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HÖLDER REGULARITY FOR NONHOMOGENEOUS ELLIPTIC SYSTEMS WITH NONLINEARITY GREATER THAN TWO

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Abstract. Regularity results for elliptic systems of second order quasilinear PDEs with nonlinear growth of order $q > 2$ are proved, extending results of [7] and [10]. In particular Hölder regularity of the solutions is obtained if the dimension n is less than or equal to $q + 2$.

Keywords: nonlinear elliptic systems, regularity up to the boundary

MSC 2000: 35J65

1. Introduction

Let Ω be a boundend open set in $\mathbb{R}^n$, $n \geqslant 2$, $N > 1$ an integer, $u = (u_1, u_2, \ldots, u_N)$ a vector function of $\mathbb{R}^N$ and $Du = (D_1 u, D_2 u, \ldots, D_n u)$. We denote by $p = (p^1, \ldots, p^n)$, with $p^i \in \mathbb{R}^N$, a typical vector of $\mathbb{R}^{nN}$. Let q be a real number $\geqslant 2$. For every $p \in \mathbb{R}^k$, $k \geqslant 1$, we set

$$V(p) = (1 + \|p\|^2)^{\frac{1}{2}} \text{ and } W(p) = V^{\frac{q-2}{2}}(p)p. \tag{1.1}$$

Let $a^i(x,p)$, $i = 1, 2, \ldots, n$, be vectors of $\mathbb{R}^N$, defined on $\Omega \times \mathbb{R}^{nN}$, of class C^1 in p and uniformly continuous in x in the following sense: for every x, $y \in \Omega$ and $p \in \mathbb{R}^{nN}$

$$\left\{ \sum_i \|a^i(x,p) - a^i(y,p)\|^2 \right\}^{\frac{1}{2}} \leqslant \omega(\|x-y\|) V^{q-2}(p) \cdot \|p\| \tag{1.2}$$

where $\omega(t)$, with $t > 0$, is a bounded, nondecreasing function, converging to zero as $t \to 0$.

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The derivatives $\partial a^i/\partial p_k^j$ are measurable in x and continuous in p.

Suppose that

$$a^i(x,0)=0, \quad \forall\, x\in\Omega. \tag{1.3}$$

Setting

$$A_{ij}^{hk}(x,p)=\frac{\partial a_h^i(x,p)}{\partial p_k^j}, \quad A_{ij}=\{A_{ij}^{hk}\}, \tag{1.4}$$

$$\tilde{A}_{ij}(x,p)=\int_0^1 A_{ij}(x,tp)\,\mathrm{d}t \tag{1.5}$$

we suppose that, $\forall\, x\in\Omega$, $\forall\, p\in\mathbb{R}^{nN}$ and $\forall\,\xi\in\mathbb{R}^{nN}$,

$$\Big\{\sum_{ij}\|A_{ij}(x,p)\|^2\Big\}^{\frac{1}{2}}\leqslant MV^{q-2}(p), \tag{1.6}$$

$$\sum_{ij}(A_{ij}(x,p)\xi^j/\xi^i)\geqslant \nu V^{q-2}(p)\|\xi\|^2, \tag{1.7}$$

where M and ν are positive constants.

By virtue of the hypothesis (1.3)

$$a^i(x,p)=\sum_j \tilde{A}_{ij}(x,p)p^j,$$

and then, by the condition (1.6),

$$|a^i(x,p)\|\leqslant MV^{q-2}(p)\|p\|. \tag{1.8}$$

Moreover let $F^0(x,u,p)$ and $F^i(x,u)$, $i=1,\dots,n$, be vectors of $\mathbb{R}^N$, defined, respectively, in $\Omega\times\mathbb{R}^N\times\mathbb{R}^{nN}$ and in $\Omega\times\mathbb{R}^N$, measurable in x, continuous in u and p, and such that

$$\|F^0(x,u,p)\|\leqslant|f_0(x)|+|b(x)|\,\|u\|+|c(x)|\,\|p\| \tag{1.9}$$

$$\|F^i(x,u)\|\leqslant|f_i(x)|+|a(x)|\,\|u\|,\ \ i=1,2,\dots,n \tag{1.10}$$

with $a(x),b(x),c(x)\in L^\infty(\Omega)$,

$$f_i(x)\in L^{\frac{q}{q-1},\frac{\mu}{q-1}}(\Omega), \quad i=1,\dots,n, \tag{1.11}$$

where $0 < \mu(q-1)^{-1} < \lambda$,[1]

$$f_0(x) \in L^{qr,\mu r}(\Omega), \tag{1.12}$$

where, for $n > q$, $r = n(n(q-1)+q)^{-1}$ and for $n = q$, qr is a number $\in (1,q)$.[2] In the present paper, let $u \in H^{1,q}(\Omega, \mathbb{R}^N)$ be a solution of the Dirichlet problem

$$\begin{cases} u - g \in H_0^{1,q}(\Omega, \mathbb{R}^N), \\ \sum_i D_i a^i(x, Du) = \sum_i D_i F^i(x,u) - F^0(x,u,Du) \quad \text{in } \Omega. \end{cases} \tag{1.13}$$

We will study the regularity of the vectors u and Du in the spaces $\mathcal{L}^{q,\mu}(\Omega)$ and, in particular, under a suitable limitation on n ($n \leqslant q+2$), the Hölder continuity in $\overline{\Omega}$ of the solution u.

For this purpose we will suppose that $\partial\Omega$ is of class C^2.

We will prove the following regularity theorem.

Theorem 1.1. *If $u \in H^{1,q}(\Omega)$ is a solution of the Dirichlet problem* (1.13), *$\partial\Omega$ is of class C^2, $g \in H^{1,q,(\frac{\mu}{q-1})}(\Omega)$, $0 < \mu(q-1)^{-1} < \lambda$, and F^0, F^i, $i = 1, \dots, n$, satisfy the conditions* (1.9)–(1.12), *then*

$$Du \in L^{q,\frac{\mu}{q-1}}(\Omega)$$

and

$$\|W(Du)\|^2_{L^{2,\frac{\mu}{q-1}}(\Omega)} \leqslant c\mathcal{K},$$

where, if $\mu \leqslant nq - 2n + q$,

$$\mathcal{K} = \sum_i \|f_i\|^{\frac{q}{q-1}}_{L^{q,\frac{\mu}{q-1}}(\Omega)} + \|Du\|^{\frac{q}{q-1}}_{L^q(\Omega)} + \|g\|^{\frac{q}{q-1}}_{H^{1,q,(\frac{\mu}{q-1})}(\Omega)} + \|W(Dg)\|^2_{L^{2,\frac{\mu}{q-1}}(\Omega)} + \|f_0\|^{\frac{q}{q-1}}_{L^{\frac{nq}{n(q-1)+q},\mu\frac{n}{n(q-1)+q}}(\Omega)}, \tag{1.14}$$

whereas, if $\mu > nq - 2n + q$,

$$\mathcal{K} = \sum_i \|f_i\|^{\frac{q}{q-1}}_{L^{q,\frac{\mu}{q-1}}(\Omega)} + \|Du\|^{\frac{q}{q-1}}_{L^{q,\frac{nq-2n+q}{q-1}}(\Omega)} + \|g\|^{\frac{q}{q-1}}_{H^{1,q,(\frac{\mu}{q-1})}(\Omega)} + \|W(Dg)\|^2_{L^{2,\frac{\mu}{q-1}}(\Omega)} + \|f_0\|^{\frac{q}{q-1}}_{L^{\frac{nq}{n(q-1)+q},\mu\frac{n}{n(q-1)+q}}(\Omega)}. \tag{1.15}$$

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[1] λ is the exponent which occurs in the fundamental estimate (3.2).

[2] For the notations see Section 2.

In particular, if $n - q < \frac{\mu}{q-1} < \lambda$, *we have*

$$u \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N) \quad \textit{with } \alpha = 1 - \frac{n(q-1) - \mu}{q(q-1)}$$

and

$$[u]^q_{\alpha,\overline{\Omega}} \leqslant c\mathcal{K}$$

where $\mathcal{K}$ *is given by* (1.14) *if* $\mu \leqslant nq - 2n + q$, *and by* (1.15) *if* $\mu > nq - 2n + q$.

Theorem 1.1 was already proved in [7] in the case of nonlinearity $q = 2$. The present paper improves the result of [7] to the case $q > 2$.

Moreover, regularity results for elliptic systems with arbitrary order equations have been considered by Widman (see [10]), who established, under less restrictive assumptions, the Hölder continuity of solutions if $n < q + \varepsilon$ with exponent $1 - \frac{n}{q} + \frac{\varepsilon}{q}$, less than the one of Theorem 1.1, because ε is only greater than zero. The Widman result allows to obtain Hölder regularity only for $n \leqslant q$, whereas in our case we obtain $n \leqslant q + 2$ (in particular $n \leqslant 4$). In a different context, J. Serrin obtained in [9] regularity results for the solutions of semilinear equations, in the case $q > 2$.

Theorem 1.1 can be extended to the case of right-hand side with more general growth conditions and to the case of principal part coefficients depending on u.

It is well known that general Hölder continuity results can not hold, as the examples [6] and [8] show.

2. Preliminary results and notations

We define

$$B(x^0, \sigma) = \{x\colon \|x - x^0\| < \sigma\}; \tag{2.1}$$

moreover, if $x^0_n = 0$,

$$B^+(x^0, \sigma) = \{x \in B(x^0, \sigma)\colon x_n > 0\}, \tag{2.2}$$

$$\Gamma(x^0, \sigma) = \{x \in B(x^0, \sigma)\colon x_n = 0\}. \tag{2.3}$$

We will simply write $B^+(\sigma)$, $\Gamma(\sigma)$ and Γ instead of $B^+(0, \sigma)$, $\Gamma(0, \sigma)$ and $\Gamma(0, 1)$, respectively.

Troughout the present paper, Ω will denote a bounded open set in $\mathbb{R}^n$ with diameter d_Ω and with $\partial\Omega$ of class C^2.

