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PRODUCTIVELY FRÉCHET SPACES

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Abstract. We solve the long standing problem of characterizing the class of strongly Fréchet spaces whose product with every strongly Fréchet space is also Fréchet.

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A topological space X is *Fréchet* if for every subset A of X, $x \in \mathrm{cl}\, A$ implies the existence of a sequence on A that converges to x. A topological space X is *strongly Fréchet*[1] if whenever $x \in \bigcap_{n \in \omega} \mathrm{cl}\, A_n$ for a decreasing sequence of subsets A_n of X, there exists a sequence $(x_n)_{n \in \omega}$ such that $x_n \in A_n$ and $x \in \lim(x_n)_{n \in \omega}$. Because of the instability of these properties under product, the quest for conditions on Fréchet spaces X and Y to ensure the Fréchetness of the product $X \times Y$ has attracted lots of attention for years [13], [1], [2], [18], [19], [20], [23], [16], [8], [17], [22], [9], [10], [7], [6]. Even the square of a compact Fréchet space need not be Fréchet [22]. It has been noticed [13] that if X and Y are two Fréchet spaces both containing a convergent free sequence, and if $X \times Y$ is Fréchet, then both X and Y are strongly Fréchet. The countable fan S_ω is an example of a Fréchet space which is not strongly Fréchet.[2] Therefore, a space whose product with every Fréchet space is Fréchet cannot contain a free convergent sequence. Actually, such spaces are exactly the *finitely generated* ones, that is, the topological spaces in which each point admits

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[1] Also called countably bisequential and characterized as Fréchet α_4 spaces.
[2] S_ω is obtained from the disjoint sum of countably many convergent sequences by identifying the limit points. The resulting set is equipped with the corresponding quotient topology.

a smallest neighborhood [14]. This class is trivial in the sense that Hausdorff finitely generated spaces are discrete.

The natural question is consequently to characterize spaces whose product with every *strongly* Fréchet space is Fréchet. It is known that a product of two strongly Fréchet spaces need not be Fréchet (e.g., [22], [2]), even not sequential[3] [24], so we have to find a convenient proper subclass of strongly Fréchet spaces. The first result in this direction is due to E. Michael [13, Proposition 4.D.4] and states that the product of a bisequential space with a strongly Fréchet space is strongly Fréchet. Two families of subsets $\mathscr{A}$ and $\mathscr{B}$ *mesh*, in symbol $\mathscr{A} \# \mathscr{B}$, if $A \cap B \neq \emptyset$ for every $A \in \mathscr{A}$ and every $B \in \mathscr{B}$. A topological space is *bisequential* [13] if whenever x is a cluster point for a filter $\mathscr{F}$, there exists a countably based filter $\mathscr{H}$ that meshes with $\mathscr{F}$ and converges to x. It is well known that a countable product of bisequential spaces is bisequential [13].

Let φ_ω denote the class of countably based filters. We introduce two types of adherence (or set of cluster points) of a filter:

$$\mathrm{adh}\,\mathscr{F} = \bigcup_{\mathscr{G} \# \mathscr{F}} \lim \mathscr{G}, \quad \mathrm{adh}_{\mathrm{First}}\,\mathscr{F} = \bigcup_{\varphi_\omega \ni \mathscr{G} \# \mathscr{F}} \lim \mathscr{G}.$$

The definitions of strongly Fréchet and bisequential spaces can be nicely rephrased in terms of these two types of adherence. Indeed, X is strongly Fréchet if and only if

(1) $$\mathrm{adh}\,\mathscr{F} \subset \mathrm{adh}_{\mathrm{First}}\,\mathscr{F},$$

for every countably based filter $\mathscr{F}$. Analogously, X is bisequential if and only if (1) holds for every filter $\mathscr{F}$. Notice that X is Fréchet if and only if (1) holds for every principal filter.

