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Miroslav Bartušek; Mariella Cecchi; Zuzana Došlá; Mauro Marini

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GLOBAL MONOTONICITY AND OSCILLATION FOR SECOND ORDER DIFFERENTIAL EQUATION

M. Bartušek, Brno, M. Cecchi, Firenze, Z. Došlá, Brno, and M. Marini, Firenze

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Abstract. Oscillatory properties of the second order nonlinear equation

$$(r(t)x')' + q(t)f(x) = 0$$

are investigated. In particular, criteria for the existence of at least one oscillatory solution and for the global monotonicity properties of nonoscillatory solutions are established. The possible coexistence of oscillatory and nonoscillatory solutions is studied too.

Keywords: second order nonlinear differential equation, oscillatory solution, nonoscillatory solution, coexistence problem

MSC 2000: 34C10

1. Introduction

In this paper we are concerned with the oscillatory behavior of solutions for the second order nonlinear differential equation

$$(r(t)x')' + q(t)f(x) = 0, \tag{1}$$

where $q\colon [t_0,\infty) \to (0,\infty)$, $r\colon [t_0,\infty) \to (0,\infty)$ are continuous and $f\colon \mathbb{R} \to \mathbb{R}$ is continuous with $f(u)u > 0$ for $u \neq 0$.

We will restrict our attention to those solutions x of (1) that exist on some interval $[t_0, b_x)$, $b_x \leqslant \infty$, and not continuable for $t \geqslant b_x$, whereby b_x may depend on the particular solution involved. A solution x is called *oscillatory* if there exists $\{t_n\}$, $t_0 \leqslant t_n < b_x$, such that $\lim\limits_{n\to\infty} t_n = b_x$ and $x(t_n) = 0$, otherwise it is called *nonoscillatory.*

Observe that any nonoscillatory solution is continuable at infinity, i.e. $b_x = \infty$, as will be pointed out in the next section.

Two problems have been widely studied in literature: the existence of nonoscillatory solutions and the oscillation of all solutions, see, e.g., [1], [2], [6], [7], [14] and [5], [15], [17], [18], respectively, and references therein. More difficult problems, especially those motivated by some physical applications, deal with: (i) the existence of at least one oscillatory solution; (ii) the global monotonicity of nonoscillatory solutions x, i.e. x and x' different from zero on the whole interval $[t_0, \infty)$; (iii) the possible coexistence of oscillatory and nonoscillatory solutions.

The investigation of these problems was started in the sixties by R. A. Moore, Z. Nehari [16], J. Kurzweil [12], M. Jasný [10] for the Emden-Fowler equation

$$x'' + q(t)|x|^{\lambda} \operatorname{sgn} x = 0 \tag{2}$$

in the case $\lambda > 1$ and some years later in the case $0 < \lambda < 1$ by K. L. Chiou [13]. We refer to [11], [18] and references therein for further contributions. For our later comparison, we recall the following result for the Emden-Fowler equation (2):

Theorem A [11, Th. 18.4]. *Assume $q(t)t^{(\lambda+3)/2}$ is nondecreasing for some $\lambda > 0$. Then* (2) *has at least one oscillatory solution. In addition, if $\lambda > 1$, then every solution of* (2) *with a zero is oscillatory.*

Later on these results have been generalized in [3], [8], [9] to equations of the form

$$x'' + f(t, x) = 0. \tag{3}$$

In particular, in [8], [9] conditions under which (3) has at least one oscillatory solution are given. Under similar assumptions, conditions ensuring oscillation of every solution of (3) with a zero are established in [3, Corollary 9].

The purpose of this paper is to solve the above posed problems (i)–(iii). In particular we show that every solution x of (1), such that $x(t_1)x'(t_1) \geqslant 0$ at some $t_1 \geqslant t_0$, is oscillatory. As a consequence, we give the affirmative answer to problems (i)–(iii), under additional assumptions. Also, we give a growth estimate for possible nonoscillatory solutions of (1). The paper is completed by an example illustrating that our results can be applied when the corresponding ones in [3], [8], [9] fail. We note that our assumption on nonlinearity requires superlinearity or sublinearity conditions neither on the whole line, nor near zero or infinity.

Our approach is based on a transformation of (1) to an equation of type (3). Since the function r in (1) may be eliminated by a change of variable, many results in literature have been stated for the case $r(t) \equiv 1$. However, the results presented here cannot be obtained directly from the quoted ones concerning the case $r(t) \equiv 1$.

2. Global monotonicity and oscillation

Two cases must be distinguished, according to the convergence or divergence of the integral $\int^{\infty}[1/r(s)]\,\mathrm{d}s$. Here the case

$$\int_{t_0}^{\infty}\frac{\mathrm{d}s}{r(s)}<\infty \tag{4}$$

is considered. The basic results for the opposite case are formulated in Section 3 without proof.

