Beatrix Bačová; Božena Dorociaková

Oscillation of nonlinear differential systems with retarded arguments

Czechoslovak Mathematical Journal, Vol. 55 (2005), No. 1, 255–262

Persistent URL: http://dml.cz/dmlcz/127974

Czechoslovak Mathematical Journal, 55 (130) (2005), 255–262

OSCILLATION OF NONLINEAR DIFFERENTIAL SYSTEMS WITH RETARDED ARGUMENTS

Beatrix Bačová and Božena Dorociaková, Žilina

(Received June 7, 2002)

Abstract. In this work we investigate some oscillatory properties of solutions of non-linear differential systems with retarded arguments. We consider the system of the form

$$\begin{aligned} y_i'(t) - p_i(t)y_{i+1}(t) &= 0, \quad i = 1, 2, \dots, n-2, \\ y_{n-1}'(t) - p_{n-1}(t)|y_n(h_n(t))|^{\alpha} \operatorname{sgn}[y_n(h_n(t))] &= 0, \\ y_n'(t) \operatorname{sgn}[y_1(h_1(t))] + p_n(t)|y_1(h_1(t))|^{\beta} &\leqslant 0, \end{aligned}$$

where $n \geqslant 3$ is odd, $\alpha > 0$, $\beta > 0$.

Keywords: nonlinear differential system, oscillatory (nonoscillatory) solution

MSC 2000: 34K15, 34K40

1. Introduction

We consider systems of nonlinear differential inequalities with retarded arguments of the form

$$\text{(S)} \qquad \begin{aligned} y_i'(t) - p_i(t)y_{i+1}(t) &= 0, \quad i = 1, 2, \dots, n-2, \\ y_{n-1}'(t) - p_{n-1}(t)|y_n(h_n(t))|^{\alpha} \operatorname{sgn}[y_n(h_n(t))] &= 0, \\ y_n'(t) \operatorname{sgn}[y_1(h_1(t))] + p_n(t)|y_1(h_1(t))|^{\beta} &\leqslant 0, \end{aligned}$$

where the following conditions are always assumed: $n \geqslant 3$ is odd, $\alpha > 0$, $\beta > 0$, $p_i\colon [a, \infty) \to [0, \infty)$, $a \in \mathbb{R}$, $i = 1, 2, \dots, n$, are continuous functions not identically

This research was supported by the grants 1/8055/01 and 1/0026/03 of Scientific Grant Agency of Ministry of Education of Slovak Republic and Slovak Academy of Sciences.

equal to zero on any subinterval of $[a,\infty)$,

$$\int_a^\infty p_i(t)\,\mathrm{d}t = \infty, \quad i = 1, 2, \dots, n-1,$$

$h_1\colon [a,\infty) \to \mathbb{R}$, $h_n\colon [a,\infty) \to \mathbb{R}$ are continuous nondecreasing functions and $h_1(t) < t$, $h_n(t) < t$ on $[a,\infty)$, $\lim\limits_{t\to\infty} h_1(t) = \lim\limits_{t\to\infty} h_n(t) = \infty$. Denote by W the set of all solutions $y(t) = (y_1(t), \dots, y_n(t))$ of the system (S) which exist on some ray $[T_y,\infty) \subset [a,\infty)$ and satisfy $\sup\Bigl\{\sum\limits_{i=1}^n |y_i(t)|\colon t \geqslant T\Bigr\} > 0$ for any $T \geqslant T_y$.

As far as the autors know there is no oscillatory result for the system (S) in the case when $n \geqslant 3$ is odd. It is to be pointed out that Theorems 1 and 2 extend the result of Theorem 3 in [4]. Moreover, Theorems 3 and 4 consider the case when $\alpha\beta = 1$, which is not treated in [4].