A. V. Arhangel'skii extended Michael's result from bisequential to $\aleph_0$-*bisequential* spaces, i.e., spaces for which each countable subset is bisequential and for every $x \in X$ and $C \subset \bigcup_{B \subset X, |B| \leqslant \aleph_0} \mathrm{cl}_{bX} B$ (where bX denotes a compactification of X) such that $x \in \mathrm{cl}_{bX} C$, there exists a countable subset D of C such that $x \in \mathrm{cl}_{bX} D$ [1].[4]

Theorem 1 [1, Proposition 6.27]. *The product of an $\aleph_0$-bisequential space and of a strongly Fréchet space is strongly Fréchet.*

Also, S. Dolecki and T. Nogura proved

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[3] A topology is *sequential* if every sequentially closed set is closed.

[4] In particular, each space of countable tightness in which the closure of each countable subset is compact and bisequential is $\aleph_0$-bisequential [1, Proposition 6.20].

Theorem 2 [6, part of Thorem 4]. *The product of a regular β_3 Fréchet q-space with a strongly Fréchet space is Fréchet.*

Recall that X is called a *q-space* if for every $x \in X$, there exists a sequence $(Q_n)_{n \in \omega}$ of neighborhoods of x such that every sequence $x_n \in Q_n$ has non empty adherence. On the other hand, a topology is β_3 [6] if for every convergent free (i.e., $x_n \neq x$) bisequence

$$x_{n,k} \to_k x_n \to_n x,$$

there exists a compact metrizable subset C of $\{x\} \cup \{x_n \colon n \in \omega\} \cup \{x_{n,k} \colon n, k \in \omega\}$ such that

$$|\{n \colon |C \cap \{x_{n,k} \colon k \in \omega\}| = \omega\}| = \omega.$$

A filter $\mathscr{F}$ on a set X is called *strongly Fréchet* [12], [4] if for every countably based filter $\mathscr{H}$ such that $\mathscr{H} \# \mathscr{F}$, there exists a countably based (equivalently a sequence) filter $\mathscr{L}$ such that $\mathscr{L} \geqslant \mathscr{F} \vee \mathscr{H}$. Notice that a space is strongly Fréchet if and only if all its neighborhood filters are strongly Fréchet. Analogously, a filter $\mathscr{F}$ on a set X is *productively Fréchet* if there exists a countably based filter $\mathscr{L} \geqslant \mathscr{F} \vee \mathscr{H}$ whenever $\mathscr{H}$ is a strongly Fréchet filter meshing with $\mathscr{F}$. A topological space is called *productively Fréchet* if all its neighborhood filters are productively Fréchet, or equivalently if (1) holds for every strongly Fréchet filter. Bisequential spaces can also be characterized by a similar property of their neighborhood filters: a filter $\mathscr{F}$ is *bisequential* if there exists a countably based filter $\mathscr{L}$ such that $\mathscr{L} \geqslant \mathscr{F}$ and $\mathscr{L} \# \mathscr{H}$ whenever $\mathscr{H}$ is a filter meshing with $\mathscr{F}$. As every countably based filter is bisequential, hence strongly Fréchet, every bisequential space is productively Fréchet and every productively Fréchet space is strongly Fréchet.

Remark 3. In view of the definitions of strongly Fréchet and productively Fréchet filters, one may be tempted to iterate the process to generate new classes of filters in the following way: $\varphi_\omega^0 = \varphi_\omega$ is the class of countably based filters, and $\mathscr{F} \in \varphi_\omega^{n+1}$ if for every $\mathscr{H} \in \varphi_\omega^n$ such that $\mathscr{H} \# \mathscr{F}$, there exists a countably based filter $\mathscr{L} \geqslant \mathscr{F} \vee \mathscr{H}$. Then the class $\varphi_\omega^1 = \varphi_\omega^3$ of strongly Fréchet filters and the class φ_ω^2 of productively Fréchet filters are the only two new classes generated.

