|G^{p^\alpha}/F^{p^\alpha}| \geqslant \aleph_0$.

Obviously, the union of cases (b1) and (b2) gives conditions (7) of the theorem. Besides, (b4) is contained in (b3) and conditions (b3) coincide with conditions (8) of the theorem.

Since $f_\alpha(S) \geqslant \aleph_0$, hence for the calculation of $f_\alpha(S)$ it is sufficient to assume that $|A| \geqslant \aleph_0$. Obviously $|A| = F_\alpha(R_p) + f_\alpha(V(RF))$ holds and since $f_\alpha(R_p) = 0$, thus $|A| = f_\alpha(V(RF))$. Therefore

$$f_\alpha(S) = \max(|G^{p^\alpha}/F^{p^\alpha}|, f_\alpha(V(RF)).$$

Taking into account that the value of $f_\alpha(V(RF)) \geqslant \aleph_0$ we obtain, by Theorem 2.4, the value of $f_\alpha(S)$ in cases (7) and (8) of the theorem. The proof is completed. $\square$

Proof of Theorem 5. Since $S^{p^\alpha}(R_tG) = S(R^{p^\alpha}{}_tG^{p^\alpha}) = S(R^{p^{\alpha+1}}{}_tG^{p^{\alpha+1}}) = S^{p^{\alpha+1}}(R_tG)$, where the first equality follows from Lemma 3.4, we have $S(R^{p^\alpha}{}_tG^{p^\alpha}) = dS$. Further, we consider 3 subcases: (i), (ii) and (iii).

(i) Let $F^{p^\alpha} \neq 1$. Then $|G^{p^\alpha}| \geqslant \aleph_0$. Since R^{p^α}, by Lemma 3.1, is a multiplicatively p-perfect ring, hence, by Lemma 4.12, applied to $S(R^{p^\alpha}{}_tG^{p^\alpha})[p]$, we conclude $\lambda = \max(|G^{p^\alpha}|, |R^{p^\alpha}|)$.

(ii) Let $F^{p^\alpha} = 1$ and $\operatorname{rad} R \neq 0$. Then $|R^{p^\alpha}(p)| = |R(p)| = |\operatorname{rad} R| \geqslant \aleph_0$ where the first equality follows from Corollary 3.2 and the second and the third inequalities from Proposition 3. Therefore $|R^{p^\alpha}(p)| \geqslant \aleph_0$ and $|R^{p^\alpha}(p)| = |\operatorname{rad} R|$. Then $|S(R^{p^\alpha}{}_tG^{p^\alpha})[p]| = |(R^{p^\alpha}{}_tG^{p^\alpha})(p)| = |R^{p^\alpha}(p)_tG^{p^\alpha}| = |R^{p^\alpha}(p)|\,|G^{p^\alpha}|$, where the second equality follows from Lemma 4.13 (b). Consequently $\lambda = \max(|G^{p^\alpha}|, |\operatorname{rad} R|)$.

(iii) Let $F^{p^\alpha} = 1$ and $\operatorname{rad} R = 0$. Then $\operatorname{rad} R^{p^\alpha} \subseteq \operatorname{rad} R = 0$, i.e. $\operatorname{rad} R^{p^\alpha} = 0$ and Lemma 4.13 (c) applied to G^{p^α} and R^{p^α} implies $S(R^{p^\alpha}{}_tG^{p^\alpha}) = 1$. Hence $\lambda = 0$. The theorem is proved. $\square$

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