Our main result concerns the existence of an oscillatory solution and the global monotonicity properties of nonoscillatory solutions. The following holds:

Theorem 1. *Let* (4) *be verified and assume that for some* $\lambda>1$

$$\frac{f(u)}{|u|^{\lambda}}\ \text{is nonincreasing on}\ (-\infty,0)\ \text{and}\ (0,\infty); \tag{5}$$

$$q(t)r(t)\Bigl(\int_t^{\infty}\frac{\mathrm{d}s}{r(s)}\Bigr)^{(3+\lambda)/2}\ \text{is nondecreasing and tends to}\ \infty\ \text{as}\ t\to\infty. \tag{6}$$

Then equation (1) *has oscillatory solutions. In addition every nonoscillatory solution x of* (1) *(if it exists) satisfies $x(t)x'(t)<0$ for every $t\geqslant t_0$ and $\lim\limits_{t\to\infty}x(t)=0$. In particular, every solution x such that $x(t_1)x'(t_1)\geqslant 0$ at some $t_1\geqslant t_0$ is oscillatory.*

The proof of Theorem 1 is accomplished by means of several lemmas. We will use the change of variable

$$s=s(t)=\Bigl(\int_t^{\infty}\frac{\mathrm{d}\tau}{r(\tau)}\Bigr)^{-1},\quad z(s)=sx(t). \tag{7}$$

A function $x(t)$ is a solution of (1) if and only if the function $z(s)=sx(t)$ is a solution of the equation

$$z''+Q(s)f\Bigl(\frac{z(s)}{s}\Bigr)=0\quad(s\geqslant a>0) \tag{8}$$

where $t=t(s)$ is the inverse of the monotone function s and

$$a=\Bigl(\int_{t_0}^{\infty}\frac{\mathrm{d}s}{r(s)}\Bigr)^{-1}>0,\quad Q(s)=r(t)q(t)\Bigl(\int_t^{\infty}\frac{\mathrm{d}\tau}{r(\tau)}\Bigr)^{3}. \tag{9}$$

Observe that, in view of the convexity, any nonoscillatory solution z of (8) is continuable at infinity. This fact implies that the same occurs for any nonoscillatory solution x of (1). In addition, for any nonoscillatory solution z of (8) we have

$$z(s)\neq 0\quad\text{for } s>s_z\implies z(s)z'(s)>0\quad\text{for } s>s_z. \tag{10}$$

Indeed, if $z(s) > 0$, then $z''(s) < 0$ and so z' is decreasing and positive (otherwise z becomes eventually negative). If $z(s) < 0$, the argument is analogous.

The following two lemmas describe asymptotic properties of nonoscillatory solutions of (8).

Lemma 1. *Assume that for some* $\lambda > 1$ *condition* (5) *is satisfied and*

$$Q(s)s^{(3-\lambda)/2} \text{ is nondecreasing for } s \geqslant a. \tag{11}$$

Then every nonoscillatory solution z *of* (8) *satisfies*

$$\lim_{s\to\infty} \frac{z(s)}{s} = 0, \quad \lim_{s\to\infty} z'(s) = 0. \tag{12}$$

In addition, if $z(s) \neq 0$ *for* $s > s_z$, *then the function* $(z'(s)s - z(s)) \operatorname{sgn} z$ *is decreasing for* $s > s_z$ *and is negative for large* s, *i.e. the function* $|z(s)|/s$ *is decreasing to zero as* $s \to \infty$.

P r o o f. Let z be a solution of (8) and, without loss of generality, assume $z(s) > 0$ for $s > s_z \geqslant a$. As already claimed, the function z' is positive decreasing and thus there exists $\lim\limits_{s\to\infty} z(s) = c_1$, $0 < c_1 \leqslant \infty$ and $\lim\limits_{s\to\infty} z'(s) = c_2$, $0 \leqslant c_2 < \infty$. Hence

$$\lim_{s\to\infty} \frac{z(s)}{s} = \lim_{s\to\infty} z'(s) = c_2, \quad 0 \leqslant c_2 < \infty.$$

