Daniyal M. Israfilov

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APPROXIMATION BY p-FABER-LAURENT RATIONAL FUNCTIONS IN THE WEIGHTED LEBESGUE SPACES

Daniyal M. Israfilov, Balikesir

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Abstract. Let $L \subset C$ be a regular Jordan curve. In this work, the approximation properties of the p-Faber-Laurent rational series expansions in the ω weighted Lebesgue spaces $L^p(L,\omega)$ are studied. Under some restrictive conditions upon the weight functions the degree of this approximation by a kth integral modulus of continuity in $L^p(L,\omega)$ spaces is estimated.

Keywords: Faber polynomial, Faber series, weighted Lebesgue space, weighted Smirnov space, k-th modulus of continuity

MSC 2000: 41A10, 41A25, 41A58, 41A30, 30E10

1. Introduction

Let L be a rectifiable Jordan curve in the complex plane C, $G := \operatorname{int} L$ and $G^- := \operatorname{ext} L$. Without loss of generality we assume that $0 \in G$. Let also $U := \{w\colon\ |w| < 1\}$, $T := \partial U$, $U^- := \{w|\colon\ |w| > 1\}$, and let φ and φ_1 be the conformal mappings of G^- and G onto U^- respectively, normalized by

$$\varphi(\infty) = \infty, \quad \lim_{z\to\infty} \varphi(z)/z > 0$$

and

$$\varphi_1(0) = \infty, \quad \lim_{z\to 0} z\varphi_1(z) > 0.$$

The inverse mappings of φ and φ_1 will be denoted by ψ and ψ_1, respectively. Later on we assume that $p \in (1,\infty)$, and denote by $L^p(L)$ and $E^p(G)$ the set of all measurable complex valued functions such that $|f|^p$ is Lebesgue integrable with respect to arclength, and the Smirnov class of analytic functions in G, respectively.

Each function $f \in E^p(G)$ has a nontangential limit almost everywhere (a.e.) on L, and if we use the same notation for the nontangential limit of f, then $f \in L^p(L)$.

For $p > 1$, $L^p(L)$ and $E^p(G)$ are Banach spaces with respect to the norm

$$\|f\|_{E^p(G)} = \|f\|_{L^p(L)} := \left(\int_L |f(z)|^p \, |\mathrm{d}z|\right)^{\frac{1}{p}}.$$

For the further properties, see [5, pp. 168–185] and [8, pp. 438–453].

The order of polynomial approximation in $E^p(G)$, $p \geqslant 1$ has been studied by several authors. In [17], Walsh and Russel gave results when L is an analytic curve. For domains with sufficiently smooth boundary, namely when L is a smooth Jordan curve and $\theta(s)$, the angle between the tangent and the positive real axis expressed as a function of arclength s, has modulus of continuity $\Omega(\theta, s)$ satisfying the Dini-smooth condition

$$(1) \qquad \int_0^\delta \frac{\Omega(\theta, s)}{s} \, \mathrm{d}s < \infty, \quad \delta > 0,$$

this problem, for $p > 1$, was studied by S. Y. Alper [1].

These results were later extended to domains with regular boundary which we define in Section 2, for $p > 1$ by V. M. Kokilashvili [13], and for $p \geqslant 1$ by J. E. Andersson [2]. Similar problems were also investigated in [10]. Let us emphasize that in these works, the Faber operator, Faber polynomials and p-Faber polynomials were commonly used and the degree of polynomial approximation in $E^p(G)$ has been studied by applying various methods of summation to the Faber series of functions in $E^p(G)$. More extensive knowledge about them can be found in [7, pp. 40–57] and [16, pp. 52–236].

In [11], for domains with a regular boundary we have constructed the approximants directly as the nth partial sums of p-Faber polynomial series of $f \in E^p(G)$, and later applying the same method in [3], we have investigated the approximation properties of the nth partial sums of p-Faber-Laurent rational series expansions in the Lebesgue spaces $L^p(L)$. The approximation properties of the p-Faber series expansions in the ω-weighted Smirnov class $E^p(G, \omega)$ of analytic functions in G whose boundary is a regular Jordan curve are studied in [12].

