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INFINITE INDEPENDENT SYSTEMS OF THE IDENTITIES OF THE ASSOCIATIVE ALGEBRA OVER AN INFINITE FIELD OF CHARACTERISTIC $p > 0$

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Abstract. In this paper some infinitely based varieties of groups are constructed and these results are transferred to the associative algebras (or Lie algebras) over an infinite field of an arbitrary positive characteristic.

Keywords: associative algebras, infinite systems of identities, Specht's problem.

MSC 2000: 16R10, 20E10

Specht's problem [1] about the finite basing of any system of identities is well known in the associative algebra theory. This problem was affirmatively solved by A. A. Kemer [2] for the case of null characteristic of the basic field. If the basic field's characteristic is positive, Specht's problem has negative solution. Essentially using results of the paper [3] A. I. Belov constructed in [4] infinitely based varieties of associative algebras over an infinite field of an arbitrary positive characteristic. (We remark that the methods of V. V. Shigolev's proofs [3] are based on direct combinatorial reasoning with algebra polynomials.) In [5] the author constructed infinite independent systems of identities of associative algebras (or Lie algebras) over an infinite field of characteristic 2, using methods completely different from those in [3]. In this paper the results from [5] are generalized to the case of an arbitrary positive characteristic.

We denote a *commutator in an algebra* by $(a, b) = ab - ba$, a *commutator in a group* by $[a, b] = a^{-1}b^{-1}ab$, the *conjugation of an element b through an element a in a group* by $b^a = a^{-1}ba$. We will also use the notation

$$(a_1, \ldots, a_{k-1}, a_k) = ((a_1, \ldots, a_{k-1}), a_k),$$
$$[a_1, \ldots, a_{k-1}, a_k] = [[a_1, \ldots, a_{k-1}], a_k].$$

Let F be an infinite field of positive characteristic p and let $\mathfrak{C}_p$ denote the variety of associative F-algebras defined by the identities

$$(1) \qquad (x, y, z) = 0, \quad x^{p^2} = 1, \quad [x, y]^p = 1, \quad [x^p, y] = 1.$$

$\mathfrak{B}$ is the variety of associative F-algebras defined by the identity

$$(2) \qquad ((x_1, x_2, x_3), (x_4, x_5, x_6), (x_7, x_8)) = 0,$$

$\mathfrak{N}_3$ is the variety of nilpotent Lie F-algebras of index not more than 3, $\mathfrak{D}$ is the variety of Lie F-algebras defined by the identity

$$(3) \qquad ((x_1x_2 \cdot x_3)(x_4x_5 \cdot x_6))(x_7x_8) = 0.$$

We also denote

$$\mu_k = ((x, y, z), (x_1, x_2), (x_3, x_4), \dots, (x_{4k-1}, x_{4k}), (x, y, z));$$
$$\nu_k = (((((xy)z)(x_1x_2))(x_3x_4)) \dots (x_{4k-1}x_{4k}))((xy)z).$$

It is proved that in the variety $\mathfrak{B} \cap \mathfrak{C}_p\mathfrak{C}_p$ the system of identities $\{\mu_k = 0\colon\ k = 1, 2, \dots\}$ is independent, i.e. no identity of this system follows from the other identities of the system (Theorem 1). We obtain as a consequence that the variety $\mathfrak{B} \cap \mathfrak{C}_p\mathfrak{C}_p$ contains a continuum of different not finitely based subvarieties and that in $\mathfrak{B}\cap\mathfrak{C}_p\mathfrak{C}_p$ there exist algebras with the unsolvable problem of words equality. It follows from the second identity in (1) that the algebras of the variety $\mathfrak{C}_p\mathfrak{C}_p$ are nil-algebras of index p^4. This is the answer to V. V. Shigolev's question [3, p. 144] about the existence of an infinite basis of the associative algebra's identities such that the degree in each variable is bounded in the aggregate.

From Theorem 1 it also follows that the system of identities $\{\nu_k = 0\colon\ k = 1, 2, \dots\}$ is independent in the variety $\mathfrak{D}\cap\mathfrak{N}_3\mathfrak{N}_3$. The identity of solvability of index 4 follows from (3). It gives the negative answer to A. M. Slinko's question [6, question 1.129] about a finitely based variety of solvable Jordan algebras in the case of a solvable variety of index 4 of special Jordan algebras over an infinite field of characteristic 2.

The varieties $\mathfrak{B} \cap \mathfrak{C}_p\mathfrak{C}_p$ and $\mathfrak{D} \cap \mathfrak{N}_3\mathfrak{N}_3$ are locally finite and locally nilpotent. As the last statement is concerned it should be mentioned that it is easy to show that any nilpotent variety of algebras (not necessarily associative) has a finite basis of identities.

Let A be an associative algebra with the identity element 1 and let B be its subalgebra satisfying the identity

$$(4) \qquad x^m = 0.$$

Then the set of elements $1-B=\{1-b\colon\ b\in B\}$ forms a group and $(1-b)^{-1}=1+b+b^2+\ldots+b^{m-1}$.

Lemma 1. *Let A be an associative algebra with the identity element 1 and let B be its subalgebra satisfying the identity (4). Then*

$$[1-u,1-v]=1+(1+u+\ldots+u^{m-1})(1+v+\ldots+v^{m-1})(u,v)$$

for $u,v\in B$.

P r o o f. We have $[1-u,1-v]=(1-u)^{-1}(1-v)^{-1}(1-u)(1-v)=(1-u)^{-1}(1-v)^{-1}(1-u)(1-v)-(1-u)^{-1}(1-v)^{-1}(1-v)(1-u)+1=1+(1-u)^{-1}(1-v)^{-1}((1-u)(1-v)-(1-v)(1-u))=1+(1-u)^{-1}(1-v)^{-1}(1-u,1-v)=1+(1-u)^{-1}(1-v)^{-1}(u,v)=1+(1+u+\ldots+u^{m-1})(1+v+\ldots+v^{m-1})(u,v)$. Lemma is proved. □

Let G be an arbitrary group, FG its *group algebra* over the field F. We recall that FG is a free F-module with the basis $\{g\colon\ g\in G\}$ and for elements of this basis the product is defined as their product in the group G. If H is a subgroup of the group G, then we denote by ωH the left ideal of the group algebra FG generated by all the elements $1-h$ $(h\in H)$. If $H=G$, then ωG is called the *augmentation ideal* of the group algebra FG.

Lemma 2 [7]**.** *Let H be a subgroup of the group G. Then*

(1) *if the elements h_i generate the subgroup H, then the elements $1-h_i$ generate the right ideal ωH;*
(2) *if $h\in G$, then $1-h\in\omega H$ if and only if $h\in H$;*
(3) *H is a normal subgroup in G if and only if ωH is a two-sided ideal of the algebra ωG;*
(4) *if H is a normal subgroup of the group G, then $F(G/H)\cong FG/\omega H$;*
(5) $\omega G=\Big\{\sum\limits_{g\in G}\lambda_g g\colon\ \sum\limits_{g\in G}\lambda_g=0\Big\}$;
(6) *the augmentation ideal ωG is generated as an F-module by elements of the form $1-g$ $(g\in G)$.*

Lemma 3 [7]**.** *Let G be a locally finite p-group and let F be a field of characteristic p. Then the augmentation ideal ωG is locally nilpotent.*

Following the ideas from [8] we consider the groups A_n, B_n and C_n. The group A_n has the representation $A_n=\langle a_1,a_2,\ldots,a_{4n}\colon\ a_i^p=1,\ \ [a_i,a_j,a_k]=1 \text{ for all } i,j,k\rangle$ where p is any natural number.

The identities

$$[xy, z] = [x, z][x, z, y][y, z], \quad [x, yz] = [x, z][x, y][x, y, z] \tag{5}$$

hold in any group. Then, using induction on the words length relative to the variables $a_1, a_2, \ldots, a_{4n}$, it is easy to show that the group A_n is nilpotent of class 2. It follows that the derived group A'_n lies in the centre of A_n. Let us now show that the identity

$$[u_1, u_2]^p = 1 \tag{6}$$

holds in the group A_n. We will prove it by induction. We have $[a_i^p, a_j] = 1$. Further, it follows from (5) that $[u_1, u_2]^p = [u_1^p, u_2]$. Suppose that $u_1 = u_2 u_3$ and that $[u_3^p, u_2] = [u_4^p, u_2] = 1$. Then $[u_1^p, u_2] = [u_1, u_2]^p = [u_3 u_4, u_2]^p = [u_3, u_2]^p [u_4, u_2]^p = [u_3^p, u_2][u_4^p, u_2] = 1$, i.e. the identity (6) is proved. It follows from (6) that the derived group A'_n is an elementary abelian p-group. As A_n/A'_n is also an elementary abelian p-group, the group A_n has the exponent p^2 and is finite, for it is the extension of a finite group with help of a finite group.

Now if $u \in A'_n$, then by (5) u can be written uniquely as the product

$$\prod_{1 \leqslant i < j \leqslant 4n} [a_i, a_j]^{\beta_{ij}},$$

where $\beta_{ij} = 0, 1, \ldots p-1$. Consider the expression

$$\prod_{1 \leqslant i < j \leqslant 4n} (1 + x_{ij})^{\beta_{ij}}.$$

Suppose that the polynomial obtained after opening the parentheses contains the monomials $\alpha_i x_{i_1 i_2} x_{i_3 i_4} \ldots x_{i_{4n-1} i_{4n}}$, where $\{i_1, i_2, \ldots, i_{4n}\} = \{1, 2, \ldots, 4n\}$. Let s_i denote the number of inversions in the permutation $i_1, i_2, \ldots, i_{4n}$, $\varrho(u) = \sum_i (-1)^{s_i} \alpha_i \pmod p$.