Suppose $c_2 > 0$ and let $\varepsilon > 0$ be such that $z(s)/s > c_2 - \varepsilon > 0$ for $s \geqslant s_0 > s_z$. Denote $K = \max z(s)/s$ for $s \in [s_0, \infty)$ and $m = \min f(u)$ for $u \in [c_2 - \varepsilon, K]$. Integrating (8) on (s_0, ∞) we obtain

$$z'(s_0) - c_2 = \int_{s_0}^{\infty} Q(\sigma) f\Bigl(\frac{z(\sigma)}{\sigma}\Bigr) \,\mathrm{d}\sigma \geqslant m \int_{s_0}^{\infty} Q(\sigma) \,\mathrm{d}\sigma,$$

which implies $\int^{\infty} Q(\sigma) \,\mathrm{d}\sigma < \infty$. Since $\lambda > 1$, by (11) we obtain

$$\int_a^s Q(\sigma) \,\mathrm{d}\sigma = \int_a^s Q(\sigma)\sigma^{(3-\lambda)/2}\sigma^{(\lambda-3)/2} \,\mathrm{d}\sigma \geqslant c \int_a^s \sigma^{(\lambda-3)/2} \,\mathrm{d}\sigma \geqslant c \int_b^s \frac{\mathrm{d}\sigma}{\sigma} \tag{13}$$

where $b = \max\{a, 1\}$ and $c = Q(a)a^{(3-\lambda)/2}$, which gives a contradiction as $s \to \infty$. Hence (12) holds. In addition we have

$$\Bigl(\frac{z(s)}{s}\Bigr)' = \frac{z'(s)s - z(s)}{s^2} \quad \text{and} \quad (z'(s)s - z(s))' = z''(s)s < 0 \quad (s > s_z),$$

i.e. the function $z'(s)s - z(s)$ is decreasing for $s > s_z$. If $z'(s)s - z(s) > 0$ for large s, then $z(s)/s$ is eventually positive increasing, which contradicts (12). Thus $z'(s)s - z(s) < 0$ for large s, i.e. $z(s)/s$ is decreasing to zero as $s \to \infty$. $\square$

Lemma 2. *Assume that for some $\lambda > 1$ conditions* (5) *and* (11) *are satisfied. Then for any nonoscillatory solution z of* (8) *such that $|z(s)|/s$ is decreasing and $z(s) \neq 0$ on $I_z = [s_z, \infty)$, $s_z \geqslant a$, the following statements hold for $s \in I_z$:*

(a) $z(s)z'(s) \geqslant 2(\lambda+1)^{-1} z(s) f(z(s)/s) Q(s) s$,

(b) $z(s) f(z(s)/s) \geqslant (1+\lambda) s^{-\lambda} \int_{s_z}^{s} f(z(\xi)/\xi) z'(\xi) \xi^{\lambda} \, \mathrm{d}\xi + (s_z/s)^{\lambda} f(z(s_z)/s_z) z(s_z)$,

(c) *the function*

$$F(s) = s[z'(s)]^2 - z(s)z'(s) + 2Q(s)s^{1-\lambda} \int_{s_z}^{s} f(z(\xi)/\xi) z'(\xi) \xi^{\lambda} \, \mathrm{d}\xi$$

is increasing.

Proof. Let z be a nonoscillatory solution of (8) and, without loss of generality, assume $z(s) > 0$ and $z'(s)s - z(s) < 0$ for $s \geqslant s_z$. Set $g(u) = f(u)/u^{\lambda}$ for $u > 0$. By (5) the function $g(z(s)/s)$ is nondecreasing for $s \geqslant s_z$.

Claim (a). In view of (10) and Lemma 1, integrating (8) on (s, ∞), $s \geqslant s_z$, we obtain

$$\begin{aligned} z'(s) &= \int_s^{\infty} Q(\xi) f\Bigl(\frac{z(\xi)}{\xi}\Bigr) \, \mathrm{d}\xi = \int_s^{\infty} Q(\xi) g\Bigl(\frac{z(\xi)}{\xi}\Bigr) \frac{z^{\lambda}(\xi)}{\xi^{\lambda}} \, \mathrm{d}\xi \\ &\geqslant g\Bigl(\frac{z(s)}{s}\Bigr) z^{\lambda}(s) \int_s^{\infty} Q(\xi) \xi^{(3-\lambda)/2} \xi^{-(3+\lambda)/2} \, \mathrm{d}\xi \\ &\geqslant f\Bigl(\frac{z(s)}{s}\Bigr) s^{\lambda} Q(s) s^{(3-\lambda)/2} \int_s^{\infty} \xi^{-(3+\lambda)/2} \, \mathrm{d}\xi \\ &= \frac{2}{\lambda+1} f\Bigl(\frac{z(s)}{s}\Bigr) Q(s) s. \end{aligned}$$