If $\mathscr{F}$ is a filter on a set X, $\mathscr{G}$ is a filter on a set Y and $\mathscr{H}$ is a filter on $X \times Y$, we denote by $\mathscr{H}\mathscr{F}$ the (possibly degenerate) filter on Y generated by the sets

$$HF = \{y \colon \exists x \in F, (x,y) \in H\},$$

for $H \in \mathscr{H}$ and $F \in \mathscr{F}$ and by $\mathscr{H}^-\mathscr{G}$ the filter on X generated by the sets

$$H^-G = \{x \colon \exists y \in G, (x,y) \in H\},$$

for $H \in \mathscr{H}$ and $G \in \mathscr{G}$. Notice that

$$\mathscr{H} \,\#\, (\mathscr{F} \times \mathscr{G}) \Longleftrightarrow \mathscr{H}\mathscr{F} \,\#\, \mathscr{G} \Longleftrightarrow \mathscr{F} \,\#\, \mathscr{H}^{-}\mathscr{G}.$$

As a convention, we consider that degenerate filters are in any class of filters we may consider.

Lemma 4. *If $\mathscr{F}$ is a strongly Fréchet filter on X and $\mathscr{H}$ is a countably based filter on $X \times Y$, then the filter $\mathscr{H}\mathscr{F}$ is strongly Fréchet.*

P r o o f . Let $\mathscr{G}$ be a countably based filter such that $\mathscr{G} \,\#\, \mathscr{H}\mathscr{F}$. Then $\mathscr{H}^{-}\mathscr{G}$ is a countably based filter that meshes with $\mathscr{F}$. As $\mathscr{F}$ is strongly Fréchet, there exists a countably based filter $\mathscr{L} \geqslant \mathscr{F} \vee \mathscr{H}^{-}\mathscr{G}$. Then, $\mathscr{H}\mathscr{L} \,\#\, \mathscr{G}$ and $\mathscr{H}\mathscr{L} \geqslant \mathscr{H}\mathscr{F}$, so that $\mathscr{H}\mathscr{L} \vee \mathscr{G}$ is a countably based filter finer than $\mathscr{H}\mathscr{F} \vee \mathscr{G}$. $\square$

Using the same argument and Lemma 4, we obtain also:

Lemma 5. *If $\mathscr{F}$ is a productively Fréchet filter on X and $\mathscr{H}$ is a countably based filter on $X \times Y$, then the filter $\mathscr{H}\mathscr{F}$ is productively Fréchet.*

Lemma 6. *If $\mathscr{F}$ is a productively Fréchet filter on X and $\mathscr{H}$ is a strongly Fréchet filter on $X \times Y$, then the filter $\mathscr{H}\mathscr{F}$ is strongly Fréchet.*

P r o o f . First assume that $\mathscr{F}$ is countably based. Let $\mathscr{G}$ be a countably based filter such that $\mathscr{G} \,\#\, \mathscr{H}\mathscr{F}$. Then $\mathscr{F} \times \mathscr{G}$ is a countably based filter meshing with $\mathscr{H}$. As $\mathscr{H}$ is strongly Fréchet, there exists a countably based filter $\mathscr{L} \geqslant \mathscr{H} \vee (\mathscr{F} \times \mathscr{G})$. The filter $\mathscr{L}\mathscr{F}$ is countably based and $\mathscr{L}\mathscr{F} \geqslant \mathscr{H}\mathscr{F}$ and $\mathscr{L}\mathscr{F} \,\#\, \mathscr{G}$, so there exists a countably based filter $\mathscr{M} \geqslant \mathscr{H}\mathscr{F} \vee \mathscr{G}$. Hence, if $\mathscr{F}$ is countably based and $\mathscr{H}$ is strongly Fréchet, then $\mathscr{H}\mathscr{F}$ is strongly Fréchet.