If $u \in L^1(\mathcal{B})$ and $\mathcal{B}$ is a measurable set with $\operatorname{meas}\mathcal{B} \neq 0$, then

$$(2.4) \qquad u_{\mathcal{B}} = \fint_{\mathcal{B}} u(x)\,\mathrm{d}x = \frac{1}{\operatorname{meas}\mathcal{B}} \int_{\mathcal{B}} u(x)\,\mathrm{d}x.$$

If $u \in L^\infty(\Omega)$, we define

$$(2.5) \qquad \|u\|_{\infty,\Omega} = \operatorname*{ess\,sup}_{\Omega} \|u(x)\|.$$

If $u \in C^{0,\alpha}(\overline{\Omega})$, $0 < \alpha \leqslant 1$, we set

$$(2.6) \qquad [u]_{\alpha,\overline{\Omega}} = \sup_{x,y\in\Omega} \frac{\|u(x) - u(y)\|}{\|x-y\|^\alpha}$$

and we will say that $u \in C^{0,\alpha}(\Omega)$ if $u \in C^{0,\alpha}(K)$ for every compact subset $K \subset \Omega$.

If $u \in L^{q,\mu}(\Omega)$, $0 \leqslant \mu \leqslant n$, or $u \in \mathcal{L}^{q,\mu}(\Omega)$, $0 \leqslant \mu \leqslant n+q$, we define, as usual (see [1])

$$(2.7) \qquad \|u\|^q_{L^{q,\mu}(\Omega)} = \sup \sigma^{-\mu} \int_{\Omega(x^0,\sigma)} \|u(x)\|^q\,\mathrm{d}x,$$

$$(2.8) \qquad [u]^q_{\mathcal{L}^{q,\mu}(\Omega)} = \sup \sigma^{-\mu} \int_{\Omega(x^0,\sigma)} \|u(x) - u_{\Omega(x^0,\sigma)}\|^q\,\mathrm{d}x,$$

where $\Omega(x^0,\sigma) = \Omega \cap B(x^0,\sigma)$ and the supremum is taken over all the $x^0 \in \Omega$ and the $\sigma \in (0, \operatorname{diam}\Omega)$.

We say that $u \in H^{1,q,(\mu)}(\Omega)$, $0 \leqslant \mu \leqslant n$, if $u \in H^{1,q}(\Omega)$ and $Du \in L^{q,\mu}(\Omega)$, and we define

$$(2.9) \qquad \|u\|_{H^{1,q,(\mu)}(\Omega)} = \|u\|_{L^{q,\mu}(\Omega)} + \|Du\|_{L^{q,\mu}(\Omega)}.$$

We recall that, if $0 \leqslant \mu \leqslant n$, then

$$G \in L^{q,\mu}(\Omega) \Leftrightarrow W(G) \in L^{2,\mu}(\Omega)$$

and the following inequality holds

$$(2.10) \qquad c\|G\|^q_{L^{q,\mu}(\Omega)} \leqslant \|W(G)\|^2_{L^{2,\mu}(\Omega)} \leqslant c\|G\|^2_{L^{q,\mu}(\Omega)}(1 + \|G\|_{L^{q,\mu}(\Omega)})^{q-2}.$$

Lemma 2.1. *There exists a positive constant $c(q)$ such that, for all $p, \overline{p} \in \mathbb{R}^k$,*

$$(2.11) \qquad \|W(p)\| + \|W(\overline{p})\| \leqslant 2W(\|p\| + \|\overline{p}\|) \leqslant c(q)\{\|W(p)\| + \|W(\overline{p})\|\}.$$

(See [3, Lemma 2.I, p. 122]).

Lemma 2.2. *If $\mu > -1$ there exist positive constants $c(\mu)$ and $C(\mu)$ such that, for every two vectors a, b in $\mathbb{R}^N$, we have*

$$c(\mu)(1 + \|a\| + \|b\|)^\mu \leqslant \int_0^1 (1 + \|a + tb\|)^\mu \,\mathrm{d}t \leqslant C(\mu)(1 + \|a\| + \|b\|)^\mu. \tag{2.12}$$

(See [3, Lemma 2.II, p. 123]).

Lemma 2.3. *Let A and C be bouded and open sets in $\mathbb{R}^n$ and τ be a mapping of class C^1, together with its inverse, from $\overline{A}$ into $\overline{C}$. Let A^* be an open set $\subset\subset A$ and $C^* = \tau(A^*)$. Then, $\forall\, q > 1$ and $\forall\, \mu \in [0, n)$, the mapping $\varphi\colon\ u \to u \circ \tau$ is a linear and continuous one together with its inverse, from $L^{q,\mu}(C^*)$ into $L^{q,\mu}(A^*)$ and from $H^{1,q,(\mu)}(C^*)$ into $H^{1,q,(\mu)}(A^*)$.*

(See [3, Lemma 2.IV and 2.V, p. 123]).

Lemma 2.4. *Let $\varphi(t)$ and $o(t)$ be nonnegative functions defined in $(0, d]$. Suppose that $\lim_{t\to 0} o(t) = 0$ and $\forall\, \sigma \in (0, d]$, $\forall\, t \in (0, 1)$,*

$$\varphi(t\sigma) \leqslant \{At^\lambda + o(\sigma)\}\varphi(\sigma) + K\sigma^\mu$$

with $0 < \mu < \lambda$, $A > 0$, and $K \geqslant 0$; then for all $\varepsilon < \lambda - \mu$ there is a $\sigma_\varepsilon \leqslant d$ such that, if $0 < \sigma \leqslant \sigma_\varepsilon$ and $t \in (0, 1)$,

$$\varphi(t\sigma) \leqslant (1 + A)t^{\lambda-\varepsilon}\varphi(\sigma) + KM(t\sigma)^\mu$$

where $M = M(A, \varepsilon, \lambda, \mu)$.

(See [3, Lemma 2.VII, p. 125]).

3. Interior local regularity results

Theorem 3.1. *If $u \in H^{1,q}(\Omega)$, $q \geqslant 2$, is a solution of the basic system*

$$\sum_i D_i a^i(Du) = 0 \quad \text{in } \Omega, \tag{3.1}$$

under the conditions (1.3), (1.6) and (1.7), then for every ball $B(\sigma) = B(x^0, \sigma) \subset\subset \Omega$ and $\forall\, t \in (0, 1)$,

$$\int_{B(t\sigma)} \|W(Du)\|^2 \,\mathrm{d}x \leqslant ct^\lambda \int_{B(\sigma)} \|W(Du)\|^2 \,\mathrm{d}x \tag{3.2}$$

where $\lambda = \min\{2+\varepsilon, n\}$, $\varepsilon = \varepsilon(\nu, M, n)$ *is a suitable number* $0 < \varepsilon < 1$ *(whose existence is ensured by* [2, *Theorem* 8.1, *p.* 90]*) and the constant* c *does not depend on* t, σ, x^0.

(See [3, Theorem 3.I, p. 128]).

Theorem 3.2. *If* $u \in H^{1,q}(\Omega)$, *with* $q \geqslant 2$, *is a solution of the basic system* (3.1), *under the hypotheses* (1.3), (1.6) *and* (1.7), *and if*

$$2 \leqslant n \leqslant q+2, \tag{3.3}$$

then, for every ball $B(\sigma) = B(x^0, \sigma) \subset\subset \Omega$ *and* $\forall\, t \in (0,1)$,

$$\int_{B(t\sigma)} \|u\|^q \,\mathrm{d}x \leqslant ct^n \Big\{ \int_{B(\sigma)} \|u\|^q \,\mathrm{d}x + \sigma^q \int_{B(\sigma)} \|W(Du)\|^2 \,\mathrm{d}x \Big\} \tag{3.4}$$

where the constant c *does not depend on* x^0, t, *and* σ.

(See [3, Theorem I.II, p. 121]).

For the implications which the fundamental estimate (3.4) has in the case $q = 2$ see [4] and [5]. In the case $q > 2$, the inequality (3.4) has analogous implications, but we will not deal with this question in this paper.

Let $u \in H^{1,q}(\Omega)$ be a solution of the Dirichlet problem

$$\begin{cases} u - g \in H_0^{1,q}(\Omega), \\ \sum_i D_i a^i(x, Du) = \sum_i D_i F^i - F^0 \quad \text{in } \Omega \end{cases}$$

where the vectors $a^i(x,p)$ satisfy the assumptions (1.2), (1.3), (1.6) and (1.7). Set

$$w = u - g.$$

The vector $w \in H_0^{1,q}(\Omega)$ is a solution of the system

$$\sum_i D_i a^i(x, Dw + Dg) = \sum_i D_i F^i(x, w+g) - F^0(x, w+g, Dw+Dg) \quad \text{in } \Omega \tag{3.5}$$

in the following sense

$$\int_\Omega \sum_i (a^i(x, Dw+Dg)/D_i\varphi)\,\mathrm{d}x = \int_\Omega \sum_i (F^i/D_i\varphi)\,\mathrm{d}x + \int_\Omega F^0/\varphi \,\mathrm{d}x \qquad \forall\, \varphi \in H_0^{1,q}(\Omega). \tag{3.6}$$

For the vector w we want to prove the following interior local regularity result.

Theorem 3.3. *If $w \in H_0^{1,q}(\Omega)$ is a solution of the system (3.5), under the assumptions (1.2), (1.3), (1.6) and (1.7), if the vectors F^0, F^i, $i = 1, \dots, n$, satisfy the conditions (1.9), (1.10), (1.11) and (1.12), and if*

$$g \in H^{1,q,(\frac{\mu}{q-1})}(\Omega), \quad 0 < \frac{\mu}{q-1} < \lambda, \tag{3.7}$$

then, for every open set $\Omega^* \subset\subset \Omega$, we have

$$Dw \in L^{q,\frac{\mu}{q-1}}(\Omega^*), \tag{3.8}$$

and the estimate

$$\|W(Dw)\|^2_{L^{2,\frac{\mu}{q-1}}(\Omega^*)} \leqslant c\Big\{\int_\Omega \|W(Dw)\|^2 \,\mathrm{d}x + \mathcal{M}\Big\} \tag{3.9}$$

holds, where, if $\mu \leqslant nq - 2n + q$, the constant c depends also on $d = \operatorname{dist}(\overline{\Omega^*}, \partial\Omega)$ and

$$\begin{aligned} \mathcal{M} = \sum_i \|f_i\|^{\frac{q}{q-1}}_{L^{q,\frac{\mu}{q-1}}(\Omega)} &+ \|Dw\|^{\frac{q}{q-1}}_{L^q(\Omega)} + \|g\|^{\frac{q}{q-1}}_{H^{1,q,(\frac{\mu}{q-1})}(\Omega)} \\ &+ \|W(Dg)\|^2_{L^{2,\frac{\mu}{q-1}}(\Omega)} + \|f_0\|^{\frac{q}{q-1}}_{L^{\frac{nq}{n(q-1)+q},\mu\frac{n}{n(q-1)+q}}(\Omega)}, \end{aligned} \tag{3.10}$$

whereas, if $\mu > nq-2n+q$, denoting by Ω^{**} an open set such that $\Omega^* \subset\subset \Omega^{**} \subset\subset \Omega$, the constant c depends also on $d^* = \operatorname{dist}(\overline{\Omega^*}, \partial\Omega^{**})$ and

$$\begin{aligned} \mathcal{M} = \sum_i \|f_i\|^{\frac{q}{q-1}}_{L^{q,\frac{\mu}{q-1}}(\Omega)} &+ \|Dw\|^{\frac{q}{q-1}}_{L^{q,\frac{nq-2n+q}{q-1}}(\Omega^{**})} + \|g\|^{\frac{q}{q-1}}_{H^{1,q,(\frac{\mu}{q-1})}(\Omega)} \\ + \|W(Dg)\|^2_{L^{2,\frac{\mu}{q-1}}(\Omega)} &+ \|f_0\|^{\frac{q}{q-1}}_{L^{\frac{nq}{n(q-1)+q},\mu\frac{n}{n(q-1)+q}}(\Omega)}. \end{aligned} \tag{3.11}$$

P r o o f. From this moment and where it is necessary, for the sake of simplicity, we set $q' = q(q-1)^{-1}$, $q'' = nq(n(q-1)+q)^{-1}$ and $q^* = nq(n-q)^{-1}$.