Claim (b). We have

$$\begin{aligned} z(s) f\Bigl(\frac{z(s)}{s}\Bigr) &- (1+\lambda) \frac{1}{s^{\lambda}} \int_{s_z}^{s} f\Bigl(\frac{z(\xi)}{\xi}\Bigr) z'(\xi) \xi^{\lambda} \, \mathrm{d}\xi \\ &= z(s) f\Bigl(\frac{z(s)}{s}\Bigr) - (1+\lambda) \frac{1}{s^{\lambda}} \int_{s_z}^{s} g\Bigl(\frac{z(\xi)}{\xi}\Bigr) z^{\lambda}(\xi) z'(\xi) \, \mathrm{d}\xi \\ &\geqslant z(s) f\Bigl(\frac{z(s)}{s}\Bigr) - (1+\lambda) \frac{1}{s^{\lambda}} g\Bigl(\frac{z(s)}{s}\Bigr) \frac{\bigl[z^{\lambda+1}(\xi)\bigr]_{s_z}^{s}}{1+\lambda} \\ &= g\Bigl(\frac{z(s)}{s}\Bigr) \frac{1}{s^{\lambda}} z^{\lambda+1}(s_z) \geqslant \Bigl(\frac{s_z}{s}\Bigr)^{\lambda} f\Bigl(\frac{z(s_z)}{s_z}\Bigr) z(s_z), \end{aligned}$$

which yields the assertion.

Claim (c). In view of claim (b) we have

$$
\begin{aligned}
F'(s) &= 2sz'z'' - zz'' + 2\big(Q(s)s^{(3-\lambda)/2}\big)' s^{-[(1+\lambda)/2]} \int_{s_z}^{s} f\Big(\frac{z(\xi)}{\xi}\Big) z'(\xi)\xi^{\lambda}\,\mathrm{d}\xi \\
&\qquad - (1+\lambda)Q(s)s^{-\lambda} \int_{s_z}^{s} f\Big(\frac{z(\xi)}{\xi}\Big) z'(\xi)\xi^{\lambda}\,\mathrm{d}\xi + 2sQ(s)f\Big(\frac{z(s)}{s}\Big)z'(s) \\
&\geqslant Q(s)\Big[z(s)f\Big(\frac{z(s)}{s}\Big) - (1+\lambda)\frac{1}{s^{\lambda}}\int_{s_z}^{s} f\Big(\frac{z(\xi)}{\xi}\Big) z'(\xi)\xi^{\lambda}\,\mathrm{d}\xi\Big] \\
&\geqslant Q(s)\Big(\frac{s_z}{s}\Big)^{\lambda} f\Big(\frac{z(s_z)}{s_z}\Big)z(s_z) > 0.
\end{aligned}
$$

□

The next result establishes upper bounds for nonoscillatory solutions of (8).

Lemma 3. *If for some $\lambda > 1$ conditions* (5) *and* (11) *hold, then every nonoscillatory solution z of* (8) *satisfies*

$$
|z(s)| \leqslant \frac{C\sqrt{s}}{\big(Q(s)s^{(3-\lambda)/2}\big)^{1/(\lambda-1)}} \quad \textit{as } s \to \infty, \tag{14}
$$

where $C > 0$ is a suitable constant. In addition, if

$$
\lim_{s\to\infty} Q(s)s^{(3-\lambda)/2} = \infty, \tag{15}
$$

then no nonoscillatory solutions z of (8) *such that*

$$
|z(s)| \geqslant c\sqrt{s} \quad \textit{for any } c > 0 \textit{ and } s \textit{ large}
$$

exist.

P r o o f. Let z be a solution of (8). In view of (10) and Lemma 1, without loss of generality, we can assume $z(s) > 0$, $z'(s) > 0$ and $z(s)/s$ decreasing for $s \geqslant T > a$. As claimed in the proof of Lemma 2, the function $g(z(s)/s)$ is nondecreasing for $s \geqslant T$. By Lemma 2 (a) we have

$$
z'(s) \geqslant \frac{2}{\lambda+1}\frac{z^{\lambda}(s)}{s^{\lambda}} g\Big(\frac{z(s)}{s}\Big)Q(s)s \geqslant cz^{\lambda}(s)Q(s)s^{1-\lambda}
$$

where $c = 2(\lambda+1)^{-1}g(z(T)/T)$. From here, by integrating on $[s,\infty)$ and using (11) and Lemma 1, we obtain

$$
\frac{1}{(\lambda-1)[z(s)]^{\lambda-1}} \geqslant c\int_{s}^{\infty} Q(t)t^{(3-\lambda)/2}t^{-(1+\lambda)/2}\,\mathrm{d}t \geqslant cQ(s)s^{(3-\lambda)/2}\frac{2}{(\lambda-1)s^{(\lambda-1)/2}},
$$

which implies the assertion. □

Assumption (15) cannot be omitted, as the following example shows.