If $u \in A_n$, let $\overline{u}$ denote the image of u under the homomorphism $A_n \to A_n/A'_n$. We define now the group B_n. It has the representation

$$B_n = \left\langle b^u, c^k \colon\ u \in A_n,\ \ k \in A_n/A'_n \right\rangle,$$

where $b \notin A_n$ and B_n satisfies the relations

$$\begin{aligned} (b^u)^p &= (c^k)^p = 1, \\ [b^u, b^v] &= 1 \quad \text{if } \overline{u} \neq \overline{v}, \\ [b^u, b^v] &= (c^{\overline{u}})^{\varrho(uv^{-1})} \quad \text{if } \overline{u} = \overline{v}, \\ [b^u, c^k] &= 1, \end{aligned} \tag{7}$$

for all $u, v \in A_n$, $k \in A_n/A_n'$. We will show later that $\varrho(u)$ is not zero for some $u \in A_n'$ and so $B_n = \langle b^u \colon u \in A_n \rangle$, $B_n' = \left\langle c^k \colon k \in A_n/A_n' \right\rangle$ and B_n' lies in the centre of B_n.

The group B_n is homomorphic image of the group $B_n^\star = \langle b^u \colon u \in A_n, \ (b^u)^p = 1, [b^{u_1}, b^{u_2}, b^{u_3}] = 1, \ \text{for all } u_1, u_2, u_3 \in A_n \rangle$. The derived group of $B_n^\star$ is an elementary abelian p-group and the elements $[b^u, b^v]$ form an independent generating set for it satisfying only the relations

$$[b^u, b^v] = 1, \quad [b^u, b^v] = [b^v, b^u]^{-1}.$$

Now, if $\overline{u} = \overline{v}$, then $uv^{-1} \in A_n'$, therefore $uv^{-1} = (vu^{-1})^{-1}$ and $(c^{\overline{u}})^{\varrho(uv^{-1})} = (c^{\overline{v}})^{(-\varrho(vu^{-1}))}$. We also have $\varrho(1) = 0$, consequently the relations of the group B_n $[b^u, b^v] = 1$ if $\overline{u} \neq \overline{v}$ and $[b^u, b^v] = (c^{\overline{u}})^{\varrho(uv^{-1})}$ if $\overline{u} = \overline{v}$ do not impose any restrictions on the group $\left\langle c^k \colon k \in A_n/A_n' \right\rangle$. Therefore B_n' is an elementary abelian p-group and the elements c^k, where $k \in A_n/A_n'$, form a set of independent generators for B_n'. Moreover, the group B_n is finite and has exponent p^2.

Define the action of A_n on B_n as follows. Let $(b^u)^v = b^{uv}$ for all $u, v \in A_n$ and let $(c^k)^u = c^{k\overline{u}}$ for all $k \in A_n/A_n'$ and all $u \in A_n$. It is straightforward to check that this action determines a monomorphism of the group A_n into the group of automorphisms of the group B_n. Form the semidirect product C_n of A_n and B_n. Since A_n, B_n are finite and of exponent p^2, C_n is finite of exponent p^4. Let γ_3 denote the subgroup of C_n generated by all commutators of the form $[u_1, u_2, u_3]$ of the group C_n. We have $\gamma_3(C_n) \subseteq B_n$, therefore $[\gamma_3(C_n), \gamma_3(C_n)] \subseteq B_n' = \langle c^k \rangle$ which is centralized by A_n' and by B_n and hence by C_n'. Therefore C_n satisfies the identity

$$\alpha(x_1, x_2, \ldots, x_8) = [[x_1, x_2, x_3], [x_4, x_5, x_6], [x_7, x_8]] = 1. \tag{8}$$

Let us now show that the inequality

$$[[b, a_1, a_2, a_3], [a_1, a_2], [a_3, a_4], \ldots, [a_{4n-1}, a_{4n}], [b, a_1, a_2]] \neq 1 \tag{9}$$

is true in the group C_n.

The group B_n is nilpotent of class 2, hence it follows from the identities (5) that for any $t_1, t_2, t_3 \in B_n$,

$$[t_1 t_2, t_3] = [t_1, t_3][t_2, t_3], [t_1, t_2 t_3] = [t_1, t_2][t_1, t_3].$$

We will use this fact without further reference.

Let $\mathbb{Z}_p$ be the ring of integers modulo p, let $\mathbb{Z}_pA'_n$ be the group ring of the group A'_n over the ring $\mathbb{Z}_n$. If $k \in \mathbb{Z}_pA'_n$ then $k = \alpha_1u_1 + \alpha_2u_2 + \ldots + \alpha_ru_r$, where $\alpha_1, \alpha_2, \ldots, \alpha_r \in \mathbb{Z}_p$, $u_1, u_2, \ldots, u_r \in A'_n$. Then we define

$$\begin{aligned}[][t_1^k, t_2] &= [(t_1^{u_1})^{\alpha_1}, t_2][(t_1^{u_2})^{\alpha_2}, t_2] \ldots [(t_1^{u_r})^{\alpha_r}, t_2] \\ &= [t_1^{u_1}, t_2]^{\alpha_1}[t_1^{u_2}, t_2]^{\alpha_2} \ldots [t_1^{u_r}, t_2]^{\alpha_r}.\end{aligned}$$

We remark that this definition does not lead to a contradiction, as B'_n is an elementary abelian p-group. If $k_1, k_2 \in \mathbb{Z}_pA'_n$, $t_1, t_2 \in B_n$, then $[t_1^{k_1+k_2}, t_2] = [t_1^{k_1}, t_2][t_1^{k_2}, t_2]$. For $u \in A'_n$, $t_1, t_2 \in B_n$ we also have $[t_1, u, t_2] = [t_1^{u-1}, t_2]$.

Extend $\varrho\colon A'_n \to \mathbb{Z}_p$ linearly to the function $\varrho\colon \mathbb{Z}_pA'_n \to \mathbb{Z}_p$. Then for $u, v \in A_n$, $k \in \mathbb{Z}_pA'_n$ we have $[b^{uk}, b^v] = 1$, if $\overline{u} \neq \overline{v}$ and $[b^{uk}, b^v] = (c^{\overline{u}})^{\varrho(ukv^{-1})}$, if $\overline{u} = \overline{v}$. Further we have

$$\begin{aligned}&[b, [a_{i_1}, a_{i_2}], [a_{i_3}, a_{i_4}], \ldots, [a_{i_{4n-1}}, a_{i_{4n}}], b] \\ &\qquad = [b^{([a_{i_1}, a_{i_2}]-1),([a_{i_3}, a_{i_4}]-1)\ldots([a_{i_{4n-1}}, a_{i_{4n}}]-1)}, b].\end{aligned}$$

But $(1+x_{i_1i_2}-1)(1+x_{i_3i_4}-1)\ldots(1+x_{i_{4n-1}i_{4n}}-1) = x_{i_1i_2}x_{i_3i_4}\ldots x_{i_{4n-1}i_{4n}}$, therefore, if s denotes the number of inversions in the permutation $i_1, i_2, \ldots, i_{4n}$, then

$$\begin{aligned}&\varrho([a_{i_1}, a_{i_2}]-1)([a_{i_3}, a_{i_4}]-1)\ldots([a_{i_{4k-1}}, a_{i_{4k}}]-1) \\ &\quad = \begin{cases} (-1)^s & \text{if } k = n \text{ and} \{i_1, i_2, \ldots, i_{4n}\} = \{1, 2, \ldots, 4n\}, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Consequently, we obtain from (7) that

$$\begin{aligned}&[b, [a_{i_1}, a_{i_2}], [a_{i_3}, a_{i_4}], \ldots, [a_{i_{4n-1}}, a_{i_{4n}}], b] \\ &\quad = \begin{cases} c^{(-1)^s} & \text{if } k = n \text{ and } \{i_1, i_2, \ldots, i_{4n}\} = \{1, 2, \ldots, 4n\}, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Further, $[b, a_1] = b^{-1}b^{a_1}$ and $[b, a_1, a_2] = b^{-a_1}bb^{-a_2}b^{a_1a_2}$, therefore

$$\begin{aligned}&[[b, a_1, a_2], [a_1, a_2], [a_3, a_4], \ldots, [a_{4n-1}, a_{4n}], [b, a_1, a_2]] \\ &\quad = [b^{-a_1}bb^{-a_2}b^{a_1a_2}, [a_1, a_2], [a_3, a_4], \ldots, [a_{4n-1}, a_{4n}], b^{-a_1}bb^{-a_2}b^{a_1a_2}] \\ &\quad = [b, [a_1, a_2], [a_3, a_4], \ldots, [a_{4n-1}, a_{4n}], b]^{-a_1}[b, [a_1, a_2], [a_3, a_4], \ldots, \\ &\qquad [a_{4n-1}, a_{4n}], b][b, [a_1, a_2], [a_3, a_4], \ldots, [a_{4n-1}, a_{4n}], b]^{-a_2} \\ &\qquad \times [b, [a_1, a_2], [a_3, a_4], \ldots, [a_{4n-1}, a_{4n}], b]^{a_1a_2} \\ &\quad = c^{-a_1}cc^{-a_2}c^{a_1a_2}.\end{aligned}$$

Earlier we have shown that $\{c^{\overline{u}}\colon\ u \in A_n\}$ is a set of independent generators for B'_n, therefore $c^{-a_1}cc^{-a_2}c^{a_1a_2} \neq 1$. Consequently, the inequality (7) is proved.