More generally, assume that $\mathscr{F}$ is productively Fréchet, that $\mathscr{H}$ is strongly Fréchet and that $\mathscr{G}$ is a countably based filter that meshes with $\mathscr{H}\mathscr{F}$. By the previous case, $\mathscr{H}^{-}\mathscr{G}$ is a strongly Fréchet filter that meshes with $\mathscr{F}$. As $\mathscr{F}$ is productively Fréchet, there exists a countably based filter $\mathscr{M} \geqslant \mathscr{F} \vee \mathscr{H}^{-}\mathscr{G}$. Again by the previous case, $\mathscr{H}\mathscr{M}$ is a strongly Fréchet filter finer than $\mathscr{H}\mathscr{F}$ and that meshes with $\mathscr{G}$. Thus, there exists a countably based filter $\mathscr{L} \geqslant \mathscr{H}\mathscr{M} \vee \mathscr{G}$ so that $\mathscr{L} \geqslant \mathscr{H}\mathscr{F} \vee \mathscr{G}$. $\square$

In the next lemma $\mathbb{D}$ and $\mathbb{J}$ denote generic classes of filters (like that of countably based, strongly Fréchet or productively Fréchet filters). $\mathbb{D}(X)$ denotes the corresponding family of filters on the set X. A class $\mathbb{D}$ of filters is called *embeddable* if the filter generated by $\mathscr{F}$ on X is in $\mathbb{D}(X)$ whenever $Y \subset X$ and $\mathscr{F} \in \mathbb{D}(Y)$.

Lemma 7. *Let $\mathbb{J}$ and $\mathbb{D}$ be two classes of filters. Assume $\mathbb{D}$ is embeddable. If $\mathcal{H}\mathcal{F} \in \mathbb{J}(Y)$ whenever $\mathcal{H} \in \mathbb{D}(X \times Y)$ and $\mathcal{F} \in \mathbb{J}(X)$, then $\mathcal{F} \vee \mathcal{G} \in \mathbb{J}(X)$ whenever $\mathcal{F} \in \mathbb{J}(X)$ and $\mathcal{G} \in \mathbb{D}(X)$.*

P r o o f. If $\mathcal{H} \in \mathbb{D}(X)$ and $\mathcal{F} \in \mathbb{J}(X)$, then the filter $\widehat{\mathcal{H}}$ generated on $X \times X$ by $\{(x,x)\colon x \in H\}_{H \in \mathcal{H}}$ is in $\mathbb{D}(X \times X)$ so that $\widehat{\mathcal{H}}\mathcal{F} = \mathcal{H} \vee \mathcal{F}$ is in $\mathbb{J}(X)$.   □

In particular, in view of Lemma 5, if $\mathcal{F}$ is a productively Fréchet filter and $A \# \mathcal{F}$, then $\mathcal{F} \vee A$ is productively Fréchet. Consequently,

Proposition 8. *A subspace of a productively Fréchet space is productively Fréchet.*

Notice that this is also true for Fréchet and strongly Fréchet (filters and) spaces.

Theorem 9. *X is productively Fréchet if and only if $X \times Y$ is Fréchet (equivalently strongly Fréchet) for every strongly Fréchet space Y.*

P r o o f. Assume that X is not productively Fréchet. Then there exists a strongly Fréchet filter $\mathcal{F}$ and a point x_0 such that $x_0 \in \mathrm{adh}\,\mathcal{F} \setminus \mathrm{adh}_{\mathrm{First}}\,\mathcal{F}$. Let Y be a copy of the underlying set of X in which all points but x_0 are isolated and $\mathcal{N}_Y(x_0) = \mathcal{F} \wedge (x_0)$. Then Y is strongly Fréchet, as each of its neighborhood filters is strongly Fréchet. There exists an ultrafilter $\mathcal{U}$ of $\mathcal{F}$ that converges to x_0 in X. Hence $\mathcal{U} \times \mathcal{U}$ converges to (x_0, x_0) in $X \times Y$ so that $(x_0, x_0) \in \mathrm{cl}_{X \times Y}\{(x,x)\colon x \neq x_0\}$. But no countably based filter (a fortiori no sequence) on $\{(x,x)\colon x \neq x_0\}$ converges to (x_0, x_0). Indeed, without loss of generality, such a filter can be assumed of the form $\mathcal{H} \times \mathcal{H}$ for a countably based filter $\mathcal{H}$. If $x_0 \in \lim_Y \mathcal{H}$, then $\mathcal{H} \geqslant \mathcal{F}$. But in this case, $x_0 \notin \lim_X \mathcal{H}$, as $x_0 \notin \mathrm{adh}_{\mathrm{First}}\,\mathcal{F}$. Hence $X \times Y$ is not Fréchet.