We prove this theorem in the case $n > q$; the proof needs only small modifications in the case $n = q$.

Fix a ball $B(\sigma) = B(x^0, \sigma)$ with $x_0 \in \Omega^*$ and $\sigma \leqslant d$.

In $B(\sigma)$ we decompose w as $v - z$, where z is a solution of the Dirichlet problem

$$\begin{cases} z \in H_0^{1,q}(B(\sigma)), \\ \sum\limits_i D_i a^i(x^0, Dz + Dw + Dg) \\ \quad = \sum\limits_i D_i[a^i(x, Dw + Dg) - F^i] + F^0 \quad \text{in } B(\sigma) \end{cases} \tag{3.12}$$

while $v \in H^{1,q}(B(\sigma))$ is a solution of the system

$$\sum_i D_i a^i(x^0, Dv + Dg) = 0 \quad \text{in } \ B(\sigma). \tag{3.13}$$

(3.12) means that $\forall\, \varphi \in H_0^{1,q}(B(\sigma))$,

$$\begin{aligned} \int_{B(\sigma)} \sum_i (a^i(x^0, Dz + Dw + Dg) - a^i(x^0, Dw + Dg)/D_i\varphi)\,\mathrm{d}x \\ = \int_{B(\sigma)} \sum_i (a^i(x, Dw + Dg) - a^i(x^0, Dw + Dg)/D_i\varphi)\,\mathrm{d}x \\ - \int_{B(\sigma)} \sum_i (F^i/D_i\varphi)\,\mathrm{d}x - \int_{B(\sigma)} F^0/\varphi\,\mathrm{d}x. \end{aligned} \tag{3.14}$$

Assuming $\varphi = z$, setting

$$B_{ij} = \int_0^1 A_{ij}(x, Dw + Dg + tDz)\,\mathrm{d}t$$

and taking into account the ellipticity condition (1.7) and Lemma 2.2, we obtain from the previous inequalities that

$$\begin{aligned} c\int_{B(\sigma)} (1 + \|Dw\| + \|Dg\| + \|Dz\|)^{q-2}\|Dz\|^2\,\mathrm{d}x \\ \leqslant \int_{B(\sigma)} \sum_i \|a^i(x, Dw + Dg) - a^i(x^0, Dw + Dg)\|\,\|Dz\|\,\mathrm{d}x \\ + \int_{B(\sigma)} \sum_i \|F^i\|\,\|D_i z\|\,\mathrm{d}x + \int_{B(\sigma)} \|F^0\|\,\|z\|\,\mathrm{d}x = A + B + C. \end{aligned} \tag{3.15}$$

On the other hand, by the hypothesis (1.2) we have: $\forall\, \varepsilon > 0$

$$\begin{aligned} A = \int_{B(\sigma)} \sum_i \|a^i(x, Dw + Dg) - a^i(x^0, Dw + Dg)\|\,\|Dz\|\,\mathrm{d}x \\ \leqslant c\omega(\sigma)\int_{B(\sigma)} V^{q-2}(Dw + Dg)\|Dw + Dg\|\,\|Dz\|\,\mathrm{d}x \\ \leqslant \varepsilon\int_{B(\sigma)} V^{q-2}(Dw + Dg)\|Dz\|^2\,\mathrm{d}x \\ + c\omega^2(\sigma)\int_{B(\sigma)} \|W(Dw + Dg)\|^2\,\mathrm{d}x. \end{aligned} \tag{3.16}$$

Now, by means of the conditions (1.9), (1.10), (1.11), (1.12) and (3.7), we have

$$\begin{aligned}(3.17)\qquad \int_{B(\sigma)} \sum_i \|f_i\|\,\|D_i z\|\,\mathrm{d}x &\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,\mathrm{d}x + c\int_{B(\sigma)} \sum_i \|f_i\|^{q'}\,\mathrm{d}x \\ &\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,\mathrm{d}x + c\sigma^{\frac{\mu}{q-1}} \sum_i \|f_i\|^{q'}_{L^{q',\frac{\mu}{q-1}}(B(\sigma))};\end{aligned}$$

$$\begin{aligned}(3.18)\qquad \int_{B(\sigma)} \|w\|\,\|Dz\|\,\mathrm{d}x &\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,\mathrm{d}x + c\int_{B(\sigma)} \|w\|^{q'}\,\mathrm{d}x \\ &\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,\mathrm{d}x + c\sigma^{\frac{nq-2n+q}{q-1}} \|w\|^{q'}_{L^{q,q}(B(\sigma))};\end{aligned}$$

$$\begin{aligned}(3.19)\qquad &\int_{B(\sigma)} \|g\|\,\|Dz\|\,\mathrm{d}x \\ &\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,\mathrm{d}x + c\int_{B(\sigma)} \|g\|^{q'}\,\mathrm{d}x \\ &\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,\mathrm{d}x + c\sigma^{n\frac{q-2}{q-1}} \left(\int_{B(\sigma)} \|g\|^q\,\mathrm{d}x\right)^{\frac{1}{q-1}} \\ &\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,\mathrm{d}x + c\sigma^{n\frac{q-2}{q-1}+\frac{\mu}{(q-1)^2}} \|g\|^{q'}_{L^{q,\frac{\mu}{q-1}}(B(\sigma))}.\end{aligned}$$

From (3.17), (3.18) and (3.19) we have,

$$\begin{aligned}(3.20)\qquad B = \int_{B(\sigma)} \sum_i \|F^i\|\,\|D_i z\|\,\mathrm{d}x &\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,\mathrm{d}x \\ &+ c\Big(\sigma^{\frac{\mu}{q-1}} \sum_i \|f_i\|^{q'}_{L^{q',\frac{\mu}{q-1}}(B(\sigma))} + \sigma^{\frac{nq-2n+q}{q-1}} \|w\|^{q'}_{L^{q,q}(B(\sigma))} \\ &\qquad + \sigma^{n\frac{q-2}{q-1}+\frac{\mu}{(q-1)^2}} \|g\|^{q'}_{L^{q,\frac{\mu}{q-1}}(B(\sigma))}\Big).\end{aligned}$$

In the same way, if we denote by q^* the Sobolev-Poincaré exponent, we have

$$\begin{aligned}(3.21)\qquad \int_{B(\sigma)} \|f^0\|\,\|z\|\,\mathrm{d}x &\leqslant \left(\int_{B(\sigma)} \|z\|^{q^*}\,\mathrm{d}x\right)^{\frac{1}{q^*}} \left(\int_{B(\sigma)} \|f^0\|^{q''}\,\mathrm{d}x\right)^{\frac{1}{q''}} \\ &\leqslant c\left(\int_{B(\sigma)} \|Dz\|^q\,\mathrm{d}x\right)^{\frac{1}{q}} \left(\int_{B(\sigma)} \|f^0\|^{q''}\,\mathrm{d}x\right)^{\frac{1}{q''}} \\ &\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,\mathrm{d}x + c\left(\int_{B(\sigma)} \|f^0\|^{\frac{nq}{n(q-1)+q}}\,\mathrm{d}x\right)^{\frac{n(q-1)+q}{n(q-1)}} \\ &\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,\mathrm{d}x + c\sigma^{\frac{\mu}{q-1}} \|f^0\|^{q'}_{L^{\frac{nq}{n(q-1)+q},\mu\frac{n}{n(q-1)+q}}(B(\sigma))};\end{aligned}$$

$$\int_{B(\sigma)} \|w\|\|z\| \,\mathrm{d}x \leqslant \Big(\int_{B(\sigma)} \|z\|^{q^*} \,\mathrm{d}x\Big)^{\frac{1}{q^*}} \Big(\int_{B(\sigma)} \|w\|^{q''} \,\mathrm{d}x\Big)^{\frac{1}{q''}} \tag{3.22}$$

$$\leqslant c\Big(\int_{B(\sigma)} \|Dz\|^{q} \,\mathrm{d}x\Big)^{\frac{1}{q}} \Big(\int_{B(\sigma)} \|w\|^{q''} \,\mathrm{d}x\Big)^{\frac{1}{q''}}$$

$$\leqslant \ \varepsilon \int_{B(\sigma)} \|Dz\|^{q} \,\mathrm{d}x + c\Big(\int_{B(\sigma)} \|w\|^{q''} \,\mathrm{d}x\Big)^{\frac{q'}{q''}}$$

$$\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^{q} \,\mathrm{d}x + c\sigma^{\frac{nq-2n+2q}{q-1}} \|w\|^{q'}_{L^{q,q}(B(\sigma))};$$

$$\int_{B(\sigma)} \|g\| \, \|z\| \,\mathrm{d}x \leqslant \Big(\int_{B(\sigma)} \|z\|^{q^*} \,\mathrm{d}x\Big)^{\frac{1}{q^*}} \Big(\int_{B(\sigma)} \|g\|^{q''} \,\mathrm{d}x\Big)^{\frac{1}{q''}} \tag{3.23}$$