Lemma 4. *Let* $t_i, t_j, t_r, t_1, t_2, \ldots, t_k \in B_n$, $u_1, u_2, \ldots, u_k \in A'_n$. *Then*

$$(10) \qquad [t_i t_j, u_1, u_2, \ldots, u_k, t_r] = [t_i, u_1, u_2, \ldots, u_k, t_r][t_j, u_1, u_2, \ldots, u_k, t_r],$$

$$(11) \qquad [t_i, u_1 t_1, u_2 t_2, \ldots, u_k t_k, t_j] = [t_i, u_1, u_2, \ldots, u_k, t_j].$$

Proof. The subgroup B_n is normal in C_n, therefore for $t \in B_n$, $w \in C_n$ we have $[t, w] = t^{-1} t^w \in B_n$ and it is nilpotent of class 2, therefore $[t_i t_j, t_r] = [t_j t_i, t_r]$. Then

$$\begin{aligned} [t_i t_j, u_1, u_2, \ldots, u_k, t_r] &= [(t_i t_j)^{(u_1-1)(u_2-1)\ldots(u_k-1)}, t_r] \\ &= [(t_i)^{(u_1-1)(u_2-1)\ldots(u_k-1)}, t_r][(t_j)^{(u_1-1)(u_2-1)\ldots(u_k-1)}, t_r] \\ &= [t_i, u_1, u_2, \ldots, u_k, t_r][t_j, u_1, u_2, \ldots, u_k, t_r], \end{aligned}$$

i.e. the equality (10) is proved.

Further, by (5),

$$\begin{aligned} [t_i, u_1 t_1, t_j] &= [[t_i, t_1][t_i, u_1][t_i, u_1, t_1], t_j] \\ &= [t_i, t_1, t_j][t_i, u_1, t_j][t_i, u_1, t_1, t_j] = [t_i, u_1, t_j]. \end{aligned}$$

Suppose that the equality (11) is true for all numbers smaller than k. Then by the induction hypothesis and (10)

$$\begin{aligned} [t_i, u_1 t_1, u_2 t_2, \ldots, u_k t_k, t_j] &= [[t_i, u_1 t_1], u_2, \ldots, u_k, t_j] \\ &= [[t_i, t_1][t_i, u_1][t_i, u_1, t_1], u_2, \ldots, u_k, t_j] \\ &= [[t_i, t_1], u_2, \ldots, u_k, t_j][[t_i, u_1], u_2, \ldots, u_k, t_j] \\ &\quad \times [[t_i, u_1, t_1], u_2, \ldots, u_k, t_j] \\ &= [t_i, u_1, u_2, \ldots, u_k, t_j], \end{aligned}$$

i.e. the equality (11) is also proved. □

Lemma 5. *Let* $t_1, t_2 \in B_n$. *Then*

$$[t_1, [a_i, a_j], [a_i, a_k], t_2] = 1$$

and

$$[t_1, [a_i, a_j], [a_k, a_l], t_2] = [t_1, [a_k, a_j], [a_i, a_l], t_2]^{-1}$$

for any i, j, k, l.

P r o o f. By (10) the expression $[t_1,[a_i,a_j],[a_i,a_k],t_2]$ is a product of factors of the form

$$[b^u,[a_i,a_j],[a_i,a_k],b^v] = [b^{u([a_i,a_j]-1)([a_i,a_k]-1)},b^v] = \begin{cases} 1 & \text{if } \overline{u}\neq\overline{v}, \\ (c^{\overline{u}})^{\varrho(u([a_i,a_j]-1)([a_i,a_k]-1)v^{-1})} & \text{if } \overline{u}=\overline{v}. \end{cases}$$

Obviously, in order to prove the equality $[t_1,[a_i,a_j],[a_i,a_k],t_2]=1$ it is enough to show that $\varrho(([a_i,a_j]-1)([a_i,a_k]-1)u)=0$ for any $u\in A'_n$. Suppose that $u=\prod[a_r,a_s]^{\beta_{rs}}$. Consider the expression $(1+x_{ij}-1)(1+x_{ik}-1)\prod(1+x_{rs})^{\beta_{rs}}$. It is obvious that any polynomial's monomial, obtained after opening the parentheses contains the product $x_{ij}x_{ik}$. Then $\varrho(([a_i,a_j]-1)([a_i,a_k]-1)u)=0$.

By analogy, in order to prove the second equality it is enough to show that

$$\varrho(([a_i,a_j]-1)([a_k,a_l]-1)u) = -\varrho(([a_k,a_j]-1)([a_i,a_l]-1)u) \quad \text{for every } u\in A'_n. \tag{12}$$

Let $u=\prod[a_r,a_s]^{\beta_{rs}}$. We have $(1+x_{ij}-1)(1+x_{kl}-1)(\prod(1+x_{rs})^{\beta_{rs}} = x_{kj}x_{il}\prod(1+x_{rs})^{\beta_{rs}}$ and $(1+x_{kj}-1)(1+x_{il}-1)\prod(1+x_{rs})^{\beta_{rs}} = x_{kj}x_{il}\prod(1+x_{rs})^{\beta_{rs}}$. As $\{i,j,k,l\}=\{k,j,i,l\}$ and these permutations differ by an odd number of inversions, both the expressions have the same number of terms of the form $x_{i_1i_2}\times x_{i_3i_4}\dots x_{i_{4n-1}i_{4n}}$, where $\{i_1,i_2,\dots,i_{4n}\}=\{1,2,\dots,4n\}$ and $\varrho(([a_i,a_j]-1)([a_k,a_l]-1)u)=-\varrho(([a_k,a_j]-1)([a_i,a_l]-1)u)$ by the definition of the mapping ϱ. The lemma is proved. □

Lemma 6. *Let $t_1,t_2\in B_n$ and $h,u,m\in A'_n$. Then*

$$[t_1,[u,hm],t_2]=[t_1,[u,h],t_2][t_1,[u,m],t_2].$$

P r o o f. We will prove the lemma by induction on the sum of the lengths of the words u, h, m written as products of the elements a_i. The result is trivial if this sum does not exceed 2. Further, taking in consideration (5), we have $[t_1,[hm],t_2]=[t_1,[u,h][u,m],t_2]=[[t_1,[u,m]][t_1,[u,h]][t_1,[u,h],[u,m]],t_2]=[[t_1,[u,m],t_2][t_1,[u,h],t_2][t_1,[u,m],[u,h],t_2]$. Therefore, in order to prove the lemma, it is enough to show that

$$[t_1,[u,m],[u,h],t_2]=1.$$

By the induction hypothesis $[t_1,[u,m],[u,h],t_2]$ is a product of factors of the form

$$[t_1,[u,h],[u,a_i],t_2]=[t_1^{([u,h]-1)([h,a_i]-1)},t_2] = [t_1^{([u,a_1]-1)([u,h]-1)},t_2]=[t_1,[u,a_i],[u,h],t_2].$$

Again by the induction hypothesis, the last expression is a product of factors of the form $[t_1, [u, a_i,], [u, a_j], t_2]$. Let $u = a_{i_1} a_{i_2} \dots a_{i_r}$. Once again by the induction hypothesis

$$\begin{aligned}
[t_1, [u, a_i], [u, a_j], t_2] = & \prod_{1 \leqslant s,t \leqslant r} [t_1, [a_{i_s}, a_i], [a_{i_t}, a_j], t_2] \\
= & \prod_{1 \leqslant s \leqslant r} [t_1, [a_{i_s}, a_i], [a_{i_s}, a_j], t_2] \\
& \times \prod_{1 \leqslant s,t \leqslant r} [t_1, [a_{i_s}, a_i], [a_{i_t}, a_j], t_2] \times [t_1, [a_{i_t}, a_i], [a_{i_s}, a_j], t_2] = 1
\end{aligned}$$

by Lemma 5. The lemma is proved. □

Lemma 7. *Let $t_1, t_2 \in B_n$ and $u_i = [w_{i_1}, w_{i_2}]$, where $w_{i_j} \in A_n$, $i = 1, 2, \dots, 2k$. Then*

$$[t_1 t_2, u_1, u_2, \dots u_{2k}, t_1 t_2] = [t_1, u_1, u_2, \dots u_{2k}, t_1][t_2, u_1, u_2, \dots u_{2k}, t_2].$$

P r o o f. Taking in consideration (10) it is sufficient to show that

$$[t_1, u_1, u_2, \dots u_{2k}, t_2] = [t_2, u_1, u_2, \dots u_{2k}, t_1]^{-1}. \tag{13}$$

We will use the fact that A_n' centralizes B_n'. Hence we have $[t_1^u, t_2] = [t_1^u, t_2]^{u^{-1}} = [t_1, t_2^{u^{-1}}]$ for $u \in A_n'$. The group B_n is nilpotent of class 2, consequently $[t_1, u_1, t_2] = [t_1^{-1} t_1^u, t_2] = [t_1^{-1}, t_2][t_1^u, t_2] = [t_1, t_2]^{-1}[t_1, t_2^{u^{-1}}] = [t_1, t_2^{-1}][t_1, t_2^{u^{-1}}] = [t_1, t_2^{-1} t_2^{u^{-1}}] = [t_1, [t_2, u^{-1}]] = [t_2, u^{-1}, t_1]^{-1}$. Further, by induction we obtain that

$$[t_1, u_1, u_2, \dots, u_{2k}, t_2] = [t_2, u_{2k}^{-1}, \dots, u_2^{-1}, u_1^{-1}, t_1]^{-1}.$$

By Lemmas 4, 6 the left- and right-hand sides of the last equality can be represented as products of factors of the form

$$[b^u, [a_{i_1}, a_{i_2}], [a_{i_3}, a_{i_4}], \dots, [a_{i_{4n-1}}, a_{i_{4n}}], b^v]$$

and

$$\begin{aligned}
&[b^u, [a_{i_{4n-1}}, a_{i_{4n}}]^{-1}, \dots, [a_{i_3}, a_{i_4}]^{-1}, [a_{i_1}, a_{i_2}]^{-1}, b^v] \\
&\qquad = [b^v, [a_{i_{4n}}, a_{i_{4n-1}}], \dots, [a_{i_4}, a_{i_3}], [a_{i_2}, a_{i_1}], b^u]^{-1},
\end{aligned}$$

respectively, where $u, v \in A_n$.