In this work, when L is a regular Jordan curve, the approximation properties of the p-Faber-Laurent rational series expansions in the ω-weighted Lebesgue spaces $L^p(L, \omega)$ are studied. Under some restrictive conditions upon weight functions the degree of this approximation is estimated by a kth ($k \geqslant 1$) integral modulus of continuity in $L^p(L, \omega)$ spaces. The results to be obtained in this work are also new in the nonweighted case $\omega = 1$.

We shall denote by c constants (in general, different in different relations) depending only on numbers that are not important for the questions of our interest.

2. New results

For the formulation of new results in detail it is necessary to introduce some definitions and auxiliary results.

Definition 1. L is called regular if there exists a number $c > 0$ such that for every $r > 0$, $\sup\{|L \cap D(z,r)|\colon z \in L\} \leqslant cr$, where $D(z,r)$ is an open disk with radius r and centered at z and $|L \cap D(z,r)|$ is the length of the set $L \cap D(z,r)$.

We denote by S the set of all regular Jordan curves in the complex plane.

Definition 2. Let ω be a weight function on L. ω is said to satisfy the Muckenhoupt A_p-condition on L if

$$\sup_{z\in L}\sup_{r>0}\biggl(\frac{1}{r}\int_{L\cap D(z,r)}\omega(\zeta)\,|\mathrm{d}\zeta|\biggr)\biggl(\frac{1}{r}\int_{L\cap D(z,r)}[\omega(\zeta)]^{-1/p-1}\,|\mathrm{d}\zeta|\biggr)^{p-1} < \infty.$$

Let us denote by $A_p(L)$ the set of all weight functions satisfying the Muckenhoupt A_p-condition on L.

For a weight function ω given on L we also define the following function spaces.

Definition 3. The set $L^p(L,\omega) := \{f \in L^1(L)\colon |f|^p\omega \in L^1(L)\}$ is called the ω-weighted L^p-space.

Definition 4. The set $E^p(G,\omega) := \{f \in E^1(G)\colon f \in L^p(L,\omega)\}$ is called the ω-weighted Smirnov space of order p of analytic functions in G.

Let $g \in L^p(T,\omega)$ and $\omega \in A_p(T)$. Since $L^p(T,\omega)$ is noninvariant with respect to the usual shift, we consider the following mean value function as a shift for $g \in L^p(T,\omega)$:

$$\sigma_h g(w) := \frac{1}{2h}\int_{-h}^{h} g(w\mathrm{e}^{it})\,\mathrm{d}t, \quad 0 < h < \pi,\ \ w \in T.$$

As follows from the continuity of the Hardy-Littlewood maximal operator in weighted $L^p(T,\omega)$ spaces, the operator σ_h is bounded in $L^p(T,\omega)$ if $\omega \in A_p(T)$ and the following inequality holds:

$$\|\sigma_h g\|_{L^p(T,\omega)} \leqslant c(p)\|g\|_{L^p(T,\omega)}, \quad 1 < p < \infty.$$

The last relation is equivalent [15] to the property

$$\lim_{h\to 0}\|\sigma_h g - g\|_{L^p(T,\omega)} = 0.$$

Starting from the last two relations we can give the following definition.

Definition 5. If $g \in L^p(T,\omega)$ and $\omega \in A_p(T)$, then the function $\Omega_{p,\omega,k}(g,\cdot)\colon [0,\infty] \to [0,\infty)$ defined by

$$\Omega_{p,\omega,k}(g,\delta) := \sup_{\substack{0<h_i\leqslant\delta \\ i=1,2,\ldots,k}} \Big\| \prod_{i=1}^{k} (E-\sigma_{h_i})g \Big\|_{L^p(T,\omega)}, \quad 1<p<\infty,$$

is called the kth integral modulus of continuity in the $L^p(T,\omega)$ space for g. Here E is the identity operator.