Conversely, assume that X is productively Fréchet and that Y is strongly Fréchet. Let $\mathcal{H}$ be a countably based filter such that $(x,y) \in \mathrm{adh}_{X \times Y}\,\mathcal{H}$. In other words, $(\mathcal{N}_X(x) \times \mathcal{N}_Y(y)) \# \mathcal{H}$. As Y is strongly Fréchet, $\mathcal{N}_Y(y)$ is a strongly Fréchet filter, so that, by Lemma 4, $\mathcal{H}^-\mathcal{N}_Y(y)$ is a strongly Fréchet filter that meshes with $\mathcal{N}_X(x)$. Hence, $x \in \mathrm{adh}_X \mathcal{H}^-\mathcal{N}_Y(y)$. As X is productively Fréchet, there exists a countably based filter $\mathcal{G}$ such that $x \in \lim_X \mathcal{G}$ and $\mathcal{G} \# \mathcal{H}^-\mathcal{N}_Y(y)$. Hence $\mathcal{H}\mathcal{G}$ is a countably based filter that meshes with the strongly Fréchet filter $\mathcal{N}_Y(y)$. Therefore, there exists a countably based filter $\mathcal{L}$ finer than $\mathcal{H}\mathcal{G} \vee \mathcal{N}_Y(y)$. The filter $\mathcal{G} \times \mathcal{L}$ is countably based, meshes with $\mathcal{H}$, and converges to (x,y) in $X \times Y$. Thus $X \times Y$ is strongly Fréchet.   □

Corollary 10. *A product of finitely many productively Fréchet spaces is productively Fréchet.*

P r o o f. Let X and Y be two productively Fréchet spaces. In view of Theorem 9 it suffices to show that $(X \times Y) \times W$ is strongly Fréchet for every strongly Fréchet space W to conclude that $X \times Y$ is productively Fréchet. As Y is productively Fréchet, by Theorem 9, $Y \times W$ is strongly Fréchet. Applying the same argument to the product of the productively Fréchet space X with the strongly Fréchet space $Y \times W$, we get that $X \times Y \times W$ is strongly Fréchet. $\qquad\square$

As a direct consequence of Theorems 9, 1 and 2, $\aleph_0$-bisequential and regular q Fréchet β_3 spaces are productively Fréchet. However, we provide a direct proof of the latter, while a direct proof of the former will be published in a forthcoming paper, because it requires the introduction of more machinery.

Proposition 11. *A regular q and β_3 Fréchet space is productively Fréchet.*

P r o o f. Let $\mathscr{F}$ be a strongly Fréchet filter and let $x \in \operatorname{adh} \mathscr{F}$. As x is a q-point, for every closed neighborhood W of x, there exists a sequence $(Q_n \cap W)$ of neighborhoods of x witnessing the definition of a q-point. As $(Q_n \cap W) \# \mathscr{F}$ and $\mathscr{F}$ is strongly Fréchet, there exists a sequence $(x_n^W) \geqslant \mathscr{F} \vee (Q_n \cap W)$, so that $\operatorname{adh}(x_n^W) \cap W \neq \emptyset$. As under the present assumptions, X is strongly Fréchet, there exists a sequence $(y_n^W) \geqslant \mathscr{F}$ such that $\lim(y_n^W) \cap W \neq \emptyset$. By regularity and Fréchetness of X, there exists a sequence $(y_p)_{p\in\omega}$ on $\bigcup\limits_{W=\operatorname{cl} W \in \mathscr{N}(x)} \lim(y_n^W) \cap W$ which converges to x. For each p, pick W_p such that $y_p \in \lim(y_n^{W_p})$. By the property β_3, there exists a compact metrizable subset C of $\{x\}\cup\{y_p\colon p \in \omega\}\cup\{y_n^{W_p}\colon n \in \omega, p \in \omega\}$ such that $|\{p\colon |C \cap \{y_n^{W_p}\colon n \in \omega\}| = \omega\}| = \omega$. Since each sequence $(y_n^W)_n \geqslant \mathscr{F}$, the countably based filter generated by $\{\{y_n^{W_p} \in C\colon n \in \omega, p \geqslant k\}\colon k \in \omega\}$ meshes with the strongly Fréchet filter $\mathscr{F}$. Therefore, there exists a sequence $(w_n)_\omega \geqslant \mathscr{F} \vee \{\{y_n^{W_p} \in C\colon n \in \omega, p \geqslant k\}\colon k \in \omega\}$. As a sequence on a compact metrizable set, $(w_n)_\omega$ has a convergent subsequence $(v_n)_{n\in\omega}$. Moreover $x \in \lim(v_n)_{n\in\omega}$ because $(v_n)_{n\in\omega} \geqslant \{\{y_n^{W_p} \in C\colon n \in \omega, p \geqslant k\}\colon k \in \omega\}$. Hence there exists a sequence finer that $\mathscr{F}$ converging to x. $\qquad\square$