Example 1. Equation

$$z'' + \frac{1}{4}s^{(\lambda-3)/2}\Big|\frac{z}{s}\Big|^{\lambda}\operatorname{sgn} z = 0, \quad \lambda > 1, \; s \geqslant 1$$

has a solution $z(s) = \sqrt{s}$, whereby the function $Q(s)s^{(3-\lambda)/2} = \frac{1}{4}$.

Remark 1. If, for some $\lambda > 1$, conditions (5), (11) and

$$Q(s) \geqslant s^{\lambda-2}h(s), \quad \lim_{s\to\infty} h(s) = \infty \tag{16}$$

hold, then no nonoscillatory solutions of (8) exist. In fact, (14) implies

$$|z(s)| \leqslant C\Big[\frac{1}{h(s)}\Big]^{1/(\lambda-1)},$$

which gives a contradiction with (10) when $s \to \infty$. In particular, conditions (16) are satisfied if

$$\lim_{s\to\infty} Q(s)s^{2-\lambda} = \infty.$$

Observe that for the Emden-Fowler nonlinearity $f(u) = |u|^{\lambda}\operatorname{sgn} u$, equation (8) takes the form $z'' + s^{-\lambda}Q(s)|z|^{\lambda}\operatorname{sgn} z = 0$ and the statement is known, see [11, Th. 18.2].

The following lemma gives the existence of oscillatory solutions and describes the global monotonicity properties of nonoscillatory solutions.

Lemma 4. *Assume that for some $\lambda > 1$ conditions* (5), (11) *and* (15) *are satisfied. Then every solution z of* (18) *such that*

$$(z'(T)T - z(T))\operatorname{sgn} z(T) \geqslant 0 \tag{17}$$

at some $T \geqslant a$ is oscillatory. Every nonoscillatory solution z_1 of (8) *satisfies*

$$z_1(s) \neq 0, \quad z_1(s)z_1'(s) > 0$$

on the whole $[a,\infty)$ and, on the same interval $[a,\infty)$, the function $|z_1(s)|/s$ is decreasing and converges to zero as $s \to \infty$.

Proof. Let z be a solution of (8) satisfying (17) at some $T \geqslant a$. Assume z is nonoscillatory. If $z(s) \neq 0$ for all $s > T$, put $\overline{T} = T$; otherwise let $\overline{T}$ be the largest zero of z. Without loss of generality assume $z(s) > 0$ for $s > \overline{T}$. By (10), $z'(s) > 0$ for $s > \overline{T}$. Consider two cases: (a) $\overline{T} = T$. By (17) we have

$$\left(\frac{z(s)}{s}\right)'_{s=\overline{T}} = \frac{z's - z}{s^2}\bigg|_{s=\overline{T}} \geqslant 0. \tag{18}$$

(b) $\overline{T} > T$. Since $z(\overline{T}) = 0$, (18) holds, too.

By Lemma 1, the function $z(s)/s$ is decreasing as $s \to \infty$. Thus there exists $\tau \geqslant \overline{T}$ such that

$$\left(\frac{z(s)}{s}\right)'_{s=\tau} = 0, \quad \left(\frac{z(s)}{s}\right)' < 0 \quad \text{for } s > \tau,$$

i.e.

$$z'(\tau)\tau - z(\tau) = 0 \quad \text{and} \quad z'(s)s - z(s) < 0 \quad \text{for } s > \tau. \tag{19}$$

Applying Lemma 2 on $[\tau, \infty)$ we obtain

$$z(s)z'(s) \geqslant \frac{2}{\lambda+1} sQ(s)z(s)f\left(\frac{z(s)}{s}\right) \geqslant \frac{2}{s^{\lambda-1}}Q(s)\int_\tau^s f\left(\frac{z(\xi)}{\xi}\right)z'(\xi)\xi^\lambda \,\mathrm{d}\xi. \tag{20}$$

Consider the function F from Lemma 2: in view of (19) we have $F(\tau) = 0$. Since F is increasing, from (20) we obtain for $s \geqslant \tau + 1$

$$\begin{aligned} 0 &= F(\tau) < F(\tau+1) \leqslant F(s) \\ &= s[z'(s)]^2 - z(s)z'(s) + 2s\frac{Q(s)}{s^\lambda}\int_\tau^s f\left(\frac{z(\xi)}{\xi}\right)z'(\xi)\xi^\lambda \,\mathrm{d}\xi \leqslant s[z'(s)]^2. \end{aligned}$$

Therefore we have for $s \geqslant \tau + 1$

$$z'(s) \geqslant \frac{\sqrt{F(\tau+1)}}{\sqrt{s}}.$$

Since, in view of (19), $[z(s)/s] > z'(s)$, we conclude

$$z(s) \geqslant \sqrt{F(\tau+1)}\sqrt{s}.$$

This contradicts Lemma 3 and hence any solution z satisfying (17) is oscillatory.