Note that the idea of defining such a modulus of continuity originates from [18]. In [9] this idea was used for investigations of the approximation problems in $L^p([0,2\pi],\omega)$ spaces. Recently, in [12], to obtain direct theorems of the approximation theory in the weighted Smirnov spaces $E^p(G,\omega)$, we have used the same idea for the case $k=1$.

It can be shown easily that $\Omega_{p,\omega,k}(g,\cdot)$ is a continuous, nonnegative and nondecreasing function satisfying the conditions

$$\lim_{\delta\to 0} \Omega_{p,\omega,k}(g,\delta) = 0, \quad \Omega_{p,\omega,k}(g_1+g_2,\cdot) \leqslant \Omega_{p,\omega,k}(g_1,\cdot) + \Omega_{p,\omega}(g_2,\cdot). \tag{2}$$

For an arbitrary function $f \in L^p(L,\omega)$ and a weight function given on L we also set

$$\begin{gathered} f_0(w) := f[\psi(w)](\psi'(w))^{1/p}, \quad f_1(w) := f[\psi_1(w)](\psi_1'(w))^{1/p} w^{2/p}, \\ \omega_0(w) := \omega[\psi(w)], \quad \omega_1(w) := \omega[\psi_1(w)]. \end{gathered} \tag{3}$$

The condition $f \in L^p(L,\omega)$, implies that $f_0 \in L^p(T,\omega_0)$ and $f_1 \in L^p(T,\omega_1)$. Then if $\omega \in A_p(L)$ and $\omega_0,\omega_1 \in A_p(T)$ we can define the weighted integral moduli of continuity $\Omega_{p,\omega,k}(f_0,\delta)$ and $\Omega_{p,\omega,k}(f_1,\delta)$, using the procedure given above.

Main result in our work is the following theorem.

Theorem 1. *Let $L \in S$ and $f \in L^p(L,\omega)$, $1<p<\infty$. If $\omega \in A_p(L)$ and $\omega_0,\omega_1 \in A_p(T)$, then for every natural number n there are a constant $c>0$ and a rational function*

$$R_n(z,f) := \sum_{k=-n}^{n} a_k^{(n)} z^k$$

such that

$$\|f - R_n(\cdot,f)\|_{L^p(L,\omega)} \leqslant c\Big[\Omega_{p,\omega_0,k}\Big(f_0,\frac{1}{n}\Big) + \Omega_{p,\omega_1,k}\Big(f_1,\frac{1}{n}\Big)\Big],$$

where the rational functions $R_n(z,f)$ are constructed as the nth partial sums of the p-Faber-Laurent series of f.

From this theorem we have the following results in the particular cases $f \in E^p(G,\omega)$ and $f \in E^p(G^-,\omega)$, respectively.

Theorem 2. *Let $L \in S$ and $f \in E^p(G,\omega)$, $1 < p < \infty$. If $\omega \in A_p(L)$ and $\omega_0 \in A_p(T)$, then for every natural number n there are a constant $c > 0$ and a polynomial*

$$P_n(z,f) := \sum_{k=0}^{n} a_k^{(n)} z^k$$

such that

$$\|f - P_n(\cdot,f)\|_{L^p(L,\omega)} \leqslant c\,\Omega_{p,\omega_0,k}\Big(f_0, \frac{1}{n}\Big),$$

where the polynomials $P_n(z,f)$ are constructed as the nth partial sums of the p-Faber series of f.

In the case $k = 1$ Theorem 2 was proved in [12].