On the other hand, a regular productively Fréchet space need not be a q-space, nor a bisequential space. Hence, Theorem 9 generalizes both Theorem 2 and [13, Proposition 4.D.4].

Example 12. (A regular productively Fréchet space which is neither bisequential nor a q-space).

The Σ-product of uncountably many copies of $\mathbb{R}$ is normal and $\aleph_0$-bisequential, hence productively Fréchet, but neither bisequential [1, Example 6.24] nor q [6, Proposition 11].

We consider a space X constructed by Nyikos [15, Remark 3.12]. The space X is productively Fréchet and not q. If we make some set-theoretical assumptions we can have that X is not α_3. Recall that a topological space is α_3 if for every stationary bisequence

$$x_{n,k} \underset{k}{\to} x,$$

there exists a sequence $(x_{n_p,k_p})_{p \in \omega}$ converging to x such that

$$|\{n \colon |\{k_p \colon n_p = n\}| = \omega\}| = \omega.$$

In particular, X is not $\aleph_0$-bisequential [1, Theorem 6.15], so that Theorem 9 also generalizes Theorem 1. We construct this example again here because we will have to work to show that it is productively Fréchet.

Given two functions $f, g \in \omega$ we write $f <^* g$ if $\{n \in \omega \colon g(n) \leqslant f(n)\}$ is finite. We let $\mathfrak{b}$ denote the bounding number of ω, i.e., the smallest cardinality of a collection $F \subseteq \omega^\omega$ such that for every $g \in \omega^\omega$ there is an $f \in F$ such that $f \not<^* g$. It is well known that one can construct a $<^*$-increasing family $\{f_\xi\}_{\xi \in \mathfrak{b}}$ of strictly increasing functions in ω^ω witnessing the definition of $\mathfrak{b}$. Given an $f \in \omega^\omega$, we let $f^{\downarrow\downarrow} = \{(n,k) \colon k < f(n)\}$. We let $\mathfrak{c}$ denote the cardinality of the real numbers.

Example 13. (A countable productively Fréchet space X which is not a q-space. Moreover, if we assume $\mathfrak{b} = \mathfrak{c}$, then X is not α_3.)

Let $K = \{f_\xi\}_{\xi \in \mathfrak{b}}$ be as above. Define a topology on $X_0 = (\omega \times \omega) \cup K$ so that: for each $f \in K$ the neighborhoods of f will be sets of the form $\{f\} \cup \{(k, f(k)) \colon k \geqslant n\}$ where $n \in \omega$ and all other points are isolated. Notice that X_0 is locally compact. We let X_1 be the one point compactification of X_0. We denote the compactifying point by ∞. Our space X will be $(\omega \times \omega) \cup \{\infty\}$ with the subspace topology from X_1.

We show that X is productively Fréchet. In view of Proposition 8, it is enough to show that X_1 is productively Fréchet. As every point in X_1 is first countable except the point ∞, it is enough to show that the neighborhood filter $\mathscr{N}$ of ∞ is productively Fréchet. Let $\mathscr{F}$ be a strongly Fréchet filter such that $\mathscr{F} \# \mathscr{N}$.