Definition 1. A solution $y \in W$ is called oscillatory (weakly oscillatory) if each component (at least one component) has arbitrarily large zeros. A solution $y \in W$ is called nonoscillatory (weakly oscillatory) if each component (at least one component) is eventually of a constant sign on some interval $[t_0,\infty)$, $t_0 \geqslant a$. We define

$$I_0 = 1$$

and

$$I_k(t,s;p_k,\dots,p_1) = \int_s^t p_k(x) I_{k-1}(x,s,p_{k-1},\dots,p_1)\,\mathrm{d}x, \quad k = 1,\dots,n-2.$$

Lemma 1. *Suppose that*

(1) $$y = (y_1,\dots,y_n) \in W$$

is a nonoscillatory solution of (S) *and*

(2) $$(-1)^{n+i} y_i(t) y_1(t) > 0 \quad \text{on } [t_0,\infty),\ t_0 \geqslant a, i = 1,\dots,n.$$

Then

(3) $$y_1(h_1(t))\operatorname{sgn}[y_1(h_1(t))] \geqslant |y_n(h_n(t))|^\alpha \int_{h_1(t)}^t p_{n-1}(x) I_{n-2}(x,h_1(t);p_{n-2},\dots,p_1)\,\mathrm{d}x$$

for all large t.

P r o o f. Let $t_0 \leqslant s \leqslant t$. It is evident that

$$y_1(s) = y_1(t) - \int_s^t y_1'(x)\,\mathrm{d}x = y_1(t) - \int_s^t p_1(x)y_2(x)\,\mathrm{d}x.$$

We calculate the second integral by parts. Denote

$$v(x) = \int_s^x p_1(\tau)\,\mathrm{d}\tau = I_1(x,s;p_1), \quad u(x) = y_2(x).$$

Then we get

$$\begin{aligned} y_1(s) &= y_1(t) - y_2(t)I_1(t,s;p_1) + \int_s^t y_2'(x)I_1(x,s;p_1)\,\mathrm{d}x \\ &= y_1(t) - y_2(t)I_1(t,s;p_1) + \int_s^t p_2(x)y_3(x)I_1(x,s;p_1)\,\mathrm{d}x. \end{aligned}$$

Applying further $(n-3)$-times the method of integration by parts to the last integral we obtain the identity

$$\begin{aligned} y_1(s) = &\sum_{j=0}^{n-2}(-1)^j y_{j+1}(t)I_j(t,s;p_j,\dots,p_1) \\ &+ \int_s^t p_{n-1}(x)|y_n(h_n(x))|^\alpha \operatorname{sgn}[y_n(h_n(x))]I_{n-2}(x,s;p_{n-2},\dots,p_1)\,\mathrm{d}x, \\ &\hfill t_0 \leqslant s \leqslant t. \end{aligned}$$

In view of (2) and the monotonicity of $y_n(t)$, we obtain for $T \geqslant t_0$ sufficiently large,

$$\begin{aligned} y_1(s)\operatorname{sgn}[y_1(s)] = &\sum_{j=0}^{n-2}(-1)^j y_{j+1}(t)\operatorname{sgn}[y_1(t)]I_j(t,s;p_j,\dots,p_1) \\ &+ \int_s^t p_{n-1}(x)|y_n(h_n(x))|^\alpha I_{n-2}(x,s;p_{n-2},\dots,p_1)\,\mathrm{d}x, \\ &\hfill T \leqslant s \leqslant t, \end{aligned}$$

$$\begin{aligned} &y_1(h_1(t))\operatorname{sgn}[y_1(h_1(t))] \\ &\qquad \geqslant |y_n(h_n(t))|^\alpha \int_{h_1(t)}^t p_{n-1}(x)I_{n-2}(x,s;p_{n-2},\dots,p_1)\,\mathrm{d}x, \quad t > T. \end{aligned}$$

The proof is complete. □

The following notation will be used:

$$\begin{gathered} \bar{p}_i(t) = \min\{p_i(s)\colon\ h_1(t) \leqslant s \leqslant t\}, \quad t \geqslant a,\ i = 1,\dots,n-1, \\ P_{n-1}(t) = \bar{p}_{n-1}(t)\bar{p}_{n-2}(t)\dots\bar{p}_1(t). \end{gathered}$$

Lemma 2. *Suppose that assumptions* (1) *and* (2) *are fulfilled. Then*

$$y_1(h_1(t))\operatorname{sgn}[y_1(h_1(t))] \geqslant \frac{(t-h_1(t))^{n-1}}{(n-1)!}P_{n-1}(t)|y_n(h_n(t))|^{\alpha} \tag{4}$$

for all large t.