Now it is obvious that in order to prove the equality (13) it is enough to show that

$$\varrho(([a_{i_{4n}}, a_{i_{4n-1}}] - 1) \dots ([a_{i_4}, a_{i_3}] - 1)([a_{i_2}, a_{i_1}] - 1)u)$$
$$= \varrho(([a_{i_1}, a_{i_2}] - 1)([a_{i_3}, a_{i_4}] - 1) \dots [a_{i_{4n-1}}, a_{i_{4n}}] - 1)u)$$

for any $u \in A_n$. This equality is proved similarly to (12), just taking into account that the permutations $i_1, i_2, \dots, i_{4k}$ and $i_{4k}, i_{4k-1}, \dots, i_1$ differ by an even number of inversions. □

Lemma 8. *The identities*

(14) $$\beta_k = \beta_k(x, y, z, u; x_1, x_2, \dots, x_{4k}) = [[x, y, z], [x_1, x_2], [x_3, x_4], \dots, [x_{4k-1}, x_{4k}], [u, y, z]] \times [[u, y, z], [x_1, x_2], [x_3, x_4], \dots, [x_{4k-1}, x_{4k}], [x, y, z]] = 1,$$

(15) $$\gamma_k = \gamma_k(x, y, z, u; x_1, x_2, \dots, x_{4k}) = [[x, y, z], [x_1, x_2], [x_3, x_4], \dots, [x_{4k-1}, x_{4k}], [x, y, u]] \times [[x, y, u], [x_1, x_2], [x_3, x_4], \dots, [x_{4k-1}, x_{4k}], [x, y, z]] = 1,$$

(16) $$\delta_k = \delta_k(x, y, z; x_1, x_2, \dots, x_{4k}) = [[x, y, z], [x_1, x_2], [x_3, x_4], \dots, [x_{4k-1}, x_{4k}], [x, y, z]] = 1$$

are true in the group C_n for $k \neq n$.

P r o o f. The subgroup B_n is normal in C_n, therefore we have $[w_1, w_2] = [u_1, u_2]t$ for any $w_1, w_2 \in C_n$ and $u_1, u_2 \in A_n$, $t \in B_n$. Further, the group A_n is nilpotent of class 2, hence $\gamma_3(C_n) \subseteq B_n$. Therefore in order to prove (14), (15) it is sufficient to show, by (10), that $[t_1, [u_1, u_2], [u_3, u_4], \dots, [u_{4k-1}, u_{4k}], t_2] = [t_2, [u_1, u_2], [u_3, u_4], \dots, [u_{4k-1}, u_{4k}], t_1]^{-1}$. This equality follows from (13).

By analogy, in order to prove the identity (16) it is sufficient to show that $[t_1, [u_1, u_2], [u_3, u_4], \dots, [u_{4k-1}, u_{4k}], t_1] = 1$. By Lemmas 5 and 7 the left-hand side of this equality is a product of factors of the form $[b^u, [a_{i_1}, a_{i_2}], [a_{i_3}, a_{i_4}], \dots, [a_{i_{4k-1}}, a_{i_{4k}}], b^u] = [b, [a_{i_1}, a_{i_2}], [a_{i_3}, a_{i_4}], \dots, [a_{i_{4k-1}}, a_{i_{4k}}], b]^u = 1$ for $k \neq n$. The lemma is proved. □

Let $\mathfrak{M}$ denote the variety of groups, defined by the identity (8), let $\mathfrak{N}_p$ be the variety of nilpotent groups of class at most 2 defined by the identity $x^{p^2} = 1$. We have shown above that $C_n \in \mathfrak{M} \cap \mathfrak{N}_p\mathfrak{N}_p$. Hence we obtain directly from (8), (9), (14)–(16) the following

Proposition 1. *The system of identities* $\{\delta_k = 1\}$, $k = 1, 2, \ldots$, *is independent in the variety of groups* $\mathfrak{M} \cap \mathfrak{N}_p\mathfrak{N}_p$.

We remark that the variety $\mathfrak{M} \cap \mathfrak{N}_p\mathfrak{N}_p$ is locally nilpotent and locally finite. We also remark that an analogous result was obtained in [8] for the case of $p = 2$.

Lemma 9. *The identity* $\delta_n = 1$ *is not a consequence of the system of identities* $\beta_k = 1$, $\gamma_k = 1$, $\delta_k = 1$ *for* $k \neq n$ *and* $\alpha(x_1, x_2, \ldots, x_8) = 1$ *in the variety of groups* $\mathfrak{M} \cap \mathfrak{N}_p\mathfrak{N}_p$.

We will further assume that F is an infinite field of characteristic $p > 0$. The group C_n is a finite p-group, therefore it follows from Lemma 3 that the augmentation ideal of ωC_n is nilpotent. Then, as was shown before Lemma 1, the set $\overline{C}_n = 1 - \omega C_n$ forms a group. Obviously, $C_n \subseteq \overline{C}_n$. Then $FC_n \subseteq F\overline{C}_n$ and $\omega C_n \subseteq \omega\overline{C}_n$. Using item (6) of Lemma 2 it is easy to show that $\omega C_n \cong \omega\overline{C}_n$. The algebra $\omega\overline{C}_n$ is nilpotent and F has characteristic p, so it is easy to show that the identity $x^{p^k} = 1$ holds in the group $\overline{C}_n$ for some k, and it follows from Lemma 1 that the group $\overline{C}_n$ is nilpotent. It follows from this that $\overline{C}_n$ contains a finite descending central series

$$\overline{C}_n = \overline{D}_1 \supset \overline{D}_2 \supset \ldots \supset \overline{D}_{r+1} = 1, \tag{17}$$

which possesses the property that all the elements of its quotient group $\overline{D}_i/\overline{D}_{i+1}$ have order p. Therefore each group $\overline{D}_i/\overline{D}_{i+1}$ is a direct product of cyclic groups of order p. We denote by $\overline{d}_{i\alpha}$ those elements of the group $\overline{D}_i$ whose images in $\overline{D}_i/\overline{D}_{i+1}$ are independent generators of the group $\overline{D}_i/\overline{D}_{i+1}$. Then each element $\overline{g} \in \overline{C}_n$ is uniquely written in the form

$$\overline{g} = \overline{d}_{1\alpha_1}^{j_1} \ldots \overline{d}_{1\alpha_m}^{j_m} \overline{d}_{2\beta_1}^{s_1} \ldots \overline{d}_{2\beta_l}^{s_l} \ldots \overline{d}_{n\gamma_1}^{t_1} \ldots \overline{d}_{n\gamma_k}^{t_k}, \tag{18}$$

where $0 < j, s, t < p$. We will assume that $\delta_1 < \delta_2 < \ldots$, where $\delta_i = \alpha_i; \beta_i; \gamma_i$.

We denote $d = 1 - \overline{d}$. Then $d \in \omega\overline{C}_n$. We will show that elements of the form

$$g = d_{a\alpha_1}^{j_1} \ldots d_{1\alpha_m}^{j_m} d_{2\beta_1}^{s_1} \ldots d_{2\beta_l}^{s_l} \ldots d_{n\gamma_1}^{t_1} \ldots d_{n\gamma_k}^{t_k} \tag{19}$$

form F-basis of the algebra $\omega\overline{C}_n$. Indeed, the sequence $u = (j_1, \ldots, j_m, s_1, \ldots, s_l, t_1, \ldots, t_k)$ will be called the *defining vector* of the element g. Suppose that the defining vectors for elements $g_1, g_2 \in \omega\overline{C}_n$ are u and v. We graphically define them in the following way. Suppose that a factor $\overline{d}_{i\alpha}$ in one of the decompositions of the form (18) of the elements $\overline{g}_1$, $\overline{g}_2$ lacks. Then we write this factor in the decomposition with the power 0. We have obtained new defining vectors $\overline{u} = (\varphi_1, \varphi_2, \ldots, \varphi_r)$, $\overline{v} = (\psi_1, \psi_2, \ldots, \psi_r)$ for the elements $\overline{g}_1$, $\overline{g}_2$. We will say that the *order* of the element g_1 is higher than the order of the element g_2 if $\varphi_i = \psi_i$ for $i = 1, 2, \ldots, s$, but $\varphi_{s+1} > \psi_{s+1}$.

Let g_k have the highest order among the elements g_i $(i = 1, 2, \ldots, t)$. Then in the notation of the polynomial $\alpha_1 g_1 + \alpha_2 g_2 + \ldots + \alpha_t g_t$ $(\alpha_i \in F)$ in terms of elements of the group $\overline{C}_n$ the coefficient of d_k is equal to $\pm\alpha_k$. This directly follows from the uniqueness of each element's notation in the form (18). Consequently, the elements (19) are linearly independent and form F-basis of the algebra $\omega\overline{C}_n$. Therefore each element of $\omega\overline{C}_n$ can be represented as a linear combination of monomials (19), moreover, this presentation is unique.