We may assume that $\mathscr{F} \# \{\omega \times \omega\}$. Otherwise, $\mathscr{F}$ has a base in $K \cup \{\infty\}$ which is a one point compactification of a discrete set which is $\aleph_0$-bisequential, by [1, Proposition 6.20]. By Lemma 4, $\mathscr{F} \vee (K \cup \{\infty\})$ is strongly Fréchet and is easily seen to mesh with $\mathscr{N}$. By the productive Fréchetness of $K \cup \{\infty\}$ we can find a countably based filter $\mathscr{L} \geqslant \mathscr{F} \vee \mathscr{N}$. So, we assume that $\mathscr{F} \# \{\omega \times \omega\}$.

Notice that the filter $\mathscr{F} \vee \{\omega \times \omega\}$ is strongly Fréchet, by Lemma 4. Moreover, we may assume that $(\mathscr{F} \vee \{\omega \times \omega\}) \# \mathscr{N}$. Otherwise, we have $\mathscr{F} \# (K \cup \{\infty\})$ and the previous argument applies.

We may also assume that $\mathscr{F} \geqslant \mathscr{C}$, where $\mathscr{C}$ is the filter generated by sets of the form $C_n = \{(k,l) \colon n \leqslant k\}$. Indeed, $\mathscr{F} \# \mathscr{C}$ and by Lemmas 4 and 7, $\mathscr{F} \vee \mathscr{C}$ is a strongly Fréchet filter meshing $\mathscr{N}$, that can be taken as our new $\mathscr{F}$.

Let $T = \{f \in K \colon f \# \mathscr{F}\}$. We consider two cases, when T is finite and when it is infinite.

Suppose T is finite. Since $\mathscr{F} \# \mathscr{N}$, we have that $\mathscr{F}^* = \mathscr{F} \vee ((\omega \times \omega) \setminus \bigcup T)$ also meshes with $\mathscr{N}$. By Lemmas 4 and 7, we also have that $\mathscr{F}^*$ is strongly Fréchet. Since $\mathscr{F}^*$ is strongly Fréchet, there is a sequence $(x_n)_{n \in \omega}$ such that $(x_n)_{n \in \omega} \geqslant \mathscr{F}^*$. By way of contradiction, assume that $\lim_{n \in \omega} x_n \neq \infty$. In this case, there is an $f \in K \setminus T$ such that $f \# (x_n)_{n \in \omega}$. Since $(x_n)_{n \in \omega} \geqslant \mathscr{F}^* \geqslant \mathscr{F}$ and $\mathscr{F} \geqslant \mathscr{C}$, we have $f \# \mathscr{F}$. However, $f \notin T$ so we have a contradiction.

Suppose now that T is infinite. Let $\{f_{\xi_n}\}_{n \in \omega}$ be the first ω-many elements of T and $\xi = \sup\{\xi_n \colon n \in \omega\}$.

For every $n \in \omega$ let $D_n = \{(k, l) \colon k, l \in \omega \colon f_{\xi_n}(k) \leqslant l < f_\xi(k)\}$. Let $\mathscr{D}$ be the filter generated by the collection $\{D_n \colon n \in \omega\}$.

Let $F \in \mathscr{F}$ and $n \in \omega$. Since $f_{\xi_{n+1}} \# \mathscr{F}$ and $\mathscr{F} \geqslant \mathscr{C}$, we have $|f_{\xi_{n+1}} \cap F| = \omega$. Since $f_{\xi_n} <^* f_{\xi_{n+1}} <^* f_\xi$, we have that $F \# D_n$. Hence, $\mathscr{F} \# \mathscr{D}$.