Theorem 3. *Let $L \in S$ and $f \in E^p(G^-,\omega)$, $1 < p < \infty$. If $\omega \in A_p(L)$ and $\omega_1 \in A_p(T)$, then for every natural number n there are a constant $c > 0$ and a rational function*

$$R_n(z,f) := \sum_{k=-n}^{0} a_k^{(n)} z^k$$

such that

$$\|f - R_n(\cdot,f)\|_{L^p(L,\omega)} \leqslant c\,\Omega_{p,\omega_1,k}\Big(f_1, \frac{1}{n}\Big),$$

where the rational functions $R_n(z,f)$ are constructed as the nth partial sums of the p-Faber-Laurent series of f.

Note that if L is a sufficiently smooth curve then the conditions $\omega \in A_p(L)$, $\omega_0 \in A_p(T)$, and $\omega_1 \in A_p(T)$ are equivalent. In particular, the following theorem holds.

Theorem 4. *Let L be a smooth boundary satisfying the condition* 1 *and $f \in L^p(L,\omega)$, $1 < p < \infty$. If $\omega \in A_p(L)$ then for every natural number n there are a constant $c > 0$ and a rational function*

$$R_n(z,f) := \sum_{k=-n}^{n} a_k^{(n)} z^k$$

such that

$$\|f - R_n(\cdot, f)\|_{L^p(L,\omega)} \leqslant c\Big[\Omega_{p,\omega_0,k}\Big(f_0, \frac{1}{n}\Big) + \Omega_{p,\omega_1,k}\Big(f_1, \frac{1}{n}\Big)\Big],$$

where the rational functions $R_n(z, f)$ are constructed as the nth partial sums of the p-Faber-Laurent series of f.

3. Construction of approximants and some auxiliary results

1. The generalized p-Faber-Laurent series

Let $f \in L^1(L)$. Then the functions f^+ and f^- defined by

$$f^+(z) = \frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_T \frac{f(\psi(w))\psi'(w)}{\psi(w) - z}\,dw, \quad z \in G, \tag{4}$$

and

$$f^-(z) = \frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_T \frac{f(\psi_1(w))\psi_1'(w)}{\psi_1(w) - z}\,dw, \quad z \in G^-, \tag{5}$$

are analytic in G and G^-, respectively, and $f^-(\infty) = 0$.

According to the celebrated Privalov's theorem [8, p. 431], if one of the functions $f^+(z)$ and $f^-(z)$ has a nontangential limit on L a.e., then Cauchy's singular integral $S_L(f)(z)$ defined as

$$\begin{aligned} S_L(f)(z_0) &:= (\mathrm{P.V.})\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta - z_0}\,d\zeta \\ &:= \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{L\cap\{\zeta\colon |\zeta - z_0|>\varepsilon\}} \frac{f(\zeta)}{\zeta - z_0}\,d\zeta, \quad z_0 \in L, \end{aligned}$$

exists a.e. on L, and also the other one of the functions $f^+(z)$ and $f^-(z)$ has a nontangential limit on L a.e. Conversely, if $S_L(f)(z)$ exists a.e. on L, then the functions $f^+(z)$ and $f^-(z)$ have nontangential limits a.e. on L. In both case, the formulae

$$f^+(z) = S_L(f)(z) + \frac{1}{2}f(z), \quad f^-(z) = S_L(f)(z) - \frac{1}{2}f(z)$$

hold a.e. on L. From this it follows that

$$f(z) = f^+(z) - f^-(z) \tag{6}$$

a.e. on L.

The mappings ψ and ψ_1 have in some deleted neighborhood of ∞ the representations

$$\psi(w) = \alpha w + \alpha_0 + \frac{\alpha_1}{w} + \frac{\alpha_2}{w^2} + \ldots + \frac{\alpha_k}{w^k} + \ldots, \quad \alpha > 0,$$

and

$$\psi_1(w) = \frac{\beta_1}{w} + \frac{\beta_2}{w^2} + \ldots + \frac{\beta_k}{w^k} + \ldots, \quad \beta_1 > 0.$$