P r o o f. In view of (3) we get

$$y_1(h_1(t))\operatorname{sgn}[y_1(h_1(t))] \geqslant |y_n(h_n(t))|^{\alpha}\bar{p}_{n-1}(t)\int_{h_1(t)}^{t} I_{n-2}(x,h_1(t);p_{n-2},\dots,p_1)\,\mathrm{d}x.$$

Integrating by parts we obtain

$$\begin{aligned} &y_1(h_1(t))\operatorname{sgn}[y_1(h_1(t))]\\ &\quad\geqslant |y_n(h_n(t))|^{\alpha}\bar{p}_{n-1}(t)\int_{h_1(t)}^{t}(t-x)p_{n-2}(x)I_{n-3}(x,h_1(t);p_{n-3},\dots,p_1)\,\mathrm{d}x\\ &\quad\geqslant \dots\geqslant |y_n(h_n(t))|^{\alpha}\bar{p}_{n-1}(t)\dots\bar{p}_1(t)\int_{h_1(t)}^{t}\frac{(t-x)^{n-2}}{(n-2)!}\,\mathrm{d}x. \end{aligned}$$

Calculating the last integral we have

$$y_1(h_1(t))\operatorname{sgn}[y_1(h_1(t))] \geqslant \frac{(t-h_1(t))^{n-1}}{(n-1)!}P_{n-1}(t)|y_n(h_n(t))|^{\alpha}, \quad t\geqslant T,$$

where T is sufficiently large. □

The next lemma follows from Theorem 3 in [4].

Lemma 3. *Suppose that* $0<\alpha\beta<1$ *and*

$$\int_T^{\infty}(h_1(t))^{(n-1)\beta}p_n(t)(P_{n-1}((h_1(t)))^{\beta}\,\mathrm{d}t=\infty, \quad T\geqslant a. \tag{5}$$

Then every nonoscillatory solution of system (S) *has the property* $\lim\limits_{t\to\infty}y_k(t)=0$, $k=1,2,\dots,n$, *and* (2) *holds.*

The next lemma is derived from Theorem 2 in [1].

Lemma 4. *Assume that* $g \in C([a,\infty),[0,\infty))$, $\delta \in C([a,\infty),\mathbb{R})$, $\lim\limits_{t\to\infty}\delta(t)=\infty$, $\delta(t)<t$ *for* $t \leqslant a$ *and*

$$\liminf_{t\to\infty}\int_{\delta(t)}^{t} g(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}}.$$

Then the functional inequality

$$y'(t)+g(t)y(\delta(t)) \leqslant 0, \quad t \geqslant a,$$

cannot have an eventually positive solution and

$$y'(t)+g(t)y(\delta(t)) \geqslant 0, \quad t \geqslant a,$$

cannot have an eventually negative solution.

The next lemma is presented in [4] as Lemma 1.

Lemma 5. *Let* $y=(y_1,\dots,y_n)\in W$ *be a weakly nonoscillatory solution of* (S), *then* y *is nonoscillatory.*

Theorem 1. *Suppose that* $0<\alpha\beta<1$, (5) *holds and*

$$\liminf_{t\to\infty}\int_{h_n(t)}^{t} p_n(s)\biggl[\int_{h_1(s)}^{s} p_{n-1}(x)I_{n-2}(x,h_1(s);p_{n-2},\dots,p_1)\,\mathrm{d}x\biggr]^{\beta}\mathrm{d}s > \frac{1}{\mathrm{e}}. \tag{6}$$