$$\leqslant c\Big(\int_{B(\sigma)} \|Dz\|^{q} \,\mathrm{d}x\Big)^{\frac{1}{q}} \Big(\int_{B(\sigma)} \|g\|^{q''} \,\mathrm{d}x\Big)^{\frac{1}{q''}}$$

$$\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^{q} \,\mathrm{d}x + c\bigg(\Big(\int_{B(\sigma)} \|g\|^{q''} \,\mathrm{d}x\Big)^{\frac{1}{q''}}\bigg)^{q'}$$

$$\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^{q} \,\mathrm{d}x + c\bigg(\Big(\int_{B(\sigma))} \|g\|^{q''} \,\mathrm{d}x\Big)^{\frac{1}{q''}} \ \sigma^{\frac{nq-2n+q}{q}}\bigg)^{\frac{q}{q-1}}$$

$$\leqslant \ \varepsilon \int_{B(\sigma)} \|Dz\|^{q} \,\mathrm{d}x + c\Big[\sigma^{\frac{\mu}{q-1}} \|g\|^{q}_{L^{q,\frac{\mu}{q-1}}(B(\sigma))} \ \sigma^{nq-2n+q}\Big]^{\frac{1}{q-1}}$$

$$\leqslant \ \varepsilon \int_{B(\sigma)} \|Dz\|^{q} \,\mathrm{d}x + c\sigma^{\frac{\mu}{(q-1)^2}+\frac{nq-2n+q}{q-1}} \|g\|^{q'}_{L^{q,\frac{\mu}{q-1}}(B(\sigma))};$$

$$\int_{B(\sigma)} \|Dw\| \, \|z\| \,\mathrm{d}x \leqslant \Big(\int_{B(\sigma)} \|Dw\|^{q''} \,\mathrm{d}x\Big)^{\frac{1}{q''}} \Big(\int_{B(\sigma)} \|z\|^{q^*} \,\mathrm{d}x\Big)^{\frac{1}{q^*}} \tag{3.24}$$

$$\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^{q} \,\mathrm{d}x + c\bigg[\Big(\int_{B(\sigma)} \|Dw\|^{q''} \,\mathrm{d}x\Big)^{\frac{1}{q''}}\bigg]^{q'}$$

$$\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^{q} \,\mathrm{d}x + c\sigma^{\frac{nq-2n+q}{q-1}} \|Dw\|^{q'}_{L^{q}(B(\sigma))};$$

$$\int_{B(\sigma)} \|z\| \, \|Dg\| \,\mathrm{d}x \leqslant \Big(\int_{B(\sigma)} \|z\|^{q^*} \,\mathrm{d}x\Big)^{\frac{1}{q^*}} \Big(\int_{B(\sigma)} \|Dg\|^{q''} \,\mathrm{d}x\Big)^{\frac{1}{q''}} \tag{3.25}$$

$$\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^{q} \,\mathrm{d}x + c\Big(\int_{B(\sigma)} \|Dg\|^{q''} \,\mathrm{d}x\Big)^{\frac{q'}{q''}}$$

$$\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^{q} \,\mathrm{d}x + c\Big[\sigma^{nq-2n+q} \int_{B(\sigma)} \|Dg\|^{q} \,\mathrm{d}x\Big]^{\frac{1}{q-1}}$$

$$\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^{q} \,\mathrm{d}x + c\Big[\sigma^{\frac{\mu}{(q-1)^2}+\frac{nq-2n+q}{q-1}} \|Dg\|^{q'}_{L^{q,\frac{\mu}{q-1}}(B(\sigma))}\Big].$$

From (3.21)–(3.25) we have

$$
\begin{aligned}
(3.26)\quad C &= \int_{B(\sigma)} \|F^0\|\|z\|\,\mathrm{d}x \\
&\leqslant \varepsilon \int_{B(\sigma)} \|Dz\|^q\,\mathrm{d}x + c\sigma^{\frac{\mu}{q-1}}\|f_0\|^{q'}_{L^{\frac{nq}{n(q-1)+q},\mu\frac{n}{n(q-1)+q}}(B(\sigma))} \\
&\quad + c\sigma^{\frac{nq-2n+q}{q-1}}\|w\|^{q'}_{L^{q,q}(B(\sigma))} + c\sigma^{\frac{\mu}{(q-1)^2}+\frac{nq-2n+q}{q-1}}\|g\|^{q'}_{L^{q,\frac{\mu}{q-1}}(B(\sigma))} \\
&\quad + c\sigma^{\frac{nq-2n+q}{q-1}}\|Dw\|^{q'}_{L^{q}(B(\sigma))} + c\sigma^{\frac{\mu}{(q-1)^2}+\frac{nq-2n+q}{q-1}}\|Dg\|^{q'}_{L^{q,\frac{\mu}{q-1}}(B(\sigma))}.
\end{aligned}
$$

From (3.16), (3.20) and (3.26) it follows that

$$
\begin{aligned}
(3.27)\quad c\int_{B(\sigma)} &(1+\|Dw\|+\|Dg\|+\|Dz\|)^{q-2}\|Dz\|^2\,\mathrm{d}x \leqslant A+B+C \\
&\leqslant \varepsilon\int_{B(\sigma)} V^{q-2}(Dw+Dg)\|Dz\|^2\,\mathrm{d}x + \varepsilon\int_{B(\sigma)}\|Dz\|^q\,\mathrm{d}x \\
&\quad + c\omega^2(\sigma)\int_{B(\sigma)}\|W(Dw+Dg)\|^2\,\mathrm{d}x \\
&\quad + c\sigma^{\frac{\mu}{q-1}}\sum_i \|f_i\|^{q'}_{L^{q',\frac{\mu}{q-1}}(B(\sigma))} + c\sigma^{\frac{nq-2n+q}{q-1}}\|w\|^{q'}_{L^{q,q}(B(\sigma))} \\
&\quad + c\sigma^{\frac{\mu}{(q-1)^2}+n\frac{q-2}{q-1}}\|g\|^{q'}_{L^{q,\frac{\mu}{q-1}}(B(\sigma))} \\
&\quad + c\sigma^{\frac{\mu}{q-1}}\|f_0\|^{q'}_{L^{\frac{nq}{n(q-1)+q},\mu\frac{n}{n(q-1)+q}}(B(\sigma))} \\
&\quad + c\sigma^{\frac{nq-2n+q}{q-1}}\|Dw\|^{q'}_{L^{q}(B(\sigma))} \\
&\quad + \sigma^{\frac{\mu}{(q-1)^2}+\frac{nq-2n+q}{q-1}}\|Dg\|^{q'}_{L^{q,\frac{\mu}{q-1}}(B(\sigma))},
\end{aligned}
$$

and, consequently,

$$
\begin{aligned}
(3.28)\quad \int_{B(\sigma)} &\|W(Dz)\|^2\,\mathrm{d}x \\
&\leqslant c\Big\{\omega^2(\sigma)\int_{B(\sigma)}\|W(Dw+Dg)\|^2\,\mathrm{d}x \\
&\quad + \sigma^{\frac{\mu}{q-1}}\sum_i \|f_i\|^{q'}_{L^{q',\frac{\mu}{q-1}}(B(\sigma))} + c\sigma^{\frac{nq-2n+q}{q-1}}\|w\|^{q'}_{L^{q,q}(B(\sigma))} \\
&\quad + \sigma^{\frac{\mu}{(q-1)^2}+n\frac{q-2}{q-1}}\|g\|^{q'}_{L^{q,\frac{\mu}{q-1}}(B(\sigma))} + \sigma^{\frac{\mu}{q-1}}\|f_0\|^{q'}_{L^{\frac{nq}{n(q-1)+q},\mu\frac{n}{n(q-1)+q}}(B(\sigma))} \\
&\quad + \sigma^{\frac{nq-2n+q}{q-1}}\|Dw\|^{q'}_{L^{q}(B(\sigma))} + \sigma^{\frac{\mu}{(q-1)^2}+n\frac{q-2}{q-1}+\frac{q}{q-1}}\|Dg\|^{q'}_{L^{q,\frac{\mu}{q-1}}(B(\sigma))}\Big\}.
\end{aligned}
$$

As for the vector $(v+g)$, it is a solution of the basic system (3.13), for and so the fundamental estimate (3.2) of Theorem 3.1 holds: $\forall\, t \in (0,1)$,

$$\int_{B(t\sigma)} \|W(Dv+Dg)\|^2 \,\mathrm{d}x \leqslant ct^{\lambda} \int_{B(\sigma)} \|W(Dv+Dg)\|^2 \,\mathrm{d}x;$$

hence (see Lemma 2.1)

$$\int_{B(t\sigma)} \|W(Dv)\|^2 \,\mathrm{d}x \leqslant ct^{\lambda} \int_{B(\sigma)} \|W(Dv)\|^2 \,\mathrm{d}x + c\int_{B(\sigma)} \|W(Dg)\|^2 \,\mathrm{d}x. \tag{3.29}$$

As $w = v - z$ in $B(\sigma)$, taking into account Lemma 2.1 it follows from (3.28) and (3.29) that $\forall\, t \in (0,1)$,

$$\begin{aligned}
\int_{B(t\sigma)} \|W(Dw)\|^2 \,\mathrm{d}x &\leqslant c\int_{B(t\sigma)} (\|W(Dv)\|^2 + \|W(Dz)\|^2)\,\mathrm{d}x \\
&\leqslant ct^{\lambda} \int_{B(\sigma)} \|W(Dw)\|^2 \,\mathrm{d}x + c\int_{B(\sigma)} \|W(Dg)\|^2 \,\mathrm{d}x \\
&\quad + c\int_{B(\sigma)} \|W(Dz)\|^2 \,\mathrm{d}x \\
&\leqslant c(t^{\lambda} + \omega^2(\sigma)) \int_{B(\sigma)} \|W(Dw)\|^2 \,\mathrm{d}x \\
&\quad + c\Bigg(\sigma^{\frac{\mu}{q-1}} \|W(Dg)\|^2_{L^{2,\frac{\mu}{q-1}}(\Omega)} \\
&\quad + \sigma^{\frac{\mu}{q-1}} \sum_i \|f_i\|^{q'}_{L^{q',\frac{\mu}{q-1}}(\Omega)} + \sigma^{\frac{nq-2n+q}{q-1}} \|w\|^{q'}_{L^{q,q}(\Omega)} \\
&\quad + \sigma^{\frac{\mu}{(q-1)^2} + n\frac{q-2}{q-1}} \Big(\|g\|^{q'}_{L^{q,\frac{\mu}{q-1}}(\Omega)} + \|Dg\|^{q'}_{L^{q,\frac{\mu}{q-1}}(\Omega)} \Big) \\
&\quad + \sigma^{\frac{\mu}{q-1}} \|f_0\|^{q'}_{L^{\frac{nq}{n(q-1)+q},\mu\frac{n}{n(q-1)+q}}(\Omega)} \\
&\quad + \sigma^{\frac{nq-2n+q}{q-1}} \|Dw\|^{q'}_{L^q(\Omega)} \Bigg) \\
&\leqslant c(t^{\lambda} + \omega^2(\sigma)) \int_{B(\sigma)} \|W(Dw)\|^2 \,\mathrm{d}x \\
&\quad + c\sigma^{\frac{\mu}{q-1} \wedge \frac{nq-2n+q}{q-1}} \mathcal{M}
\end{aligned} \tag{3.30}$$

where

$$\begin{aligned}
\mathcal{M} = \sum_i \|f_i\|^{q'}_{L^{q',\frac{\mu}{q-1}}(\Omega)} &+ \|f_0\|^{q'}_{L^{\frac{nq}{n(q-1)+q},\mu\frac{n}{n(q-1)+q}}(\Omega)} \\
&+ \|g\|^{q'}_{H^{1,q,\frac{\mu}{q-1}}(\Omega)} + \|Dw\|^{q'}_{L^q(\Omega)} + \|W(Dg)\|^2_{L^{2,\frac{\mu}{q-1}}(\Omega)}.
\end{aligned} \tag{3.31}$$