The number $j_1 + \ldots + j_m + p(s_1 + \ldots + s_l) + \ldots + p^{r-1}(t_1 + \ldots + t_k)$ will be called the *weight* of the monomial (19) of the algebra $\omega\overline{C}_n$. The weight of the polynomial's lowest monomial will be called the polynomial's weight. We denote by D_s the submodule of the F-module $D = \omega\overline{C}_n$, generated by the monomials from D whose weights are not less than s. We will show that the inclusion

$$D_s \cdot D_t \subseteq D_{s+t} \tag{20}$$

is true in the algebra D. Indeed, consider a monomial of the general form

$$d_{i_1\alpha_1}^{j_1} d_{i_2\alpha_2}^{j_2} \ldots d_{i_{s+t}\alpha_{s+t}}^{s+t} \quad (0 < j_i < p) \tag{21}$$

from $D_s \cdot D_t$. If the powers of the same basic element are situated side by side in (21), for example $d_{i\alpha}^{j_m} d_{i\alpha}^{j_{m+1}}$ and $j_m + j_{m+1} < p$, then we substitute the expression $d_{i\alpha}^{j_m + j_{m+1}}$ for this pair. The total weight does not change. But if $j_m + j_{m+1} = j + p$, then the product $d_{i\alpha}^{j_m} d_{i\alpha}^{j_{m+1}}$ will be represented in the form $d_{i\alpha}^{j} d_{i\alpha}^{p}$. As $d_{i\alpha}^{p}$ enters higher members of the central series (17), the sum of the monomials that belong to these higher members can be substituted for $d_{i\alpha}^{p}$. The total weight does not change. Let now $d_{i_m\alpha_m}^{j_m} d_{i_{m+1}\alpha-m+1}^{j_{m+1}}$ be such elements that their orders are inverse in the normal form (19). Then for the product $d_{i_m\alpha_m}^{j_m} d_{i_{m+1}\alpha_{m+1}}^{j_{m+1}}$ we substitute the expression

$$d_\beta d_\alpha = d_\alpha d_\beta + d_\alpha [d_\beta, d_\alpha] + d_\beta [d_\beta, d_\alpha] - [d_\beta, d_\alpha] - d_\alpha d_\beta [d_\beta, d_\alpha], \tag{22}$$

where $d_\alpha = d_{i_m\alpha_m}^{j_m}$, $d_\beta = d_{i_{m+1}\alpha_{m+1}}^{j_{m+1}}$, $[d_\beta, d_\alpha] = d_\beta^{-1} d_\alpha^{-1} d_\beta d_\alpha$. As (17) is the central series of the group $\overline{C}_n$, the commutator $[d_\beta, d_\alpha]$ will be contained in a member of a higher number than d_α and d_β. Therefore the weights of the members from the right-hand side of the equality (22) is not less than $d_\beta d_\alpha$. Using the stated rules, we are able to express each product of the form (21) as a product of higher order, and the members' weight does not diminish.

The group C_n is finite. Hence there are only a finite number of factors of the form $\overline{d}_{i\alpha}$ in (18). Therefore each monomial of (21) can be reduced (by a finite number of transformations) to a polynomial whose every monomial has the form (19). Then the weight of the monomial's product is not less than the sums of these polynomials' weights. Consequently, the inclusion (20) is proved.

From (20) and the above proved statement about the basis of the algebra D it follows that D_i is an ideal of the algebra D and that $D^i \subseteq D_i$, where D^i means the ith power of the algebra D. Let $\overline{D}_i$ be the ith member of the central series (17). We will show that

$$\overline{D}_i \subseteq 1 - D_i \tag{23}$$

for $i = 1, 2, \ldots, r+1$. In order to prove (23) we will show that $\overline{D}_i \subseteq 1 - D^i$. We have $\overline{D} = \overline{D}_1 \subseteq 1 - D^1$. Suppose that $\overline{D}_i \subseteq 1 - D^i$ and let $\overline{d}_i = 1 - d_i \in 1 - D^i$, $\overline{d} = 1 - d \in 1 - D$. Then $1 - [\overline{d}_i, \overline{d}] = \overline{d}_i^{-1} d^{-1}(\overline{d}\overline{d}_i - \overline{d}_i\overline{d}) = 1 - \overline{d}_i^{-1}\overline{d}(\overline{d}, \overline{d}_i) = 1 - \overline{d}_i\overline{d}(d, d_i) \in 1 - D^{i+1}$, i.e. the inclusion (23) is proved.

By the construction, $C_n/B_n \cong A_n$. Then by item 4) of Lemma 2 $FC_n/\omega B_n \cong FA_n$. Earlier we have shown that $FC_n = F\overline{C}_n$. It is obvious that $\overline{B}_n = 1 - \omega B_n$ will be the kernel of the homomorphism induced on the group $\overline{C}_n$ by the homomorphism $F\overline{C}_n \to F\overline{C}_n/\omega B_n$. Therefore $\overline{B}_n$ is a normal subgroup of the group $\overline{C}_n$. Further, by item 5) of Lemma 2 the homomorphism $F\overline{C}_n \to F\overline{C}_n/\omega B_n$ preserves the sum of the polynomials' coefficients. Therefore, again by item 5) of Lemma 2, it follows from the relation $F\overline{C}_n/\omega B_n \cong FA_n$ that $\omega\overline{C}_n/\omega B_n \cong \omega A_n$, where ωA_n is the augmentation ideal of the group algebra FA_n. Now let us show that these relations imply that $\overline{C}_n/\overline{B}_n \cong \overline{A}_n$, where $\overline{A}_n = 1 - A_n$. Consider a homomorphism $\alpha\colon F\overline{C}_n \to FA_n$. Let $\overline{c} = 1 - c$, where $c \in \omega\overline{C}_n$. Then it follows from the relation $\omega\overline{C}_n/\omega B_n \cong \omega A_n$ that $\alpha(c + \omega B_b) = a$, where $a \in \omega A_n$. If e is the identity element of the group A_n, then $\alpha(\overline{c}\overline{B}_n) = \alpha((1-e)(1-\omega B_n)) = \alpha(1 - c - \omega B_n + c \cdot \omega B_n) = \alpha(1 - (c + \omega B_n)) = \alpha 1 - \alpha(c + \omega B_n) = e - a \in \overline{A}_n$. It means that the homomorphism α maps the group $\overline{C}_n$ into the group $\overline{A}_n$. But if $\overline{a} = e - a \in \overline{A}_n \subseteq \overline{C}_n$, then $\alpha\overline{a} = \alpha(e - a) = \alpha e - \alpha a = \alpha e - \alpha(c + \omega B_n) = \alpha e - \alpha(c + \omega B_n - c \cdot \omega B_n) = \alpha(1 - c - \omega B_n + c \cdot \omega B_n) = \alpha(1-c)(1 - \omega B_n) = \alpha(\overline{c}\overline{B}_n)$. It means that α is an epimorphism. Therefore $\overline{C}_n/\overline{B}_n \cong \overline{A}_n$. It follows that $\omega\overline{C}_n/\omega\overline{B}_n \cong \omega\overline{A}_n$. Further, as $F\overline{C}_n = FC_n$, it is easy to show that $\omega\overline{B}_n = \omega B_n$, $\omega\overline{A}_n = \omega A_n$. Therefore

$$\overline{C}_n = 1 - \omega C_n, \quad \overline{B}_n = 1 - \omega B_n, \quad \overline{A}_n = 1 - \omega A_n. \tag{24}$$

We denote $t = j_1 + \ldots + j_m + p(s_1 + \ldots + s_l)$. The set $(\omega\overline{B}_n)_t$ is an ideal of the algebra $\omega\overline{B}_n$, and $\omega\overline{B}_n$ is an ideal of the algebra $\omega\overline{C}_n$. It easily follows that $(\omega\overline{B}_n)_t$ will be an ideal of the algebra $\omega\overline{C}_n$, too. Consider the homomorphism $\varphi\colon \omega\overline{C}_n \to \omega\overline{C}_n/(\omega\overline{B}_n)_t$. Let $\varphi(\omega\overline{C}_n) = U_n$, $\varphi(\omega\overline{B}_n) = V_n$, $\varphi(\omega\overline{A}_n) = W_n$, $\varphi\overline{C}_n = \overline{U}_n$, $\varphi\overline{B}_n =$

$\overline{V}_n$, $\varphi\overline{A}_n = \overline{W}_n$. It follows from the relation $\omega\overline{C}_n/\omega\overline{B}_n \cong \omega\overline{A}_n$ that $U_n/V_n \cong W_n$. Now consider the homomorphism ψ: $U_n \to W_n \to W_n/(W_n)_t$ and let $\psi U_n = L_n$, $\psi V_n = M_n$, $\psi W_n = K_n$. The series (17) of the group $\overline{C}_n$ induces the central series $\overline{L}_n = (\overline{L}_n)_1 \supseteq (\overline{L}_n)_2 \supseteq \dots$ of the group $\overline{L}_n$, which in turn induces respectively the central series $\overline{M}_n = (\overline{M}_n)_1 \supseteq (\overline{M}_n)_2 \supseteq \dots$ and $\overline{K}_n = (\overline{K}_n)_1 \supseteq (\overline{K}_n)_2 \supseteq \dots$ of the subgroups $\overline{M}_n$ and $\overline{K}_n$, where $(\overline{M}_n)_i = \overline{M}_n \cap (\overline{L}_n)_i$, $(\overline{K}_n)_i = \overline{K}_n \cap (\overline{L}_n)_i$. The quotient groups $(\overline{M}_n)_i/(\overline{M}_n)_{i+1}$, $(\overline{K}_n)_i/(\overline{K}_n)_{i+1}$ are elementary abelian p-groups. Then it follows from the definition of series (17), homomorphisms φ, ψ and (23) that $(\overline{M}_n)_3 = 1$, $(\overline{K}_n)_3 = 1$ and that the derived groups $(\overline{M}_n)'$, $(\overline{K}_n)'$ are also elementary abelian p-groups. Therefore the groups $\overline{M}_n$, $\overline{K}_n$ are nilpotent of class 2 and have exponent p^2, i.e. they belong to the variety $\mathfrak{N}_p$. Then it follows from the relations $\omega\overline{C}_n/\omega\overline{B}_n \cong \omega\overline{A}_n$ and $\overline{C}_n/\overline{B}_n \cong \overline{A}_n$ that

$$(25) \qquad L_n/M_n \cong K_n, \quad \overline{L}_n/\overline{M}_n \cong \overline{K}_n \quad \overline{M}_n, \overline{K}_n \in \mathfrak{N}_p, \quad \overline{L}_n \in \mathfrak{N}_p\mathfrak{N}_p.$$