Then all solutions of system (S) *are oscillatory.*

P r o o f. Assume that the system (S) has a solution $y=(y_1,\dots,y_n)\in W$ at least one component of which is eventually of constant sign. Then by Lemma 5 y is nonoscillatory. We may suppose that $y_1(t)>0$ for $t\geqslant t_0\geqslant a$. By Lemma 3 the solution y has the property

$$\lim_{t\to\infty} y_k(t)=0, \quad k=1,2,\dots,n$$

and(2) holds. Applying Lemma 1 to the nth inequality of the system (S) we obtain

$$y_n'(t)+y_n^{\alpha\beta}(h_n(t))p_n(t)\biggl[\int_{h_1(t)}^{t} p_{n-1}(x)I_{n-2}(x,h_1(t);p_{n-2},\dots,p_1)\,\mathrm{d}x\biggr]^{\beta} \leqslant 0, \qquad t\geqslant T\geqslant t_0.$$

With regard to the facts that $0<\alpha\beta<1$ and $\lim\limits_{t\to\infty} y_n(t)=0$, we get

$$y_n'(t)+p_n(t)\biggl[\int_{h_1(t)}^{t} p_{n-1}(x)I_{n-2}(x,h_1(t);p_{n-2},\dots,p_1)\,\mathrm{d}x\biggr]^{\beta} y_n(h_n(t)) \leqslant 0, \qquad t\geqslant T, \tag{7}$$

where T is sufficiently large. By Lemma 4 the inequality (7) cannot have a positive solution. This contradicts the fact that $y_n(t) > 0$ for $t \geqslant T$. The proof is complete. □

Theorem 2. *Suppose that* $0 < \alpha\beta < 1$, (5) *holds and*

$$\liminf_{t\to\infty} \int_{h_n(t)}^{t} (s - h_1(s))^{(n-1)\beta} P_{n-1}^{\beta}(s) p_n(s)\, \mathrm{d}s > \frac{[(n-1)!]^{\beta}}{\mathrm{e}}. \tag{8}$$

Then all solutions of system (S) *are oscillatory.*

P r o o f. Assume that the system (S) has a solution $y = (y_1, \ldots, y_n) \in W$ at least one component of which is nonoscillatory. Then by Lemma 5 y is nonoscillatory. We may suppose that $y_1(t) > 0$ for $t \geqslant t_0 \geqslant a$. Due to Lemma 3 the solution y has the property $\lim\limits_{t\to\infty} y_k(t) = 0$, $k = 1, 2, \ldots, n$, and (2) holds. Applying (4) to the nth inequality of (S) we get

$$y_n'(t) + \frac{(t - h_1(t))^{(n-1)\beta}}{[(n-1)!]^{\beta}} P_{n-1}^{\beta}(t) p_n(t) y_n^{\alpha\beta}(h_n(t)) \leqslant 0, \quad t \geqslant T \geqslant t_0.$$

By virtue of the conditions $0 < \alpha\beta < 1$ and $\lim\limits_{t\to\infty} y_n(t) = 0$, we obtain

$$y_n'(t) + \frac{(t - h_1(t))^{(n-1)\beta}}{[(n-1)!]^{\beta}} P_{n-1}^{\beta}(t) p_n(t) y_n(h_n(t)) \leqslant 0, \quad t \geqslant T \tag{9}$$

where T is sufficiently large.

By Lemma 4 the inequality (9) cannot have a positive solution. This is a contradiction with property (2). □

The next lemma follows from Lemma 2 and Lemma 5 in [4].

Lemma 6. *Suppose that the assumption* (1) *of Lemma* 1 *is fulfilled. Then there exists* $l \in \{1, 2, \ldots, n\}$, l *is odd and* $t_0 \geqslant a$ *such that*

$$y_i(t) y_1(t) > 0 \quad \text{on } [t_0, \infty) \quad \text{for } i = 1, 2, \ldots, l, \tag{10}$$

$$(-1)^{n+i} y_i(t) y_1(t) > 0 \quad \text{on } [t_0, \infty) \quad \text{for } i = l+1, \ldots, n, \tag{11}$$

and

$$|y_i(t/2)| \geqslant c_i t^{n-i} P_{n-1}^{i}(t) |y_n(t)|^{\alpha} \quad \text{for } t \geqslant t_0, \quad i = 1, 2, \ldots, l-1, \tag{12}$$

where

$$c_i = \frac{2^{-2(n-i)}}{(n-1)!(n-i)!}, \quad i = 1, 2, \ldots, n-1,$$

$$P_{n-1}^i(t) = \overline{p}_{n-1}(t)\overline{p}_{n-2}(t) \ldots \overline{p}_i(t) \quad \textit{for } i = 1, 2, \ldots, n-1,$$