From Lemma 2.4 and (3.30) it follows that $\forall\,\varepsilon<\lambda-\left(\frac{\mu}{q-1}\wedge\frac{nq-2n+q}{q-1}\right)\exists\,\sigma_\varepsilon\leqslant d$ such that $\forall\,\sigma<\sigma_\varepsilon$ and $\forall\,t\in(0,1)$,

$$\int_{B(t\sigma)}\|W(Dw)\|^2\,\mathrm{d}x\leqslant(1+c)t^{\frac{\mu}{q-1}\wedge\frac{nq-2n+q}{q-1}}\int_{B(\sigma)}\|W(Dw)\|^2\,\mathrm{d}x + c\mathcal{M}(t\sigma)^{\frac{\mu}{q-1}\wedge\frac{nq-2n+q}{q-1}} \tag{3.32}$$

where the constant c does not depend on $\mathcal{M}$. This implies that, $\forall\,\sigma<\sigma_\varepsilon$,

$$\int_{B(\sigma)}\|W(Dw)\|^2\,\mathrm{d}x \leqslant c\sigma^{\frac{\mu}{q-1}\wedge\frac{nq-2n+q}{q-1}}\left\{\sigma_\varepsilon^{-(\frac{\mu}{q-1}\wedge\frac{nq-2n+q}{q-1})}\int_\Omega\|W(Dw)\|^2\,\mathrm{d}x+\mathcal{M}\right\}, \tag{3.33}$$

and, consequently,

$$\|W(Dw)\|^2_{L^{2,\frac{\mu}{q-1}\wedge\frac{nq-2n+q}{q-1}}(\Omega^*)}\leqslant c\left\{\int_\Omega\|W(Dw)\|^2\,\mathrm{d}x+\mathcal{M}\right\}. \tag{3.34}$$

Thus if $\mu\leqslant nq-2n+q$, the theorem is proved.

If $\mu>nq-2n+q$ we fix $\Omega^*\subset\subset\Omega^{**}\subset\subset\Omega$. From (3.30), taking into account [2, Lemma 3.III, p. 23], and Lemma 2.1, we have

$$\begin{aligned} W(Dw)&\in L^{2,\frac{nq-2n+q}{q-1}}(\Omega^{**}),\\ Dw&\in L^{q,\frac{nq-2n+q}{q-1}}(\Omega^{**}),\\ w&\in L^{q,\frac{q^2+nq-2n}{q-1}}(\Omega^{**}). \end{aligned} \tag{3.35}$$

From these data, repeating the same arguments as above, we see that $\forall\,t\in(0,1)$ and $\forall\,\sigma\in(0,d^*)$ (where $d^*=\operatorname{dist}(\overline{\Omega^*},\partial\Omega^{**})$)

$$\begin{aligned}\int_{B(t\sigma)}\|W(Dw)\|^2\,\mathrm{d}x&\leqslant c(t^\lambda+\omega^2(\sigma))\int_{B(\sigma)}\|W(Dw)\|^2\,\mathrm{d}x\\ &\quad+c\Bigg(\sigma^{\frac{\mu}{q-1}}\|W(Dg)\|^2_{L^{2,\frac{\mu}{q-1}}(\Omega)}+\sigma^{\frac{\mu}{q-1}}\sum_i\|f_i\|^{q'}_{L^{q',\frac{\mu}{q-1}}(\Omega)}\\ &\quad+\sigma^{\frac{q^2(n+1)-2nq}{(q-1)^2}}\|w\|^{q'}_{L^{q,\frac{q^2+nq-2n}{q-1}}(\Omega^{**})}+\sigma^{\frac{q^2(n+2)-2nq-q}{(q-1)^2}}\|w\|^{q'}_{L^{q,\frac{q^2+nq-2n}{q-1}}(\Omega^{**})}\\ &\quad+\sigma^{\frac{\mu}{(q-1)^2}+n\frac{q-2}{q-1}}\|g\|^{q'}_{H^{1,q,(\frac{\mu}{q-1})}(\Omega)}+\sigma^{\frac{\mu}{q-1}}\|f_0\|^{q'}_{L^{\frac{nq}{n(q-1)+q},\mu\frac{n}{n(q-1)+q}}(\Omega)}\\ &\quad+\sigma^{\frac{q^2(n+1)-2nq}{(q-1)^2}}\|Dw\|^{q'}_{L^{q,\frac{nq-2n+q}{q-1}}(\Omega^{**})}\Bigg)\\ &\leqslant c(t^\lambda+\omega^2(\sigma))\int_{B(\sigma)}\|W(Dw)\|^2\,\mathrm{d}x+c\sigma^{\frac{\mu}{q-1}\wedge\frac{q^2(n+1)-2nq}{(q-1)^2}}\mathcal{M},\end{aligned}$$

where

$$\mathcal{M} = \sum_i \|f_i\|^{q'}_{L^{q',\frac{\mu}{q-1}}(\Omega)} + \|Dw\|^{q'}_{L^{q,\frac{nq-2n+q}{q-1}}(\Omega^{**})} + \|f_0\|^{q'}_{L^{\frac{nq}{n(q-1)+q},\mu\frac{n}{n(q-1)+q}}(\Omega)} + \|g\|^{q'}_{H^{1,q,\frac{\mu}{q-1}}(\Omega)} + \|W(Dg)\|^2_{L^{2,\frac{\mu}{q-1}}(\Omega)}.$$

Now, by virtue of (3.3), since $\frac{q^2(n+1)-2nq}{(q-1)^2} > n \geqslant \lambda > \frac{\mu}{q-1}$, Theorem 3.3 is completely proved. □

Theorem 3.4. *If $w \in H_0^{1,q}(\Omega)$ is a solution of the Dirichlet problem*

$$\sum_i D_i a^i(x, Dw + Dg) = \sum_i D_i F^i(x, w+g) - F^0(x, w+g, Dw+Dg) \ \text{in}\ \Omega, \tag{3.36}$$

under the assumptions (1.3), (1.6) *and* (1.7), *if the vectors F^0, F^i, $i = 1, \dots, n$ satisfy conditions* (1.9)–(1.12), *and if $g \in H^{1,q,(\frac{\mu}{q-1})}(\Omega)$, $0 < \frac{\mu}{q-1} < \lambda$, then we have the following estimate*

$$\int_\Omega \|W(Dw)\|^2 \,\mathrm{d}x \leqslant c\mathcal{M} \tag{3.37}$$

where $\mathcal{M}$ is given by (3.10) *if $\mu \leqslant nq - 2n + q$, and by* (3.11) *if $\mu > nq - 2n + q$.*

P r o o f. (3.36) means that $\forall \varphi \in H_0^{1,q}(\Omega)$,

$$\int_\Omega \sum_i (a^i(x, Dw+Dg) - a^i(x, Dg)/D_i\varphi) \,\mathrm{d}x = \int_\Omega \sum_i (F^i - a^i(x, Dg)/D_i\varphi) \,\mathrm{d}x - \int_\Omega F^0/\varphi \,\mathrm{d}x. \tag{3.38}$$

Take $\varphi = w$ in (3.38). Setting

$$\mathcal{B}_{ij} = \int_0^1 A_{ij}(x, Dg + tDw) \,\mathrm{d}t,$$

repeating the same arguments as in Theorem 3.3 in order to obtain (3.15), and taking into account the ellipticity condition (1.7) and Lemma 2.2, we obtain

$$c\int_\Omega (1 + \|Dg\| + \|Dw\|)^{q-2} \|Dw\|^2 \,\mathrm{d}x \leqslant \int_\Omega \sum_{ij} (B_{ij} D_j w / D_i w) \,\mathrm{d}x. \tag{3.39}$$

By the condition (1.8) we have

$$\int_\Omega \sum_i \|a^i(x,Dg)\|\,\|Dw\|\,\mathrm{d}x \leqslant \varepsilon \int_\Omega V^{q-2}(Dg)\|Dw\|^2\,\mathrm{d}x + c\int_\Omega \|W(Dg)\|^2\,\mathrm{d}x. \tag{3.40}$$

Now, by virtue of the conditions (1.9)–(1.12), from (3.17)–(3.25), where z and Dz are replaced with w and Dw, and from (3.40), we obtain

$$\int_\Omega \|W(Dw)\|^2\,\mathrm{d}x \leqslant c\mathcal{M}$$

where, if $\mu \leqslant nq-2n+q$, $\mathcal{M}$ is given by (3.10), whereas if $\mu > nq-2n+q$, taking into account (3.35), $CalM$ is given by (3.11). □

4. Regularity at the boundary

Let $a^i(x,p)$ be vectors of $\mathbb{N}$, defined in $\Lambda^+ = B^+(1)\times\mathbb{R}^{nN}$, of class C^1 in p and $F^0(x,u,p)$, $F^i(x,u)$, $i=1,2,\dots,n$, be vectors of $\mathbb{R}^N$ defined, respectively, in $B^+(1)\times\mathbb{R}^N\times\mathbb{R}^{nN}$ and $B^+(1)\times\mathbb{R}^N$, measurable in x, continuous in u and p.