In particular, every solution with a zero is oscillatory. Therefore, any nonoscillatory solution z_1 satisfies $z_1(s) \neq 0$ for $s \geqslant a$ and thus, in view of (10), $z_1(s)z_1'(s) > 0$ for $s \geqslant a$. By Lemma 1 the function $(z_1'(s)s - z_1(s))\operatorname{sgn} z_1$ is decreasing for $s > a$ and negative for large s. Since any solution satisfying (17) is oscillatory, the function $(z_1'(s)s - z_1(s))\operatorname{sgn} z_1$ must be negative for every $s \geqslant a$, i.e. the function $|z_1(s)|/s$ is decreasing on $[a, \infty)$. $\square$

P r o o f of Theorem 1. Let x be a solution of (1) and consider the transformation (7). Obviously, this transformation preserves zeros of solutions. Since, in view of (7),

$$Q(s)s^{(3-\lambda)/2} = q(t)r(t)\biggl(\int_t^\infty \frac{\mathrm{d}\tau}{r(\tau)}\biggr)^{(3+\lambda)/2}, \tag{21}$$

assumptions (11) and (15) for equation (8) are equivalent to the assumption (6) for equation (1). Applying Lemmas 1 and 4 to (8) and going back to (1) we obtain that every solution x of (1) with a zero is oscillatory and any nonoscillatory solution x of (1) satisfies $\lim\limits_{t\to\infty} x(t) = 0$.

It remains to prove that every nonoscillatory solution x of (1) satisfies $x(t)x'(t) < 0$ for $t \geqslant t_0$. Without loss of generality assume $x(t) > 0$ for $t \geqslant t_0$. Then $z(s) > 0$ is a nonoscillatory solution of (8) and, by Lemma 4, $z(s)/s$ is decreasing on $[a,\infty)$. The change of variable (7) gives

$$\dot{s} = \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{\bigl(\int_t^\infty 1/r\bigr)^2}\frac{1}{r(t)} = \frac{s^2}{r(t)}, \qquad x' = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\dot{z}(s)s - z(s)}{r(t)},$$

from where $x'(t) < 0$ follows for all $t \geqslant t_0$. □

3. Applications and extensions

As we have already mentioned, the problem of existence of an oscillatory solution and the one of the global monotonicity of nonoscillatory solutions are closely related to the possible coexistence of such kinds of solutions. In this section we give some applications of our results to this problem.

We recall the following existence result that can be obtained, e.g., from [11], [14], [18].

Lemma 5. *Let* (4) *be satisfied and assume there exist a nondecreasing function g and positive numbers β, K, ε such that*

$$|f(u)| \leqslant g(u) \quad \textit{for } |u| \leqslant \varepsilon, \tag{22}$$

$$\int^\infty q(t)g\biggl(\int_t^\infty \frac{\mathrm{d}s}{r(s)}\biggr)\,\mathrm{d}t < \infty. \tag{23}$$

Then there exists a nonoscillatory solution x of (1) *such that*

$$x(t) = \int_t^\infty \frac{\mathrm{d}\tau}{r(\tau)}[\alpha + o(1)] \tag{24}$$

where $\alpha \neq 0$ is an arbitrary constant.

The first application follows from Theorem 1 and Lemma 5.

Corollary 1. *Assume conditions* (4), (5), (6), (22) *and* (23) *are satisfied. Then every solution of* (1) *with a zero is oscillatory and there exists a nonoscillatory solution x of* (1) *such that* $\lim_{t\to\infty} x(t) = 0$ *and* $x(t)x'(t) < 0$ *on the whole* $[t_0, \infty)$.

Example 3 below illustrates Corollary 1. The second application follows from Lemma 3 and establishes an *a-priori* estimate for nonoscillatory solutions of (1).

Theorem 2. *Assume, for some $\lambda > 1$, that conditions* (4), (5) *are satisfied and*

$$q(t)r(t)\biggl(\int_t^\infty \frac{\mathrm{d}s}{r(s)}\biggr)^{(3+\lambda)/2} \quad \text{is nondecreasing on } [t_0,\infty).$$

Then every nonoscillatory solution x of (1) *satisfies for large t*

$$|x(t)| \leqslant c\biggl(q(t)r(t)\biggl(\int_t^\infty \frac{\mathrm{d}s}{r(s)}\biggr)^{2-\lambda}\biggr)^{-1/(\lambda-1)}$$

where c is a suitable constant.