Now let us show that the homomorphism φ: $F\overline{C}_n \to F\overline{C}_n/(\omega\overline{B}_n)_t$ induces isomorphisms on the subgroups A_n, B_n of the group C_n. Indeed, let H be a normal subgroup of the group B_n corresponding to the induced homomorphism η. We have to show that $H = 1$. Assume the contrary. Let the element $1 \neq h \in H$ have the form (18). With help of the identity $1 - xy = 1 - x + 1 - y - (1-x)(1-y)$ we write the element $1 - h$ as a linear combination of monomials of the form (19). By the construction, the groups A_n, B_n are nilpotent of class 2, the derived groups A_n', B_n' and the quotient groups A_n/A_n', B_n/B_n' are elementary abelian p-groups. Then no monomial of the form (19) from the decomposition of the element $1 - h$ has a weight greater than t. It means that $1 - h$ does not belong to the ideal $(\overline{B}_n)_t$. On the other hand, by item 2) of Lemma 2 the element $1 - h$ belongs to the ideal ωH corresponding to the homomorphism η. We obtain a contradiction as it is obvious that $\omega H \subseteq H \cap (\overline{B}_n)_t$. Consequently, $H = 1$, i.e. η is an isomorphism of the subgroup B_n. The isomorphisms $\varphi A_n \cong A_n$ and $\psi(\varphi B_n) \cong B_n$, $\psi(\varphi A_n) \cong A_n$ can be proved by analogy. Therefore

$$(26) \qquad \psi(\varphi C_n) \cong C_n.$$

Let us denote elements of the group $\overline{L}_n$ by l and let E be the subgroup of the group $\overline{L}_n$, generated by all the expressions

$$(27) \qquad \alpha(l_1, l_2, \dots, l_8), \quad \beta_k(l_i, l_j, l_s, l_t; l_1, l_2, \dots, l_{4k}),$$
$$\gamma_k(i_i, l_j, l_s, l_t; l_1, l_2, \dots, l_{4k}), \quad \delta_k(l_i, l_j, l_s; l_1, l_2, \dots, l_{4k})$$

for $k \neq n$. Obviously, the subgroup E is normal in $\overline{L}_n$ and the identities (8), (14)–(16) hold in $\overline{L}_n/E$. By d, g_i we denote the images of the elements b, a_i under the homomorphism $\overline{C}_n \to \overline{L}_n/E = \overline{T}_n$. We have shown earlier that $\overline{L}_n \in \mathfrak{M} \cap \mathfrak{N}_p\mathfrak{N}_p$. Then $\overline{T}_n \in \mathfrak{M} \cap \mathfrak{N}_p\mathfrak{N}_p$, too. By Lemma 9 the identity $\delta_n = 1$ is not a consequence of the system of identities (8), (14)–(16) in the variety of groups $\mathfrak{M} \cap \mathfrak{N}_p\mathfrak{N}_p$. So it follows from (9) and (26) that the inequality

$$\delta_n(d, g_1, g_2; g_1, g_2, \ldots, g_{4n}) \neq 1 \tag{28}$$

is true in the group $\overline{T}_n$.

The subgroup E is normal in $\overline{L}_n$. Then by item 3) of Lemma 2, ωE will be an ideal of the algebra $F\overline{L}_n$. Consider a homomorphism $\varphi\colon F\overline{L}_n \to F\overline{L}_n/\omega E$. We denote $\varphi L_n = T_n$, $\varphi M_n = S_n$, $\varphi K_n = R_n$, $\varphi\overline{L}_n = \overline{T}_n$, $\varphi\overline{M}_n = \overline{S}_n$, $\varphi\overline{K}_n = R_n$. Earlier we have proved the following properties for the groups $\overline{M}_n$, $\overline{K}_n$: a) the groups $\overline{M}_n$, $\overline{K}_n$ belong to the variety $\mathfrak{N}_p$; b) the derived groups $\overline{M}'_n$, $\overline{K}'_n$ are elementary abelian p-groups. Then the relations

$$\overline{R}_n, \overline{S}_n \in \mathfrak{N}_p, \quad T_n/S_n \cong R_n, \quad \overline{T}_n/\overline{S}_n \cong R_n \tag{29}$$

follow from a) and (25). It follows from the properties a), b) that the identities $[x,y]^p = 1$, $[x,y,z] = 1$ hold in the groups $\overline{M}_n$, $\overline{K}_n$. Taking in consideration (5), the identity $[x^p, y] = 1$ follows from them. Therefore the identities

$$[x,y]^p = 1, \quad [x^p, y] = 1 \tag{30}$$

hold in the groups $\overline{R}_n$, $\overline{S}_n$.

Earlier we have shown that the algebra $\omega\overline{C}_n$ is nilpotent. Then the algebra T_n is also nilpotent, say, of index m. We denote the elements of the group C_n by u_i, and the images of the elements u_i under the homomorphism $\overline{C}_n \to \overline{T}_n$ by v_i. We introduce the notation

$$\begin{aligned} w_i &= 1 - v_i, \\ \{x,y\} &= -(1 + x + \ldots + x^{m-1})(1 + y + \ldots + y^{m-1})(x,y), \\ \{x_1, \ldots, x_{i-1}, x_i\} &= \{\{x_1, \ldots, x_{i-1}\}, x_i\}. \end{aligned}$$

It follows from Lemma 1 that $[v_i, v_j] = 1 - \{w_i, w_j\}$, and this implies directly that

$$[v_1, v_2, \ldots, v_i] = 1 - \{w_1, w_2, \ldots, w_i\}. \tag{31}$$

We also denote

$$
\begin{aligned}
&\theta(x_1, x_2, \ldots, x_8)\\
&\quad = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_7, x_8\}\},\\
&\eta_k(x_s, x_t, x_i, x_j; x_1, x_2, \ldots, x_{4k})\\
&\quad = \{\{x_s, x_i, x_j\}, \{x_1, x_2\}, \ldots, \{x_{4k-1}, x_{4k}\}, \{x_t, x_i, x_j\}\}\\
&\qquad - \{\{x_t, x_i, x_j\}, \{\{x_s, x_i, x_j\}, \{x_1, x_2\}, \ldots, \{x_{4k-1}, x_{4k}\}\}\},\\
&\xi_k(x_s, x_t, x_i, x_j; x_1, x_2, \ldots, x_{4k})\\
&\quad = \{\{x_i, x_j, x_s,\}, \{x_1, x_2\}, \ldots, \{x_{4k-1}, x_{4k}\}, \{x_i, x_j, x_t\}\}\\
&\qquad - \{\{x_i, x_j, x_t\}, \{\{x_i, x_j, x_s\}, \{x_1, x_2\}, \ldots, \{x_{4k-1}, x_{4k}\}\}\},\\
&\lambda_k(x_i, x_j, x_s; x_1, x_2, \ldots, x_{4k})\\
&\quad = \{\{x_i, x_j, x_s\}, \{x_1, x_2\}, \ldots, \{x_{4k-1}, x_{4k}\}, \{x_i, x_j, x_s\}\}.
\end{aligned}
$$

Now let us show that the equalities

$$
\begin{aligned}
&\theta = \theta(\alpha_1 w_1, \alpha_2 w_2, \ldots, \alpha_8 w_8) = 0,\\
(32)\qquad &\eta_k = \eta_k(\alpha_s w_s, \alpha_t w_t, \alpha_i w_i, \alpha_j w_j; \alpha_1 w_1, \alpha_2 w_, \ldots, \alpha_{4k} w_{4k}) = 0,\\
&\xi_k = \xi_k(\alpha_s w_s, \alpha_t w_t, \alpha_i w_i, \alpha_j w_j; \alpha_1 w_1, \alpha_2 w_2, \ldots, \alpha_{4k} w_{4k}) = 0,\\
&\lambda_k = \lambda_k(\alpha_i w_i, \alpha_j w_j, \alpha_s w_s; \alpha_1 w_1, \alpha_2, w_2, \ldots, \alpha_{4k} w_{4k}) = 0
\end{aligned}
$$

hold in the algebra T_n. Indeed, in (27) we substitute $l_i = \alpha_i(1 - u_i)$, where $\alpha_i \in F$, and let the image of the expression obtained for $\eta_k(l_i, l_j, l_s, l_t; l_1, l_2, \ldots, l_{4k})$ under the homomorphism $\overline{L}_n \to \overline{L}_n/E$ have the form $\varphi_k \varrho_k$. Then the equality $\varphi_k \varrho_k = 1$ or $\varphi_k = \varrho_k^{-1}$ is true in the group $\overline{L}_n/E = \overline{T}_n$, where φ_k, ϱ_k are commutator expressions of the group $\overline{T}_n$. With help of the identity $[u, v] = [v, u]^{-1}$ we represent ϱ^{-1} in the form ψ_k, in which the arrangement of parentheses $[\,,\,]$ in ψ_k coincides with the arrangement of parentheses $\{\,,\,\}$ in the second member of η_k. The parentheses arrangements in φ_k and in the first member of η_k coincide. Now we apply the equality (31) for φ_k, ψ_k. Suppose that $\varphi_k = 1 - \overline{\varphi}_k$, $\psi_k = 1 - \overline{\psi}_k$. As the equality $\varphi_k = \psi_k$ holds in the group $\overline{L}_n/E$, it follows from the relation $\omega(\overline{L}_n/E) \cong \omega\overline{L}_n/\omega E$ that the equality $\overline{\varphi}_k - \overline{\psi}_k = 0$ holds in the algebra T_n. But $\overline{\varphi}_k - \overline{\psi}_k = \eta_k$. Therefore the equality $\eta_k = 0$ holds in the algebra T_n. By analogy we obtain the validity of the equalities $\theta = 0$, $\xi_k = 0$, $\lambda_k = 0$ in the algebra T_n.