Since $\mathscr{F}$ is strongly Fréchet, there exists a sequence $(x_n)_{n \in \omega} \geqslant \mathscr{F} \vee \mathscr{D}$. We claim that $\lim x_n = \infty$. If not, then there is $\beta \in \mathfrak{b}$ such that $\{x_n \colon n \in \omega\} \cap f_\beta$ is infinite. Since $\mathscr{D}$ has a base consisting of subsets of $f_\xi^{\downarrow\downarrow}$, we have $\beta < \xi$. Since, $(x_n)_{n \in \omega} \geqslant \mathscr{F}$, we have that $f_\beta \# \mathscr{F}$. So, $f_\beta = f_{\xi_n}$ for some $n \in \omega$. Since $(x_n)_{n \in \omega} \geqslant \mathscr{D}$, we have that $\{x_n \colon n \in \omega\} \cap f_\beta$ is finite, a contradiction. Thus, $\lim x_n = \infty$.

Therefore, X_1 is productively Fréchet.

We show that X is not q. Let $\{U_n\}_{n \in \omega}$ be a nested sequence of neighborhoods of ∞. Without loss of generality, we may assume that $X \setminus U_n$ is contained in the union of a finite collection L of graphs of functions from K together with some finite collection of points. Pick $f \in K$ such that $g <^* f$ for all $g \in L$. Consider $\{(k, f(k))\}_{k \in \omega}$ as a sequence. Since $g <^* f$ for all $g \in L$, there is for every $n \in \omega$ a $k_n \in \omega$ such that $(k_n, f(k_n)) \in U_n$. Clearly, $\{(k_n, f(k_n)\}_{n \in \omega}$ has no point of adherence in X. Thus, X is not q.

The simple argument that $\mathfrak{b} = \mathfrak{c}$ implies that X is not α_3 is given in [15, Remark 3.12].

Fréchet spaces are exactly the hereditarily quotient images of metrizable spaces; strongly Fréchet spaces are the countably biquotient images of metrizable spaces and bisequential spaces are biquotient images of metrizable spaces (see [13] and [3] for details). Analogously, productively Fréchet spaces can be characterized as images of metrizable spaces under a nice class of quotient maps, intermediate between countably biquotient and biquotient maps.

A continuous surjection $f \colon X \to Y$ is called φ_ω^1-*biquotient* if for every strongly Fréchet filter $\mathscr{H}$ on Y

$$y \in \operatorname{adh}_Y \mathscr{H} \implies f^- y \cap \operatorname{adh}_X f^- \mathscr{H} \neq \emptyset.$$

In view of the convergence-theoretic interpretation [3, Theorem 1.2] of quotient maps, of [14, Theorem 4.8] and of [14, Theorem 7.1], we have:

Proposition 14. *Let $f\colon X \to Y$ be a continuous surjection. The following are equivalent*

1. *f is φ_ω^1-biquotient;*
2. *$y \in \mathrm{adh}_Y \mathscr{H} \implies f^- y \cap \mathrm{adh}_X f^- \mathscr{H} \neq \emptyset$, for every strongly Fréchet filter $\mathscr{H}$;*
3. *for every φ_ω^1-cover [5] $\mathscr{S}$ of $f^- y$, there exists a finite subfamily $\mathscr{R} \subset \mathscr{S}$ such that*
   $$y \in \mathrm{int}\Big(\bigcup_{R \in \mathscr{R}} f(R)\Big);$$
4. *$f \times \mathrm{Id}_Z$ is hereditarily quotient for every atomic [6] strongly Fréchet topological space Z;*
5. *$f \times g$ is countably biquotient for every biquotient map with strongly Fréchet range.*

Since productively Fréchet spaces are exactly topological spaces that satisfy (1) for every strongly Fréchet filter, the following is an immediate consequence of [3, Theorems 5.2 and 1.2].

Theorem 15. *A space Y is productively Fréchet if and only if there exists a φ_ω^1-biquotient map $f\colon X \to Y$ with metrizable domain X.*

Problem 16. Is the product of countably many productively Fréchet spaces productively Fréchet?

Problem 17. Is there a ZFC example of a countable productively Fréchet space that is not bisequential?

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[5] An open cover $\mathscr{S}$ is called φ_ω^1-*cover*, if the family of complements of elements of the ideal generated by $\mathscr{S}$ form a strongly Fréchet filter.

[6] A topology is called *atomic* if all but one point are isolated.

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