The following examples illustrate our results. In particular, Example 3 illustrates a case of coexistence of oscillatory and globally nonoscillatory solutions and shows that the quoted results in [8], [9] cannot be applied.

Example 2. Consider the generalized Emden-Fowler equation

$$(t^2x')' + t^\delta |x|^\gamma \operatorname{sgn} x = 0. \tag{25}$$

When $0 < \gamma \leqslant 1$ and $\delta > 0$, then choosing $1 < \lambda < 1 + 2\delta$ we obtain from Theorem 1 the existence of oscillatory solutions and the global monotonicity property of nonoscillatory solutions. When $\gamma > 1$ and $\delta > (\gamma - 1)/2$, then by virtue of Theorem 1 again, the same result holds. Note that the existence of an oscillatory solution of (25) follows also from other known results (see, e.g. [11]).

Example 3. Consider the equation

$$(t^2x')' + t^{1/4+\varepsilon} f(x) = 0 \quad (t \geqslant 1) \tag{26}$$

where $\varepsilon > 0$ and

$$f(u) = \begin{cases} |u|^{4/3} \operatorname{sgn} u & \text{for } |u| \leqslant 1, \\ \sqrt{|u|} \operatorname{sgn} u & \text{for } |u| \geqslant 1. \end{cases}$$

It is easy to verify that the assumptions of Theorem 1 are satisfied with $\lambda = 3/2$ and any $\varepsilon > 0$. Hence (26) has an oscillatory solution and, as we will point out in the next section, such a solution is nontrivial in any neighbourhood of infinity. In addition, for $\varepsilon < 1/12$ Corollary 1 is applicable and thus for $\varepsilon \in (0, 1/12)$ the coexistence of oscillatory and nonoscillatory solutions of (26) takes place.

Let us show that the quoted result from [8], [9] cannot be applied. First we show that condition of Izjumova

$$\liminf_{t\to\infty,\ |x|\to\infty} \frac{t^{3/2}|f(t,t^{1/2}x)|}{x} > 1 \tag{27}$$

is not verified for equation (26). To check this, we apply transformation (7): in this case $s = t$, $s \geqslant 1$ and (26) is reduced to

$$z'' + s^{-3/4+\varepsilon} f\Big(\frac{z}{s}\Big) = 0 \quad (s \geqslant 1). \tag{28}$$

For equation (28) assumption (27) becomes

$$\liminf_{s\to\infty,\ |u|\to\infty} \frac{s^{3/2}s^{-3/4+\varepsilon} f(u/\sqrt{s})}{u} = \liminf_{s\to\infty,\ |u|\to\infty} \frac{s^{3/4+\varepsilon} f(u/\sqrt{s})}{u} > 1. \tag{29}$$

Choosing $u = s^2$ and $\varepsilon < 1/2$, condition (29) is not verified. Similarly, the condition of [8]

$$\liminf_{|x|\to 0} \frac{t_0^{3/2} f(t_0, t_0^{1/2}x)}{x} > 1/4 \tag{30}$$

is not satisfied for equation (26) because for x in a right neighboorhood of zero we have

$$\frac{t^{3/2} f(t, t^{1/2}x)}{x} = t^{\varepsilon} t^{29/12} x^{1/3}.$$

In a similar way one can check that the results of [3] cannot be applied in this case, either.

We close this section by considering the case

$$\int_{t_0}^{\infty} \frac{\mathrm{d}s}{r(s)} = \infty. \tag{31}$$

Using the well-known transformation

$$\sigma = \sigma(t) = \int_0^t \frac{\mathrm{d}\tau}{r(\tau)}, \quad z(\sigma) = x(t(\sigma))$$

and proceeding in a similar way as in Section 2, we obtain the following result, which extends Theorem A.

Theorem 3. *Let* (31) *be verified and assume that for some* $\lambda > 1$

$$\frac{f(u)}{|u|^{\lambda}} \text{ is nondecreasing on } (-\infty, 0) \text{ and } (0, \infty), \tag{32}$$

$$q(t)r(t)\left(\int_0^t \frac{\mathrm{d}s}{r(s)}\right)^{(3+\lambda)/2} \text{ is nondecreasing and tends to } \infty \text{ as } t \to \infty. \tag{33}$$

Then (1) *has oscillatory solutions. On the other hand, every nonoscillatory solution* x *(if it exists) satisfies* $x(t)x'(t) > 0$ *for all* $t \geqslant t_0$ *and*

$$\lim_{t\to\infty} x(t)\left(\int_{t_0}^t \frac{\mathrm{d}\tau}{r(\tau)}\right)^{-1} = 0.$$

In particular, every solution x *such that* $x(t_1)x'(t_1) \geqslant 0$ *at some* $t_1 \geqslant t_0$ *is oscillatory.*