Theorem 4.1. *If $u\in H^{1,q}(B^+(1))$, $q\geqslant 2$, is a solution of the problem*

$$\begin{cases} \sum_i D_i a^i(Du) = 0 & \text{in } B^+(1),\\ u=0 & \text{on } \Gamma \end{cases} \tag{4.1}$$

under the hypotheses (1.3), (1.6) *and* (1.7), *where Ω is replaced by $B^+(1)$, then, $\forall\,\sigma\leqslant 1$ and $\forall\,t\in(0,1)$,*

$$\int_{B^+(t\sigma)} \|W(Du)\|^2\,\mathrm{d}x \leqslant ct^\lambda \int_{B^+(\sigma)} \|W(Du)\|^2\,\mathrm{d}x \tag{4.2}$$

where the constant c does not depend on t, σ and $\lambda=\min\{2+\varepsilon,n\}$ (with $\varepsilon\neq n-2$).

(See [3, Theorem 6.II, p. 141]).

The fundamental estimate (4.2) enables us to obtain the following boundary local regularity theorem:

Theorem 4.2. *If $w \in H^{1,q}(B^+(1))$ is a solution of the problem*

$$(4.3)\qquad \begin{cases} w = 0 \quad \text{on } \Gamma, \\ \sum\limits_i D_i a^i(x, Dw + Dg) = \sum\limits_i D_i F^i - F^0 \qquad \text{in } B^+(1). \end{cases}$$

under the assumptions (1.2), (1.3), (1.6) *and* (1.7),[3] *if the vectors* F^0, F^i, $i = 1, 2, \ldots, n$, *satisfy the conditions* (1.9), (1.10), (1.11) *and* (1.12), *and if*

$$(4.4)\qquad g \in H^{1,q,(\frac{\mu}{q-1})}(B^+(1)), \quad 0 < \frac{\mu}{q-1} < \lambda,$$

then, for every $R < 1$, we have

$$(4.5)\qquad Dw \in L^{q,\frac{\mu}{q-1}}(B^+(R))$$

and the estimate

$$(4.6)\qquad \|W(Dw)\|^2_{L^{2,\frac{\mu}{q-1}}(B^+(R))} \leqslant c\Big\{\int_{B^+(1)} \|W(Dw)\|^2 \,\mathrm{d}x + \mathcal{M}'\Big\}$$

holds, where, if $\mu \leqslant nq - 2n + q$, the constant c depends also on $d = 1 - R$ and

$$(4.7)\qquad \begin{aligned} \mathcal{M}' = \|W(Dg)\|^2_{L^{2,\frac{\mu}{q-1}}(B^+(1))} &+ \sum_i \|f_i\|^{q'}_{L^{q,\frac{\mu}{q-1}}(B^+(1))} \\ + \|f_0\|^{q'}_{L^{\frac{nq}{n(q-1)+q},\mu\frac{n}{n(q-1)+q}}(B^+(1))} &+ \|Dw\|^{q'}_{L^q(B^+(1))} \\ + \|g\|^{q'}_{H^{1,q,(\frac{\mu}{q-1})}(B^+(1))}, \end{aligned}$$

whereas, if $\mu > nq - 2n + q$, denoting by R^* a number such that $0 < R < R^* < 1$, the constant c depends also on $d^* = R^* - R$ and

$$(4.8)\qquad \begin{aligned} \mathcal{M}' = \|W(Dg)\|^2_{L^{2,\frac{\mu}{q-1}}(B^+(1))} &+ \sum_i \|f_i\|^{q'}_{L^{q,\frac{\mu}{q-1}}(B^+(1))} \\ + \|f_0\|^{q'}_{L^{\frac{nq}{n(q-1)+q},\mu\frac{n}{n(q-1)+q}}(B^+(1))} &+ \|Dw\|^{q'}_{L^{q,\frac{nq-2n+q}{q-1}}(B^+(R^*))} \\ + \|g\|^{q'}_{H^{1,q,(\frac{\mu}{q-1})}(B^+(1))}. \end{aligned}$$

P r o o f. In this proof, as we will see, we argue as in Theorem 3.3, but with suitable modifications.

[3] In the assumptions (1.2), (1.3), (1.6), (1.7), (1.9), (1.10), (1.11) and (1.12), Ω is replaced by $B^+(1)$.

Fix R, $0 < R < 1$. In any hemisfere $B^+(x^0, \sigma)$, with $\sigma < 1 - R$ and $x^0 \in \Gamma(R)$, we write $w = v - z$, where z is a solution of the Dirichlet problem

$$(4.9) \qquad \begin{cases} z \in H_0^{1,q}(B^+(x^0, \sigma)), \\ \sum_i D_i a^i(x^0, Dz + Dw + Dg) \\ \quad = \sum_i D_i[a^i(x, Dw + Dg) - F^i] + F^0 \quad \text{in } B^+(x^0, \sigma), \end{cases}$$

whereas $v \in H^{1,q}(B^+(x^0, \sigma))$ is a solution of the problem

$$(4.10) \qquad \begin{cases} v = 0 \quad \text{on } \Gamma(x^0, \sigma), \\ \sum_i D_i a^i(x^0, Dv + Dg) = 0 \quad \text{in } B^+(x^0, \sigma). \end{cases}$$

Arguing as in the proof of Theorem 3.3 in order to obtain (3.28), we get the following estimate for the vector z

$$\begin{aligned} (4.11) \quad & \int_{B^+(x^0,\sigma)} \|W(Dz)\|^2 \, \mathrm{d}x \\ & \leqslant c \Big\{ \omega^2(\sigma) \int_{B^+(x^0,\sigma)} \|W(Dw + Dg)\|^2 \, \mathrm{d}x \\ & + \sigma^{\frac{\mu}{q-1}} \sum_i \|f_i\|^{q'}_{L^{\frac{q}{q-1}, \frac{\mu}{q-1}}(B^+(x^0,\sigma))} + \sigma^{\frac{nq-2n+q}{q-1}} \|w\|^{q'}_{L^{q,q}(B^+(x^0,\sigma))} \\ & + \sigma^{\frac{nq-2n+2q}{q-1}} \|w\|^{q'}_{L^{q,q}(B^+(x^0,\sigma))} + \sigma^{\frac{nq-2n+q}{q-1}} \|Dw\|^{q'}_{L^{q}(B^+(x^0,\sigma))} \\ & + \sigma^{\frac{\mu}{(q-1)^2} + \frac{n(q-2)}{q-1}} \|g\|^{q'}_{L^{q,\frac{\mu}{q-1}}(B^+(x^0,\sigma))} \\ & + \sigma^{\frac{\mu}{(q-1)^2} + \frac{n(q-2)}{q-1}} \|Dg\|^{q'}_{L^{q,\frac{\mu}{q-1}}(B^+(x^0,\sigma))} \\ & + \sigma^{\frac{\mu}{q-1}} \|f_0\|^{q'}_{L^{\frac{nq}{n(q-1)+q}, \mu\frac{n}{n(q-1)+q}}(B^+(x^0,\sigma))} \Big\}. \end{aligned}$$

For the vector v, we have the following estimate (see [3, Appendix 2, p. 148]): $\forall\, t \in (0, 1)$ and $\forall\, \varepsilon > 0$,

$$\begin{aligned} (4.12) \quad & \int_{B^+(x^0,t\sigma)} \|W(Dv)\|^2 \, \mathrm{d}x \\ & \leqslant c t^{\lambda-\varepsilon} \int_{B^+(x^0,\sigma)} \|W(Dv)\|^2 \, \mathrm{d}x + c(\varepsilon) \sigma^{\frac{\mu}{q-1}} \|W(Dg)\|^2_{L^{2,\frac{\mu}{q-1}}(B^+(1))}. \end{aligned}$$

As $w = v - z$ in $B^+(x^0,\sigma)$, by virtue of Lemma 2.1, from (4.11) and (4.12) it follows that: $\forall\, t \in (0,1)$ and $\forall\, \varepsilon > 0$,

$$\begin{aligned}(4.13)\quad \int_{B^+(x^0,t\sigma)} \|W(Dw)\|^2\,\mathrm{d}x &\leqslant c\int_{B^+(x^0,t\sigma)} \|W(Dv)\|^2\,\mathrm{d}x + c\int_{B^+(x^0,t\sigma)} \|W(Dz)\|^2\,\mathrm{d}x \\ &\leqslant c(t^{\lambda-\varepsilon} + \omega^2(\sigma))\int_{B^+(x^0,\sigma)} \|W(Dw)\|^2\,\mathrm{d}x + \sigma^{\frac{\mu}{q-1}\wedge\frac{nq-2n+q}{q-1}}\mathcal{M}',\end{aligned}$$

where $\omega^2(\sigma)$ tends to zero with σ and

$$\begin{aligned}\mathcal{M}' = \sum_i \|f_i\|^{q'}_{L^{q',\frac{\mu}{q-1}}(B^+(1))} + \|f_0\|^{q'}_{L^{\frac{nq}{n(q-1)+q},\mu\frac{q}{n(q-1)+q}}(B^+(1))} \\ + \|g\|^{q'}_{H^{1,q,\frac{\mu}{q-1}}(B^+(1))} + \|Dw\|^{q'}_{L^q(B^+(1))} + \|W(Dg)\|^2_{L^{2,\frac{\mu}{q-1}}(B^+(1))}.\end{aligned}$$

After this inequality, with the same procedure as in Theorem 3.3, we see that $\forall\,\varrho < (\lambda-\varepsilon) - (\frac{\mu}{q-1}\wedge\frac{nq-2n+q}{q-1})\ \exists\,\sigma_\varrho \leqslant 1 - R$ such that $\forall\,\sigma < \sigma_\varrho$ and $\forall\, t \in (0,1)$

$$\begin{aligned}\int_{B^+(x^0,t\sigma)} \|W(Dw)\|^2\,\mathrm{d}x \leqslant (1+c)t^{\lambda-\varepsilon-\varrho}\int_{B^+(1)} \|W(Dw)\|^2\,\mathrm{d}x \\ + c\mathcal{M}'(t\sigma)^{\frac{\mu}{q-1}\wedge\frac{nq-2n+q}{q-1}},\end{aligned}$$

and, consequently, $\forall\,\sigma < \sigma_\varrho$

$$\begin{aligned}(4.14)\quad \int_{B^+(x^0,\sigma)} \|W(Dw)\|^2\,\mathrm{d}x \\ \leqslant c\sigma^{\frac{\mu}{q-1}\wedge\frac{nq-2n+q}{q-1}}\Big\{\sigma_\varrho^{-(\frac{\mu}{q-1}\wedge\frac{nq-2n+q}{q-1})}\int_{B^+(1)} \|W(Dw)\|^2\,\mathrm{d}x + \mathcal{M}'\Big\}.\end{aligned}$$

Now we consider the case $x^0 \in B^+(R)$ with $x^0_n > 0$.