Let $f = f(x_1, x_2, \ldots, x_t)$ be one of the polynomials

$$
\begin{gathered}
\theta(x_1, x_2, \ldots, x_8), \quad \eta_k(x_s, x_t, x_i, x_j; x_1, x_x, \ldots, x_{4k}),\\
\xi(x_s, x_t, x_i, x_j; x_1, x_x, \ldots, x_{4k}), \quad \lambda_k(x_i, x_j, x_s; x_1, x_x, \ldots, x_{4k}).
\end{gathered}
$$

By the definition of $\{\ ,\ \}$ we pass to the operations $(+)$, $(\cdot)$ in f and for the polynomial obtained we introduce in a natural way the notions of *degree in every variable* x_i, *degree* and *homogeneity* of polynomials. Let us represent f in the form $f = f_0 + f_1 + \ldots + f_{r_1}$, where f_i is the sum of all the monomials of the polynomial f that have the degree i in x_1. Let $w_1, w_2, \ldots, w_t$ be elements of the algebra ωT_n determined above. Using abbreviations we write $f(w)$ instead of $f(w_1, w_2, \ldots, w_t)$. If $\alpha \in F$, then $f(\alpha w_1, w_2, \ldots, w_t) = f_0(w) + \alpha f_1(w) + \alpha^2 f_2(w) + \ldots + \alpha^{r_1} f_{r_1}(w)$. Let $\alpha_1, \alpha_2, \ldots, \alpha_{r_1}$ be arbitrary elements from F. Then by (32) we get a system consisting of r_1 equations

$$f_0(w) + \alpha_i f_1(w) + \ldots + \alpha_i^{r_1} f_{r_1}(w) = 0$$

with variables $f_0(w), f_1(w), \ldots, f_{r_1}(w)$. By [9], $d_1 f_j(w) = 0$, where d_1 is the determinant of this system. The field F is infinite. Then we can choose such $\alpha_1, \alpha_2, \ldots, \alpha_{r_1}$ that $d_1 \neq 0$. That is why $f_j(w) = 0$. Doing the same operation with the polynomials f_{j_i} and variable x_2 and so on, we finally get the following statement.

Lemma 10. *Let $f = f_1(x_1, x_2, \ldots, x_t) + \ldots + f_i(x_1, x_2, \ldots, x_t) + \ldots + f_r(x_1, x_2, \ldots, x_t)$ be the decomposition of the polynomial f into homogeneous components $f_i(x_1, x_2, \ldots, x_t)$ and let $w_1, w_2, \ldots, w_t$ be the elements of the algebra ωT_n determined above. Then $f_i(w_1, w_2, \ldots, w_t) = 0$.*

In particular, examining the homogeneous components of the least degree in each of the cases (32), we obtain that the equalities

$$\begin{aligned}
&((w_1, w_2, w_3), (w_4, w_5, w_6), (w_7, w_8)) = 0,\\
&((w_i, w_j, w_s), (w_1, w_2), \ldots, (w_{4k-1}, w_{4k}), (w_t, w_j, w_s))\\
&\qquad - ((w_i, w_j, w_s), ((w_t, w_j, w_s), (w_1, w_2), \ldots, (w_{4k-1}, w_{4k}))) = 0,\\
&((w_i, w_j, w_s), (w_1, w_2), \ldots, (w_{4k-1}, w_{4k}), (w_i, w_j, w_t))\\
&\qquad - ((w_i, w_j, w_s), ((w_i, w_j, w_t), (w_1, w_2), \ldots, (w_{4k-1}, w_{4k}))) = 0,\\
&((w_i, w_j, w_s), (w_1, w_2), \ldots, (w_{4k-1}, w_{4k}), (w_i, w_j, w_s)) = 0
\end{aligned}$$

are valid in the algebra T_n for $k \neq n$.

By item (6) of Lemma 2 the augmentation ideal ωC_n is generated as an F-module by elements of the form $1 - u_i$. Then the F-module T_n is generated by the elements w_i, i.e. any element h from T_n has the decomposition $h = \alpha_1 w_1 + \ldots + \alpha_s w_s$. The statement can be proved taking into account the identity $(x, y) = -(y, x)$ and using induction on the length s, from the last equalities it is easy to prove the statement.

Lemma 11. *The identities* (2) *and* $\mu_k = 0$ *hold in the algebra* T_n *for* $k \neq n$.

Lemma 12. *The algebra T_n belongs to the variety $\mathfrak{B} \cap \mathfrak{C}_p\mathfrak{C}_p$.*

P r o o f. The groups $\overline{R}_n$, $\overline{S}_n$ are epimorphic images of the groups $\overline{A}_n$, $\overline{B}_n$, and algebras R_n, S_n are respectively the images of the algebras $\omega\overline{A}_n$, $\omega\overline{B}_n$. Then it follows from (24) that

$$\overline{R}_n = 1 - R_n, \quad \overline{S}_n = 1 - S_n, \quad \overline{T}_n = 1 - T_n. \tag{33}$$

Let h be an arbitrary element of the algebra R_n (or S_n) and let $q = 1 - h$. Then it follows from (33) that $q \in \overline{R}_n$ (or $q \in \overline{S}_n$), and it follows from (29) that $q^{p^2} = 1$. We have $h^{p^2} = (1-q)^{p^2} = 1 + \sum_{i=1}^{p^2-1} \binom{i}{p^2-1}(-1)^i q^i + (-1)^{p^2} q^{p^2} = 1 + (-1)^{p^2}$ since all binomial coefficients can be divided by p. If $p = 2$, then $1 + (-1)^{p^2} = 1 + 1 = 0$, as F is a field of characteristic 2. But if $p \neq 2$, then $1 + (-1)^{p^2} = 1 - 1 = 0$. Consequently, $h^{p^2} = 0$, i.e., the algebras R_n, S_n satisfy the identity $x^{p^2} = 0$.

The groups $\overline{R}_n$, $\overline{S}_n$ satisfy the identity (30) and by (29) are nilpotent of the index 2. Then, considering (33), similarly to the proof of the of identity (2) in the algebra T_n (Lemma 11), it is shown that the algebras R_n, S_n satisfy the identities $(x, y, z) = 0$, $(x, y)^p = 0$, $(x^p, y) = 0$. Consequently, $R_n, S_n \in \mathfrak{C}_p$ and by (29) $T_n \in \mathfrak{C}_p\mathfrak{C}_p$. It follows from Lemma 11 that $T_n \in \mathfrak{B}$, therefore $T_n \in \mathfrak{B} \cap \mathfrak{C}_p\mathfrak{C}_p$. Lemma is proved. □

Let $(A, \cdot, +)$ be an arbitrary associative algebra. It is known that the algebra's A operations of taking the commutator $(u, v) = uv - vu$ and addition $(+)$ turn A into a Lie algebra. We denote it by $\Lambda(A)$.

Now we use the notation introduced before the relation (28). Let G be a subgroup of the group $\overline{T}_n$ generated by the set $X = \{d, g_1, g_2, \dots, g_{4n}\}$ and let A be a subalgebra of the augmentation ideal $\omega\overline{T}_n$ generated by the set $\{y_i \colon\ y_i = 1 - x_i\}$. We have shown earlier (after the relation (28)) that the algebra $\omega\overline{C}_n$ is nilpotent. Then the algebra $\overline{T}_n$ is nilpotent. In particular, the algebra A is also nilpotent. Then for every monomial $v \in A$ there exists such a number m that $v \in A^m \backslash A^{m+1}$. The number m will be called the weight of the monomial v. A polynomial that consists of monomials of the weight m will be called *homogeneous* of the weight m. Let U be a word of group G from the generating set X. We pass in U to the generators y_i of the algebra A, with help of the relation $x_i = 1 - y_i$. Assume that U has the decomposition

$$U = 1 - (u_m + u_{m+1} + \dots, u_r) \tag{34}$$

in A, where u_i is a homogeneous polynomial from A of the weight i and u_m is a polynomial of the smallest weight. We define a mapping $\delta \colon G \to A$ by $\delta(U) = 0$ if $U = 1$, and $\delta(U) = u_m$ otherwise.

Lemma 13. *Let U, V be words ($\neq 1$) of the group G from the generating set X and let $\delta(U) = u_m$, $\delta(V) = v_k$. Then for every integer l*

$$\delta(U^l) = lu_m. \tag{35}$$

If $m < k$, then

$$\delta(UV) = u_k. \tag{36}$$

If $m = k$ and $u_k + v_k \neq 0$, then

$$\delta(UV) = u_k + v_k. \tag{37}$$

If $m = k$ and $u_k + v_k = 0$, then $UV = 1$ or $\delta(UV)$ lies in A^t, where $t > m$. If $(u_m, v_k) \neq 0$, then

$$\delta([U,V]) = (u_m, v_k). \tag{38}$$

If $(u_m, v_k) = 0$, then $[U,V] = 1$ or $\delta([U,V])$ lies in A^t, where $t > m + k$.