Hence the functions

$$\frac{(\psi'(w))^{1-\frac{1}{p}}}{\psi(w) - z}, \quad z \in G$$

and

$$\frac{w^{-2/p}(\psi_1'(w))^{1-\frac{1}{p}}}{\psi_1(w) - z}, \quad z \in G^-$$

are analytic in the domain U^- and have a simple zero and a zero of order 2 at ∞, respectively. Therefore, they have expansions

$$\frac{(\psi'(w))^{1-\frac{1}{p}}}{\psi(w) - z} = \sum_{k=0}^{\infty} \frac{F_{k,p}(z)}{w^{k+1}}, \quad z \in G \text{ and } w \in U^-, \tag{7}$$

and

$$\frac{w^{-2/p}(\psi_1'(w))^{1-\frac{1}{p}}}{\psi_1(w) - z} = \sum_{k=1}^{\infty} -\frac{\widetilde{F}_{k,p}(1/z)}{w^{k+1}}, \quad z \in G^- \text{ and } w \in U^-, \tag{8}$$

where $F_{k,p}(z)$ and $\widetilde{F}_{k,p}(1/z)$ are the p-Faber polynomials of degree k with respect to z and $1/z$ for the continuums $\overline{G}$ and $\overline{C} \setminus G$, respectively (see also [16, pp. 255–257] for $p = \infty$).

Note that the functions $\varphi^k(\varphi')^{1/p}$ and $\varphi_1^{k-2/p}(\varphi_1')^{1/p}$ have poles of order k at the points ∞ and $z = 0$, respectively. Therefore, the polynomial $F_{k,p}(z)$ can alternatively be defined as the polynomial part of the Laurent expansion of $\varphi^k(\varphi')^{1/p}$ in some neighbourhood of the point ∞. Similarly, the principle part of the Laurent expansion of $\varphi_1^{k-2/p}(\varphi_1')^{1/p}$ in some neighbourhood of the point $z = 0$ defines the polynomial $\widetilde{F}_{k,p}(1/z)$. Moreover, the following relations hold:

$$[\varphi(z)]^k(\varphi'(z))^{1/p} = F_{k,p}(z) + E_{k,p}(z), \quad z \in G^-,$$
$$[\varphi_1(z)]^{k-2/p}(\varphi_1'(z))^{1/p} = \widetilde{F}_{k,p}(1/z) + \widetilde{E}_{k,p}(z), \quad z \in G \setminus \{0\},$$

where the functions $E_{k,p}(z)$ and $\widetilde{E}_{k,p}(z)$ are analytic in G^- and in G, respectively.

We shall also exploit the integral representations

$$F_{k,p}(z) = \varphi^k(z)(\varphi'(z))^{\frac{1}{p}} + \frac{1}{2\pi i}\int_L \frac{\varphi^k(\zeta)(\varphi'(\zeta))^{\frac{1}{p}}}{\zeta - z}\,d\zeta, \quad z \in G^- \tag{9}$$

and

$$\widetilde{F}_{k,p}\Big(\frac{1}{z}\Big) = [\varphi_1(z)]^{k-\frac{2}{p}}(\varphi_1'(z))^{\frac{1}{p}} - \frac{1}{2\pi i}\int_L \frac{[\varphi_1(\zeta)]^{k-\frac{2}{p}}(\varphi_1'(\zeta))^{\frac{1}{p}}}{\zeta - z}\,d\zeta, \quad z \in G \setminus 0, \tag{10}$$

which are proved similarly as in the classical case $p = \infty$ (see for example, [14, pp. 114–118]).