Fixing σ, $0 < \sigma < \frac{1}{2}\sigma_\varrho$, we distinguish two cases: if $x^0_n \leqslant \sigma$, then

$$B(x^0,\sigma) \cap B^+(R) \subset B^+(\overline{x^0}, 2\sigma)$$

where $\overline{x^0} = (x^0_1,\ldots,x^0_{n-1},0)$ and so, by virtue of (4.1), we have the estimate

$$\begin{aligned}(4.15)\quad \int_{B(x^0,\sigma)\cap B^+(R)} \|W(Dw)\|^2\,\mathrm{d}x \leqslant c\sigma^{\frac{\mu}{q-1}\wedge\frac{nq-2n+q}{q-1}} \\ \times\Big\{\mathcal{M}' + \sigma_\varrho^{-(\frac{\mu}{q-1}\wedge\frac{nq-2n+q}{q-1})}\int_{B^+(1)} \|W(Dw)\|^2\,\mathrm{d}x\Big\}.\end{aligned}$$

Otherwise, if $x_n^0 > \sigma$, then $B(x^0, \sigma)$ is an interior ball of $B^+(1)$; thus, using the interior regularity result, we have

$$\int_{B(x^0,\sigma)} \|W(Dw)\|^2 \, dx \leqslant c\sigma^{\frac{\mu}{q-1} \wedge \frac{nq-2n+q}{q-1}}$$
$$\times \Big\{ \mathcal{M}' + \sigma_\varrho^{-(\frac{\mu}{q-1} \wedge \frac{nq-2n+q}{q-1})} \int_{B^+(1)} \|W(Dw)\|^2 \, dx \Big\}.$$

In any case, if $\sigma < \sigma_\varrho$ and $x^0 \in B^+(R)$, we can assert that

$$Dw \in L^{q, \frac{\mu}{q-1} \wedge \frac{nq-2n+q}{q-1}}(B^+(R)).$$

Now, if $\mu \leqslant nq - 2n + q$ the theorem is proved, whereas, if $\mu > nq - 2n + q$, denoting by R^* a number such that $0 < R < R^* < 1$ and arguing as in Theorem 3.3, we have

$$\|W(Dw)\|^2_{L^{2,\frac{\mu}{q-1}}(B^+(x^0,\sigma))} \leqslant c\Big\{ \int_{B^+(1)} \|W(Dw)\|^2 \, dx + \mathcal{M}' \Big\}$$

with

$$\begin{aligned} \mathcal{M}' = {} & \|W(Dg)\|^2_{L^{2,\frac{\mu}{q-1}}(B^+(1))} + \sum_i \|f_i\|^{q'}_{L^{q,\frac{\mu}{q-1}}(B^+(1))} \\ & + \|f_0\|^{q'}_{L^{\frac{nq}{n(q-1)+q}, \mu\frac{n}{n(q-1)+q}}(B^+(1))} + \|Dw\|^{q'}_{L^{q,\frac{nq-2n+q}{q-1}}(B^+(R^*))} \\ & + \|g\|^{q'}_{H^{1,q,(\frac{\mu}{q-1})}(B^+(1))}. \end{aligned}$$

Now Theorem 4.2 is completely proved. □

5. A global regularity result

Let $u \in H^{1,q}(\Omega, \mathbb{R}^N)$ be a solution of the Dirichlet problem

$$\text{(5.1)} \qquad \begin{cases} u - g \in H_0^{1,q}(\Omega, \mathbb{R}^N), \\ \sum\limits_i D_i a^i(x, Du) = \sum\limits_i D_i F^i(x, u) - F^0(x, u, Du) \quad \text{in } \Omega, \end{cases}$$

where $\partial\Omega$ is of class C^2, $g \in H^{1,q,(\frac{\mu}{q-1})}(\Omega)$ and F^0, F^i, $i = 1, \ldots, n$ satisfy the assumptions (1.9), (1.10), (1.11) and (1.12). The vectors $a^i(x, p)$ satisfy conditions (1.2), (1.3), (1.6) and (1.7).

Assuming $w = u - g$, the problem (5.1) may be written in the equivalent form

$$\text{(5.2)} \qquad \begin{cases} w \in H_0^{1,q}(\Omega), \\ \sum\limits_i D_i a^i(x, Dw + Dg) = \sum\limits_i D_i F^i - F^0 \quad \text{in } \Omega. \end{cases}$$

We remark that, since $\partial\Omega$ is of class C^2, if $x^0 \in \partial\Omega$, there is an open neighborhood $\mathcal{B}$ of x^0 and a mapping τ of class C^2, together with its inverse, such that $\overline{B(0,1)} = \tau(\overline{\mathcal{B}})$, $B^+(1) = \tau(\Omega \cap \mathcal{B})$ and $\Gamma = \tau(\partial\Omega \cap \mathcal{B})$.

A solution w to the problem (5.2) satisfies, in particular, the system

$$\text{(5.3)} \quad \sum_i \int_{\Omega\cap\mathcal{B}} (a^i(x, Dw + Dg) D_i\varphi)\,\mathrm{d}x = \sum_i \int_{\Omega\cap\mathcal{B}} (F^i(x, w+g)/D_i\varphi)\,\mathrm{d}x + \int_{\Omega\cap\mathcal{B}} F^0(x, w+g, Dw+Dg)\varphi\,\mathrm{d}x \quad \forall\,\varphi \in H_0^{1,q}(\Omega\cap\mathcal{B}, \mathbb{R}^N).$$

We set

$$\frac{\partial\tau(x)}{\partial x} = \Big\{\frac{\partial\tau_i(x)}{\partial x_j}\Big\}, \quad J(x) = \Big|\det\frac{\partial\tau(x)}{\partial x}\Big|,$$

and, for all $y \in B(0,1)$ and $p \in \mathbb{R}$, we define

$$\text{(5.4)} \quad \begin{aligned} \alpha_{ij}(y) &= \Big(\frac{\partial\tau_i}{\partial x_j}\Big)(\tau^{-1}(y)); \quad \beta_{hi}(y) = \Big(\frac{\partial\tau_h}{\partial x_i}\frac{1}{J}\Big)(\tau^{-1}(y)),\\ q^j(y,p) &= \sum_{i=1}^n \alpha_{ij}(y)p^i, \qquad q = (q^1, q^2, \dots, q^n),\\ \mathcal{A}^h(y,p) &= \sum_{i=1}^n a^i(\tau^{-1}(y), q(y,p))\beta_{hi}(y),\\ \mathcal{F}^h(y,u) &= \sum_{i=1}^n F^i(\tau^{-1}(y), u)\beta_{hi}(y),\\ \mathcal{F}^0(y,u,p) &= F^0(\tau^{-1}(y), u, q(y,p))\frac{1}{J(\tau^{-1}(y))}. \end{aligned}$$

Clearly q^i and $\mathcal{A}^h$ are vectors of $\mathbb{R}^N$ defined in $B(0,1)\times\mathbb{R}^{nN}$; $\mathcal{F}^h$ are vectors of $\mathbb{R}^N$ defined in $B(0,1)\times\mathbb{R}^N$ and $\mathcal{F}^0$ is a vector of $\mathbb{R}^N$ defined in $B(0,1)\times\mathbb{R}^N\times\mathbb{R}^{nN}$; moreover α_{ij} and β_{hi} are functions of class $C^1(\overline{B(0,1)})$.

It is not difficult to deduce from (5.4) that also the vectors $\mathcal{A}^h(y,p)$, $\mathcal{F}^h(y,u)$, $\mathcal{F}^0(y,u,p)$ satisfy the same conditions as $a^i(x,p)$, $F^i(x,u)$, $F^0(x,u,p)$ in which the constants and the coefficients are multiplied by a suitable positive constant $c(\tau)$ and F^i, F^0 are replaced by $\mathcal{F}^h$, $\mathcal{F}^0$. Then, setting

$$\text{(5.5)} \quad \begin{aligned} \tilde u(y) &= u(\tau^{-1}(y)) &&\text{and so}\quad u(x) = \tilde u(\tau(x)),\\ \tilde w(y) &= w(\tau^{-1}(y)) &&\text{and so}\quad w(x) = \tilde w(\tau(x)),\\ \tilde g(y) &= g(\tau^{-1}(y)) &&\text{and so}\quad g(x) = \tilde g(\tau(x)),\\ \tilde\varphi(y) &= \varphi(\tau^{-1}(y)) &&\text{and so}\quad \varphi(x) = \tilde\varphi(\tau(x)), \end{aligned}$$

hence

$$D_i w(x) = \sum_{h=1}^{n} D_h \tilde{w}(\tau(x)) \cdot D_i \tau_h(x), \tag{5.6}$$

$$D_i g(x) = \sum_{h=1}^{n} D_h \tilde{g}(\tau(x)) \cdot D_i \tau_h(x),$$

$$D_i \varphi(x) = \sum_{h=1}^{n} D_h \tilde{\varphi}(\tau(x)) \cdot D_i \tau_h(x),$$

from (5.3), and taking into account (5.5) and (5.6) we get

$$\begin{aligned}
&\sum_{i=1}^{n} \int_{B^+(1)} \Big[a^i(\tau^{-1}(y), \sum_{j=1}^{n} \alpha_{j_1}(y)(D_j \tilde{w}(y) + D_j \tilde{g}(y)), \dots, \\
&\qquad \sum_{j=1}^{n} \alpha_{j_n}(y)(D_j \tilde{w}(y) + D_j \tilde{g}(y)) \Big/ \sum_{h=1}^{n} \beta_{hi}(y) D_h \tilde{\varphi}(y) \Big] \,\mathrm{d}y \\
&= \sum_{i=1}^{n} \int_{B^+(1)} \Big[F^i(\tau^{-1}(y), \tilde{w}(y) + \tilde{g}(y)) \Big/ \sum_{h=1}^{n} \beta_{hi}(y) D_h \tilde{\varphi}(y) \Big] \,\mathrm{d}y \\
&\quad + \int_{B^+(1)} \Big[F^0(\tau^{-1}(y), \tilde{w}(y) + \tilde{g}(y), \sum_{j=1}^{n} \alpha_{j_1}(y)(D_j \tilde{w}(y) + D_j \tilde{g}(y)), \dots, \\
&\qquad \sum_{j=1}^{n} \alpha_{j_n}(y)(D_j \tilde{w}(y) + D_j \tilde{g}(y)))/\tilde{\varphi}(y) \Big] \frac{1}{J(\tau^{-1}(y))} \,\mathrm{d}y.
\end{aligned}$$