Proof. We denote $u_m + m_{m+1} + \dots, u_r = u$. Then $U = 1 - u$. We use the decomposition $(1-u)^l = \sum_{t=0}^{l} (-1)^t \binom{l}{t} u^t$, where $\binom{l}{t} = \frac{l(l-1)\dots(l-t+1)}{t!}$, for the proof of (35). As $u \in A$, all nonconstant members of the smallest weight of the element $(1-u)^l$ belong to $-lu$. Hence (35) is proved.

The assertions (36), (37) follow from the multiplication rules, and the other assertions follow from Lemma 1. □

We denote $D_k = \{g \in G\colon 1 - g \in (\omega G)^k\}$. It is easy to see that D_k is the kernel of the homomorphism induced on the group G by the natural homomorphism $FG \to FG/(\omega G)^k$. This follows from Lemma 13.

Lemma 14. *If G_m is the m-th member of the lower central series of the group G, then $G_m \subseteq D_m$.*

Proof. We will use induction on m. We have $G_1 = G = D_1$. Suppose that $G_m \subseteq D_m$ and let $a \in G_m$, $u \in G$. Then $[a,u] = 1$, or $\delta([a,u]$ has weight not less than $m+1$, as $\delta(a)$ has weight not less than m. In any case $[a,u] \in D_{m+1}$ and therefore $G_{m+1} \subseteq D_{m+1}$. The lemma is proved. □

By the construction, the group C_n is finite of exponent p^4. Then the group G, being a subgroup of the homomorphic image of the group C_n, is also finite of exponent p^4. Therefore it is nilpotent. Following [10], we link the lower central series $G = G_1 \supset G_2 \supset \ldots \supset G_s = 1$ of the group G with the Lie algebra $L(G)$. It is the direct sum of modules $B_i = G_i/G_{i+1}$, $i = 1, 2, \ldots, s-1$, in which the multiplication $[\,,\,]$ is defined in the following way. Let $b_i \in B_i$, $b_j \in B_j$ and let $g_i \in G_i$, $g_j \in G_j$ be such elements that the mappings

$$G_i \to G_i/G_{i+1}, \quad G_j \to G_j/G_{j+1}$$

transfer g_i into b_i and g_j in b_j. Then the product $[b, b_j]$ is defined as the element from G_{i+j}/G_{i+j+1} containing the commutator $[g_i, g_j]$. The null element of the algebra $L(G)$ will be $1 + \ldots + 1$, where 1 is the identity element of B_i.

The commutator $\beta^k(x_1, x_2, \ldots, x_k)$ is naturally defined for the elements $x_i \in X$, where β^k is an arrangement of parentheses [and] [10]. The group G is nilpotent. Hence there exists such a number $\mu(k)$ that $\beta^k(x_1, x_2, \ldots, x_k) \in G_{\mu(k)}/G_{\mu(k)+1}$.

Proposition 2. *Let G and A be the algebras considered above. Then the mapping $x_i G_2 \to y_i$ induces the monomorphism of the Lie algebra $L(G)$ into the Lie algebra $A \subseteq \Lambda(\omega G)$. The monomorphism is determined in the following way:*

Let $\beta^k(x_1, x_2, \ldots, x_k)$, where $x_i \in X$, be a commutator of the group G with some parentheses arrangement of β^k and let $\beta^k(x_1, x_2, \ldots, x_k) \in G_{\mu(k)} \backslash G_{\mu(k)+1}$. Then the mapping

$$\beta^k(x_1, x_2, \ldots, x_k) G_{\mu(k)+1} \to \beta^k(y_1, y_2, \ldots, x_k)$$

is a monomorphism of the quotient group $G_{\mu(k)}/G_{\mu(k)+1}$ in the additive group $\Lambda_{\mu(k)}$ (A), where $\Lambda_{\mu(k)}(A)$ is the submodule of the module $\Lambda(A)$ that consists of homogeneous polynomials of the weight $\mu(k)$ and the parentheses arrangement β^k means the multiplication in $\Lambda(A)$.

P r o o f. By the definition of the multiplication operation in the algebra $L(G)$, and also by the link between the operation of taking the commutator in the group G_k/G_{k+1} and the multiplication in the algebra $\Lambda(\omega G)$, the expression $\beta^k(x_{i_1}, x_{i_2}, \ldots, x_k)$ obviously turns into an element $\beta^k(y_{i_1}, y_{y_2}, \ldots, y_k)$ of the algebra $\Lambda(A)$.

Further, an arbitrary element U from $G_k \backslash G_{k+1}$, under the mapping $x_i \to y_i$, is transferred into the element of the algebra A of the form

$$1 + u_k + u_{k+1} + \ldots + u_t,$$

where u_i has the weight i or equals zero, and $i > k$ by Lemma 14. This lemma also shows that the equality

$$\delta(UG_k) = \delta(U) = u_k$$

determines a mapping δ_k of the group $C_k = G_k/G_{k+1}$ into the set of homogeneous elements of the weight k of the algebra A. Modulo members of the lower central series, the multiplication in the group G coincides with the addition in the algebra $L(G)$. Therefore the identity $x^{p^4} = 1$ of the group G does not influence the characteristic p of the field F, and it follows from (34)–(38) that δ_k is a linear mapping C_k in A^k. By [10] the commutators of the form $[x_1, x_2, \ldots, x_k]$ generate the subgroup G_k, therefore the mapping

$$\delta(V) = \delta_1(v_1) + \delta_2(v_2) + \ldots + \delta_k(v_k) + \ldots$$

is a linear mapping of the $\mathbb{Z}_p$-module $L(G)$ into the $\mathbb{Z}_p$-module A, where $\mathbb{Z}_p$ means the ring of integers modulo p. Consequently, the mapping $x_iG_2 \to y_i$ induces a homomorphism of the Lie algebra $L(G)$ in A.

By [10] the subgroup G_2 generated by all the commutators of the group G is contained in the Frattini subgroup. Therefore the mapping $x_iG_2 \to y_i$ is one-to-one. If a, b are elements from G, then it follows from Lemma 1 that $[a, b] = 1 - a^{-1}b^{-1}(a, b)$. Therefore, if $[a, b] \neq 1$ then $(a, b) \neq 0$. Now it is easy to show by induction that if $\beta^k(x_{i_1},\ x_{i_2}, \ldots, x_{i_k}) \neq 1$, then $\beta^k(y_{i_1}, y_{i_2}, \ldots, x_k) \neq 0$. Then it follows from (38) that the mapping $x_iG_2 \to y_i$ induces a monomorphism of the Lie algebra $L(G)$ into the Lie algebra A. The proposition is proved. □

By (33) we have $\overline{T}_n = 1 - T_n$. Then it follows from the definition of the augmentation ideal $\omega\overline{T}_n$ that $\omega\overline{T}_n \subseteq T_n$. Then $A \subseteq T_n$ and (28) together with Proposition 2 yields

Lemma 15. *The identity $\mu_n = 0$ does not hold in the algebra T_n.*

Now we directly obtain from Lemmas 3, 11 and 15:

Theorem 1. *In the variety $\mathfrak{B} \cap \mathfrak{C}_p\mathfrak{C}_p$ of associative algebras over an infinite field of characteristic $p > 0$ the system of identities $M = \{\mu_k = 0\colon\ k = 1, 2, \ldots\}$ is independent.*

Different subsets from M determine different varieties, hence Theorem 1 implies

Corollary 1. *The variety $\mathfrak{B} \cap \mathfrak{C}_p\mathfrak{C}_p$ contains a continuum of different not finitely based subvarieties.*

Corollary 2. *In the variety $\mathfrak{B} \cap \mathfrak{C}_p\mathfrak{C}_p$ there exists an algebra determined by an enumerable set of identity relations in which the words problem is unsolvable.*

P r o o f. Let S be an enumerable and unsolvable set of numbers. Consider the algebra of the variety $\mathfrak{B} \cap \mathfrak{C}_p\mathfrak{C}_p$ determined by the identity relations $\{\mu_k = 0\}$ for $n \in S$. It is obvious that each relation of the algebra A is an identity relation. By Theorem 1 an arbitrary identity from $\{\mu_k = 0\}$ for given n is fulfilled in A if and only if $n \in S$. Consequently, in A the problem of words equality is not solvable. □

It is known that if on the additive F-module T_n we introduce multiplication $(\cdot)$: $x \cdot y = xy - yx$, then the resulting algebra will be special Jordan and since F is a field of characteristic 2, it will be Lie, too. Then, from Theorem 1 and (2) we get

Corollary 3. *In the variety $\mathfrak{D} \cap \mathfrak{N}_3\mathfrak{N}_3$ of Lie algebras (special Jordan algebras) over an infinite field of characteristic $p > 0$ (over an infinite field of characteristic 2) the system of identities $\{\nu_k = 0\colon\ k = 1, 2, \ldots\}$ is independent.*

As in the case of Corollaries 1, 2, this implies

Corollary 4. *The variety $\mathfrak{D} \cap \mathfrak{N}_3\mathfrak{N}_3$ contains a continuum of different not finitely based subvarieties and in $\mathfrak{D} \cap \mathfrak{N}_3\mathfrak{N}_3$ there exists an algebra determined by an enumerable set of identity relations, where the words problem is unsolvable.*

Note, eventually, that in the case of Lie algebras Corollary 3 does not pretend to novelty. Infinite systems of identities for the varieties of Lie algebras over the field are given in [11], [12].

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