$$P_{n-1}^1(t) = P_{n-1}(t).$$

Remark. The inequality (10) implies

$$|y_i(t)| \geqslant |y_i(t/2)| \quad \text{for } i = 1, 2, \ldots, l-1.$$

Hence(12) can be written in the form

$$|y_i(t)| \geqslant c_i t^{n-i} P_{n-1}^i(t) |y_n(t)|^\alpha \quad \text{for } t \geqslant t_0,\ i = 1, \ldots, l-1. \tag{13}$$

Theorem 3. *Suppose that* $\alpha\beta = 1$, (6) *holds and*

$$\liminf_{t\to\infty} \int_{h_1(t)}^t [h_1(s)]^{(n-1)\beta} [P_{n-1}(h_1(s))]^\beta p_n(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}c_1^\beta}. \tag{14}$$

Then all solutions of the system (S) *are oscillatory.*

Proof. Assume that the system (S) has a solution $y = (y_1, \ldots, y_n) \in W$ at least one component of which is nonoscillatory. Then by Lemma 5 the solution y is nonoscillatory. We may assume that $y_1(t) > 0$ for $t \geqslant t_0 \geqslant a$ and $y_1(h_1(t)) > 0$ for $t \geqslant t_1 \geqslant t_0$. Then the nth inequality of (S) implies that $y_n'(t) \leqslant 0$ for $t \geqslant t_1$ and it is not identically zero on any subinterval of $[t_1, \infty]$. As $y_1(t) > 0$ and $y_n'(t) \leqslant 0$ for $t \geqslant t_1$, then by Lemma 6 we get (10), (11), and (12) or (13).

Let $l \geqslant 2$. From (13) we have for $i = 1$,

$$y_1(t) \geqslant c_1 t^{n-1} P_{n-1}(t) y_n^\alpha(t), \quad t \geqslant t_2 \geqslant t_1.$$

Then the nth inequality of system (S) implies

$$y_n'(t) + c_1^\beta [h_1(t)]^{(n-1)\beta} [P_{n-1}(h_1(t))]^\beta p_n(t) [y_n(h_1(t))] \leqslant 0, \quad t \geqslant t_3 \geqslant t_2.$$

This inequality by Lemma 4 cannot have an eventually positive solution $y_n(t)$, which is a contradiction. The case when $l = 1$ is also impossible. This case can be treated as in the proof of Theorem 1. So the proof is complete. □

Theorem 4. *Suppose that* $\alpha\beta = 1$ *and* (5), (8), (14) *hold. Then all solutions of system* (S) *are oscillatory.*

The result of the theorem follows from Theorems 3 and 2.

References

[1] *R. G. Koplatadze and T. A. Chanturia*: On the oscillatory and monotone solutions of the first order differential equations with deviating arguments. J. Diff. Equations *8* (1982), 1463–1465. (In Russian.)

[2] *I. Foltynska and J. Werbowski*: On the oscillatory behaviour of solution of system of differential equation with deviating arguments. Colloquia Math. Soc. J. B., Qualitative theory of Diff. Eq. Szeged *30* (1979), 243–256.

[3] *Y. Kitamura and T. Kusano*: On the oscillation of a class of nonlinear differential systems with deviating argument. J. Math. Annal Appl. *66* (1978), 20–36.

[4] *P. Marušiak*: On the oscillation of nonlinear differential systems with retarded arguments. Math. Slovaca *34* (1984), 73–88.

[5] *P. Marušiak and R. Olach*: Functional Differential Equations. University of Žilina, EDIS, Žilina, 2000. (In Slovak.)

Authors' addresses: B. Bačová, Department of Mathematics, Faculty of Sciences, University of Žilina, Slovak Republic, e-mail: beatrix.bacova@fpv.utc.sk; B. Dorociaková, Department of Applied Mathematics, Faculty of Mechanical Engineering, University of Žilina, Slovak Republic, e-mail: dorociak@kam.utc.sk.