Hence $\tilde{w}(y)$ is a solution of the problem

$$\begin{cases} \tilde{w}(y) \in H^{1,q}(B^+(1)), \\ \tilde{w} = 0 \quad \text{on } \Gamma \\ \sum_h D_h \mathcal{A}^h(y, D\tilde{w} + D\tilde{g}) = \sum_h D_h \mathcal{F}^h - \mathcal{F}^0 \quad \text{in } B^+(1). \end{cases} \tag{5.7}$$

It follows by Lemma 2.3 that, $\forall R^* < 1$, $\tilde{g} \in H^{1,q,(\frac{\mu}{q-1})}(B^+(R^*))$, $W(D\tilde{g}) \in L^{2,\frac{\mu}{q-1}}(B^+(R^*))$, and

$$\begin{aligned}
\|\tilde{g}\|_{H^{1,q,(\frac{\mu}{q-1})}(B^+(R^*))} &\leqslant c(\tau,\mu) \|g\|_{H^{1,q,(\frac{\mu}{q-1})}(\Omega \cap \mathcal{B})}, \\
\|W(D\tilde{g})\|_{L^{2,\frac{\mu}{q-1}}(B^+(R^*))} &\leqslant c(\tau,\mu) \|W(Dg)\|_{L^{2,\frac{\mu}{q-1}}(\Omega \cap \mathcal{B})}.
\end{aligned}$$

Then we may apply Theorem 4.2, obtaining $\forall R$, $0 < R < R^* < 1$, $D\tilde{w} \in L^{q,\frac{\mu}{q-1}}(B^+(R))$, and, since $\tilde{w} = \tilde{u} - \tilde{g}$,

$$\|W(D\tilde{u})\|^2_{L^{2,\frac{\mu}{q-1}}(B^+(R))} \leqslant c \left\{ \int_{B^+(1)} \|W(D\tilde{u})\|^2 \,\mathrm{d}x + \mathcal{M}' \right\} \tag{5.8}$$

where, if $\mu \leqslant nq - 2n + q$,

$$\mathcal{M}' = \|W(D\tilde{g})\|^2_{L^{2,\frac{\mu}{q-1}}(B^+(R^*))} + \sum_i \|f_i\|^{q'}_{L^{q,\frac{\mu}{q-1}}(B^+(R^*))} + \|f_0\|^{q'}_{L^{\frac{nq}{n(q-1)+q},\mu\frac{n}{n(q-1)+q}}(B^+(R^*))} + \|D\tilde{u}\|^{q'}_{L^q(B^+(R^*))} + \|\tilde{g}\|^{q'}_{H^{1,q,(\frac{\mu}{q-1})}(B^+(R^*))},$$

whereas, if $\mu > nq - 2n + q$, denoting by R^{**} a number $0 < R < R^{**} < R^* < 1$,

$$\mathcal{M}' = \|W(D\tilde{g})\|^2_{L^{2,\frac{\mu}{q-1}}(B^+(R^*))} + \sum_i \|f_i\|^{q'}_{L^{q,\frac{\mu}{q-1}}(B^+(R^*))} + \|f_0\|^{q'}_{L^{\frac{nq}{n(q-1)+q},\mu\frac{n}{n(q-1)+q}}(B^+(R^*))} + \|D\tilde{u}\|^{q'}_{L^{q,\frac{nq-2n+q}{q-1}}(B^+(R^{**}))} + \|\tilde{g}\|^{q'}_{H^{1,q,(\frac{\mu}{q-1})}(B^+(R^*))}.$$

Denote by $\mathcal{B}(R)$ the inverse image of $B(0,R)$. It follows by Lemma 2.3 and from (5.8) that

$$\|W(Du)\|^2_{L^{2,\frac{\mu}{q-1}}(\Omega\cap\mathcal{B}(R))} \leqslant c\left\{\int_\Omega \|W(Du)\|^2\,\mathrm{d}x + \overline{\mathcal{M}}\right\}, \tag{5.9}$$

where, if $\mu \leqslant nq - 2n + q$,

$$\overline{\mathcal{M}} = \sum_i \|f_i\|^{q'}_{L^{q,\frac{\mu}{q-1}}(\Omega)} + \|Du\|^{q'}_{L^q(\Omega)} + \|g\|^{q'}_{H^{1,q,(\frac{\mu}{q-1})}(\Omega)} + \|W(Dg)\|^2_{L^{2,\frac{\mu}{q-1}}(\Omega)} + \|f_0\|^{q'}_{L^{\frac{nq}{n(q-1)+q},\mu\frac{n}{n(q-1)+q}}(\Omega)}, \tag{5.10}$$

whereas, if $\mu > nq - 2n + q$, denoting by R^{**} a number $0 < R < R^{**} < R^* < 1$,

$$\overline{\mathcal{M}} = \sum_i \|f_i\|^{q'}_{L^{q,\frac{\mu}{q-1}}(\Omega)} + \|Du\|^{q'}_{L^{q,\frac{nq-2n+q}{q-1}}(\Omega\cap\mathcal{B}(R^{**}))} + \|g\|^{q'}_{H^{1,q,(\frac{\mu}{q-1})}(\Omega)} + \|W(Dg)\|^2_{L^{2,\frac{\mu}{q-1}}(\Omega)} + \|f_0\|^{q'}_{L^{\frac{nq}{n(q-1)+q},\mu\frac{n}{n(q-1)+q}}(\Omega)}. \tag{5.11}$$

Since $\partial\Omega$ is compact, only a finite number of neighborhoods $\mathcal{B}$, said $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_m$, is needed to cover $\partial\Omega$. For each $\mathcal{B}_j$, we can suppose that R is close enough to 1 such that $\mathcal{B}_1(R), \dots, \mathcal{B}_m(R)$ still cover $\partial\Omega$.

Then there exists an open set $\Omega^* \subset\subset \Omega$ such that Ω^*, $\mathcal{B}_1(R), \dots, \mathcal{B}_m(R)$ cover $\overline{\Omega}$. Theorem 3.3 and the estimate (3.9), with $w = u - g$, may be applied to the open set Ω^*. Then we obtain

$$\|W(Du)\|^2_{L^{2,\frac{\mu}{q-1}}(\Omega^*)} \leqslant c\left\{\int_\Omega \|W(Du)\|^2\,\mathrm{d}x + \overline{\overline{\mathcal{M}}}\right\}, \tag{5.12}$$

where, if $\mu \leqslant nq - 2n + q$, $\overline{\overline{\mathcal{M}}}$ is given by the right-hand side of (5.10), whereas, if $\mu > nq - 2n + q$, denoting by Ω^{**} an open set such that $\Omega^* \subset\subset \Omega^{**} \subset\subset \Omega$,

$$\overline{\overline{\mathcal{M}}} = \sum_i \|f_i\|^{q'}_{L^{q,\frac{\mu}{q-1}}(\Omega)} + \|Du\|^{q'}_{L^{q,\frac{nq-2n+q}{q-1}}(\Omega^{**})} + \|g\|^{q'}_{H^{1,q,\frac{\mu}{q-1}}(\Omega)} + \|W(Dg)\|^2_{L^{2,\frac{\mu}{q-1}}(\Omega)} + \|f_0\|^{q'}_{L^{\frac{nq}{n(q-1)+q},\mu\frac{n}{n(q-1)+q}}(\Omega)}. \tag{5.13}$$

Finally, by virtue of the estimate (3.37), with $w = u - g$, and bearing in mind that the inequality (5.9) holds for each

$$\Omega \cap \mathcal{B}_j(R), \quad j = 1, \ldots, m,$$

we may conclude that

$$W(Du) \in L^{2,\frac{\mu}{q-1}}(\Omega),$$

and the following inequality is true

$$\|W(Du)\|^2_{L^{2,\frac{\mu}{q-1}}(\Omega)} \leqslant c\mathcal{K}, \tag{5.14}$$

where, if $\mu \leqslant nq - 2n + q$, $\mathcal{K}$ is given by the right-hand side of (5.10), whereas, if $\mu > nq - 2n + q$, denoting by Ω^{**} an open set such that $\Omega^* \subset\subset \Omega^{**} \subset\subset \Omega$, by R^{**} a number $0 < R < R^{**} < R^* < 1$, and supposing, as it is possible, that Ω^{**}, $\mathcal{B}_1(R^{**})$,…, $\mathcal{B}_m(R^{**})$ still cover $\overline{\Omega}$,

$$\mathcal{K} = \sum_i \|f_i\|^{q'}_{L^{q,\frac{\mu}{q-1}}(\Omega)} + \|g\|^{q'}_{H^{1,q,\frac{\mu}{q-1}}(\Omega)} + \|W(Dg)\|^2_{L^{2,\frac{\mu}{q-1}}(\Omega)} + \|f_0\|^{q'}_{L^{\frac{nq}{n(q-1)+q},\mu\frac{n}{n(q-1)+q}}(\Omega)} + \|Du\|^{q'}_{L^{q,\frac{nq-2n+q}{q-1}}(\Omega)}. \tag{5.15}$$

Now, from (5.14), if $n - q < \mu(q-1)^{-1} < \lambda$ and Ω is of class C^2, we see that

$$u \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N) \quad \text{with} \quad \alpha = 1 - \frac{n(q-1) - \mu}{q(q-1)}, \tag{5.16}$$

and it follows the estimate

$$[u]^q_{\alpha,\overline{\Omega}} \leqslant c(n)\|W(Du)\|^2_{L^{2,\frac{\mu}{q-1}}(\Omega)} \leqslant c\mathcal{K}$$

where $\mathcal{K}$ is given by the right-hand side of (5.10) if $\mu \leqslant nq-2n+q$ and holds by (5.15) if $\mu > nq - 2n + q$.

Thus Theorem 1.1 is completely proved. □

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