We define the coefficients a_k and $\widetilde{a}_k$ starting from the relations (4), (3), (7) and relations (5), (3), (8), respectively, by

$$a_k = a_k(f) := \frac{1}{2\pi i}\int_T \frac{f_0(w)}{w^{k+1}}\,dw, \quad k = 0, 1, 2, \ldots, \tag{11}$$

and

$$\widetilde{a}_k = \widetilde{a}_k(f) := \frac{1}{2\pi i}\int_T \frac{f_1(w)}{w^{k+1}}\,dw, \quad k = 1, 2, \ldots. \tag{12}$$

Then taking the relation (6) into account we can associate a formal series

$$\sum_{k=0}^{\infty} a_k F_{k,p}(z) + \sum_{k=1}^{\infty} \widetilde{a}_k \widetilde{F}_{k,p}(1/z)$$

with the function $f \in L^1(L)$, i.e.,

$$f(z) \sim \sum_{k=0}^{\infty} a_k F_{k,p}(z) + \sum_{k=1}^{\infty} \widetilde{a}_k \widetilde{F}_{k,p}(1/z).$$

This formal series is called the p-Faber-Laurent series of f, and the coefficients a_k and $\widetilde{a}_k$ are said to be the p-Faber-Laurent coefficients of f.

We will also use the following lemma which was proved in [12].

Lemma 1. *If $L \in S$ and $\omega \in A_p(L)$, then $f^+ \in E^p(G, \omega)$ and $f^- \in E^p(G^-, \omega)$ for each $f \in L^p(L, \omega)$.*

Since $f_0 \in L^p(T, \omega_0)$ and $f_1 \in L^p(T, \omega_1)$, under the conditions $\omega_0, \omega_1 \in A_p(T)$ we have by Lemma 1 that

$$f_0^+ \in E^p(U, \omega_0), \quad f_0^- \in E^p(U^-, \omega_0), \tag{13}$$

$$f_1^+ \in E^p(U, \omega_1), \quad f_1^- \in E^p(U^-, \omega_1). \tag{14}$$

Moreover, $f_0^-(\infty) = f_1^-(\infty) = 0$, and by the celebrated Privalov's theorem

$$f_0(w) = f_0^+(w) - f_0^-(w) \tag{15}$$

and

$$f_1(w) = f_1^+(w) - f_1^-(w) \tag{16}$$

a.e. on T. Using the relations (13) and (15) in (11), and (14) and (16) in (12) we conclude that the coefficients a_k, $k = 0, 1, 2, \ldots$, and $\widetilde{a}_k$, $k = 1, 2, \ldots$, are also the Taylor coefficients of the functions $f_0^+ \in E^p(U, \omega_0)$ and $f_1^+ \in E^p(U, \omega_1)$, respectively.

2. Singular integrals and modulus of continuity

As was noted in [6, p. 89], for the Cauchy singular integral the following result, which is analogously deduced from [4], holds.

Theorem 5. *Let $L \in S$, $1 < p < \infty$, and let ω be a weight function on L. The inequality*

$$\|S_L(f)\|_{L^p(L,\omega)} \leqslant c\|f\|_{L^p(L,\omega)}$$

holds for every $f \in L^p(L, \omega)$ if and only if $\omega \in A_p(L)$.

Lemma 2. *Let $g \in L^p(T, \omega)$ and let $\omega \in A_p(T)$. Then*

$$\sigma_{h_1,h_2,\ldots,h_k}[S_T(g)](w) = S_T[\sigma_{h_1,h_2,\ldots,h_k}(g)](w)$$

for every natural number k.