Observe that, in this case, a completely different proof of the statement that every solution with a zero is oscillatory, has been presented in [3, Corollary 9]. Under additional assumptions ensuring the existence of bounded nonoscillatory solutions (see, e.g., [11, Corollary 8.2]) we can obtain conditions for the coexistence of oscillatory and nonoscillatory solutions. For instance, the following holds:

Corollary 2. *Assume conditions* (31), (32), (33) *are satisfied and*

$$\int^{\infty} r(t)q(t)\left(\int_0^t \frac{\mathrm{d}\tau}{r(\tau)}\right) \mathrm{d}t < \infty.$$

Then every solution of (1) *with a zero is oscillatory and there exists a bounded nonoscillatory solution* x *of* (1) *such that* $x(t)x'(t) > 0$ *on the whole* $[t_0, \infty)$.

4. Concluding remarks

Here we discuss the continuability at infinity and the uniqueness problem. As we have already claimed, nonoscillatory solutions of (1) are continuable at infinity. Hence the only possible noncontinuable solutions of (1) are the oscillatory ones. For the Emden-Fowler equation (2), if the function $q(t)r(t)$ is locally of bounded variation, then every solution of (1) is continuable at infinity (see e.g. [3], [4], [18]).

Another "pathological" type of oscillatory solutions, which may arise, are solutions identically equal to zero in a neighbourhood of infinity. These solutions do not exist when an additional uniqueness assumption on f is supposed, for instance the Osgood condition, i.e. $|f(u)| \leqslant \omega(|u|)$ for $|u| \leqslant \varepsilon$ where ω is a nondecreasing function such that $\int_0^\varepsilon 1/\omega(u) = \infty$. As concerns the problem of continuability of oscillatory solutions, we recall [3], where this problem is deeply discussed for the equation

$$x'' + xF(x^2, t) = 0.$$

However, conditions stated in [3] seem to be rather difficult to apply for equation (1), under our assumptions. Hence we offer here the following criterion for continuability of solutions of (1).

Proposition 1. *Assume conditions* (4), (5) *are satisfied. If* $q(t)r(t)$ *is absolute continuous on* $[t_0, \infty)$, f *is absolute continuous on* $\mathbb{R}$ *and for any* $c \neq 0$ *the function*

$$q(t)r(t)\left(\int_t^\infty \frac{\mathrm{d}s}{r(s)}\right)^3 \left| f\left(c\int_t^\infty \frac{\mathrm{d}s}{r(s)}\right)\right| \tag{34}$$

is nonincreasing, then all solutions of (1) *are continuable at infinity.*

P r o o f. For the reduced equation (8), the condition (34) becomes

$$Q(s)\left|f\left(\frac{c}{s}\right)\right| \text{ is nonincreasing for } s \geqslant a > 0 \text{ and } c \neq 0.$$

For any solution z of (8) consider the function

$$F(s) = \int_0^{z(s)} Q(s) f\left(\frac{\tau}{s}\right) \mathrm{d}\tau + \frac{1}{2} z'^2(s) \geqslant \frac{1}{2} z'^2(s).$$

We have $F(s) \geqslant 0$ and

$$F'(s) = \int_0^{z(s)} \operatorname{sgn} z(\tau) \frac{\mathrm{d}}{\mathrm{d}s}\left[Q(s)\left|f\left(\frac{\tau}{s}\right)\right|\right] \mathrm{d}\tau \leqslant 0.$$

It means that F is bounded, so z' is bounded too and z is a continuable at infinity solution. By using the transformation (7), we obtain the assertion. $\square$

Applying Proposition 1 to the equation (26) in Example 3 we can conclude that all solutions of this equation are defined on $[1, \infty)$ and this equation has oscillatory solutions defined on $[1, \infty)$ and nontrivial on any neighbourhood of infinity.

When (31) is assumed, then a result similar to that in Proposition 1 holds; the details are left to the reader.

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Authors' addresses: M. Bartušek, Department of Mathematics, Masaryk University, Janáčkovo nám. 2a, 662 95 Brno, Czech Republic e-mail: `bartusek@math.muni.cz`; M. Cecchi, Department of Electronics and Telecomunications, University of Florence, Via S. Marta 3, 501 39 Firenze, Italy, e-mail: `cecchi@det.unifi.it`; Z. Došlá, Department of Mathematics, Masaryk University, Janáčkovo nám. 2a, 662 95 Brno, Czech Republic, e-mail: `dosla@math.muni.cz`; M. Marini, Department of Electronics and Telecomunications, University of Florence, Via S. Marta 3, 501 39 Firenze, Italy, e-mail: `marini@ing.unifi.it`.