P r o o f. Let $k = 1$. Applying the Fubini theorem we have

$$\begin{aligned}
[S_T(g)]_h(w) &= \frac{1}{2h}\int_{-h}^{h} S_T(g(w\mathrm{e}^{\mathrm{i}\theta}))\,\mathrm{d}\theta \\
&= \frac{1}{2h}\int_{-h}^{h} \frac{1}{2\pi\mathrm{i}}\biggl((\mathrm{P.V.})\int_T \frac{g(\tau)\,\mathrm{d}\tau}{\tau - w\mathrm{e}^{\mathrm{i}\theta}}\biggr)\,\mathrm{d}\theta \\
&= \frac{1}{2h}\int_{-h}^{h} \frac{1}{2\pi\mathrm{i}}\biggl((\mathrm{P.V.})\int_T \frac{g(\tau\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\tau}{\tau\mathrm{e}^{\mathrm{i}\theta} - w\mathrm{e}^{\mathrm{i}\theta}}\biggr)\,\mathrm{d}\theta \\
&= \frac{1}{2h}\int_{-h}^{h} \frac{1}{2\pi\mathrm{i}}\biggl((\mathrm{P.V.})\int_T \frac{g(\tau\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\tau}{\tau - w}\biggr)\,\mathrm{d}\theta \\
&= \frac{1}{2\pi\mathrm{i}}(\mathrm{P.V.})\int_T \frac{(1/2h)\int_{-h}^{h} g(\tau\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta}{\tau - w}\,\mathrm{d}\tau \\
&= \frac{1}{2\pi\mathrm{i}}(\mathrm{P.V.})\int_T \frac{g_h(\tau)}{\tau - w}\,\mathrm{d}\tau = [S_T(g_h)](w).
\end{aligned}$$

Now let the lemma hold for $n = k - 1$, i.e.

$$\sigma_{h_1,h_2,\dots,h_{k-1}}[S_T(g)](w) = S_T[\sigma_{h_1,h_2,\dots,h_{k-1}}(g)](w).$$

Then

$$\begin{aligned}\sigma_{h_1,h_2,\dots,h_k}[S_T(g)](w) &= \sigma_{h_k}\{\sigma_{h_1,h_2,\dots,h_{k-1}}[S_T(g)]\}(w)\\ &= \sigma_{h_k}[S_T\sigma_{h_1,h_2,\dots,h_{k-1}}(g)](w)\\ &= S_T[\sigma_{h_1,h_2,\dots,h_k}(g)](w),\end{aligned}$$

and the lemma is proved by the induction method. □

Lemma 3. *Let $g \in L^p(T,\omega)$ and let $\omega \in A_p(T)$. Then*

$$\Omega_{p,\omega,k}(S_T(g),\cdot) \leqslant c\,\Omega_{p,\omega,k}(g,\cdot).$$

Proof. Again we use the induction method. Let $k = 1$. Using Lemma 2 and Theorem 5 we obtain

$$\|S_T(g) - \sigma_{h_1}[S_T(g)]\|_{L^p(T,\omega)} = \|S_T(g - \sigma_{h_1}g)\|_{L^p(T,\omega)} \leqslant c\|g - \sigma_{h_1}g\|_{L^p(T,\omega)}.$$

This inequality implies that

$$\Omega_{p,\omega,1}(S_T(g),\cdot) \leqslant c\,\Omega_{p,\omega,1}(g,\cdot).$$

Let the lemma hold for $n = k - 1$, i.e.,

$$\Omega_{p,\omega,k-1}(S_T(g),\cdot) \leqslant c\,\Omega_{p,\omega,k-1}(g,\cdot).$$

Then applying Lemma 2 and the last inequality successively we obtain

$$\begin{aligned}\Omega_{p,\omega,k}(S_T(g),\delta) &= \sup_{\substack{0<h_i\leqslant\delta\\ i=1,2,\dots,k}} \Big\|\prod_{i=1}^{k}(E - \sigma_{h_i})S_Tg\Big\|_{L^p(T,\omega)}\\ &= \sup_{\substack{0<h_i\leqslant\delta\\ i=1,2,\dots,k}} \Big\|\prod_{i=2}^{k}(E - \sigma_{h_i})(E - \sigma_{h_1})S_Tg\Big\|_{L^p(T,\omega)}\end{aligned}$$

$$= \sup_{\substack{0<h_i\leqslant\delta\\ i=1,2,\dots,k}} \Big\|\prod_{i=2}^{k}(E-\sigma_{h_i})S_T(g-\sigma_{h_1}g)\Big\|_{L^p(T,\omega)}$$

$$\leqslant c \sup_{\substack{0<h_i\leqslant\delta\\ i=1,2,\dots,k}} \Big\|\prod_{i=2}^{k}(E-\sigma_{h_i})(g-\sigma_{h_1}g)\Big\|_{L^p(T,\omega)}$$

$$= c \sup_{\substack{0<h_i\leqslant\delta\\ i=1,2,\dots,k}} \Big\|\prod_{i=2}^{k}(E-\sigma_{h_i})(E-\sigma_{h_1})(g)\Big\|_{L^p(T,\omega)}$$

$$= c \sup_{\substack{0<h_i\leqslant\delta\\ i=1,2,\dots,k}} \Big\|\prod_{i=1}^{k}(E-\sigma_{h_i})(g)\Big\|_{L^p(T,\omega)} = c\,\Omega_{p,\omega,k}(g,\delta).$$

□

Lemma 4. *If $g\in L^p(T,\omega)$ and $\omega\in A_p(T)$, then*

$$\Omega_{p,\omega,k}(g^+,\cdot)\leqslant\Big(c+\frac{1}{2}\Big)\Omega_{p,\omega,k}(g,\cdot).$$

P r o o f. Taking into account the relation

$$g^+=\frac{g}{2}+S_Tg$$

which holds a.e. on T, by virtue of Lemma 3 and the property (2) we obtain the proof of Lemma 4. □

Lemma 5. *Let $g\in E^p(U,\omega)$ and $\omega\in A_p(T)$. If*

$$\sum_{k=0}^{n}\alpha_k(g)w^k$$

is the nth partial sum of the Taylor series of g at the origin, then there exists a constant $c>0$ such that

$$\Big\|g(w)-\sum_{k=0}^{n}\alpha_k(g)w^k\Big\|_{L^p(T,\omega)}\leqslant c\,\Omega_{p,\omega,k}\Big(g,\frac{1}{n}\Big)$$

for every natural number n.

P r o o f. In the case $k=1$ this lemma was shown in [12, Lemma 9]. For $k>1$ the proof proceeds analogously. □

4. Proof of the new results

P r o o f of Theorem 1. We shall prove that the rational function

$$R_n(f,z) := \sum_{k=0}^{n} a_k F_{k,p(z)} + \sum_{k=1}^{n} \widetilde{a}_k \widetilde{F}_{k,p}(1/z)$$

satisfies the necessary inequality from Theorem 1.

In view of the relation

$$f(z) = f^+(z) - f^-(z),$$

which holds a.e. on L, it suffices to establish the inequalities

$$\Big\| f^-(z) + \sum_{k=1}^{n} \widetilde{a}_k \widetilde{F}_{k,p}(1/z) \Big\|_{L^p(L,\omega)} \leqslant c\, \Omega_{p,\omega_1,k}\Big(f_1, \frac{1}{n}\Big) \tag{17}$$

and

$$\Big\| f^+(z) - \sum_{k=0}^{n} a_k F_{k,p}(z) \Big\|_{L^p(L,\omega)} \leqslant c\, \Omega_{p,\omega_0,k}\Big(f_0, \frac{1}{n}\Big). \tag{18}$$

Putting $\varphi(z)$ and $\varphi_1(z)$ instead of w in the notation (3) of the functions $f_0(w)$ and $f_1(w)$, respectively, and using the relations (15) and (16), we obtain

$$f(z) = [f_0^+(\varphi(z)) - f_0^-(\varphi(z))](\varphi'(z))^{1/p} \tag{19}$$

and

$$f(z) = [f_1^+(\varphi_1(z)) - f_1^-(\varphi_1(z))](\varphi_1(z))^{-2/p}(\varphi_1'(z))^{1/p} \tag{20}$$

a.e. on L.

First we prove the estimate (17). Let us take a $z' \in G$. Using the relations (10) and (20) we obtain

$$\begin{aligned}\sum_{k=1}^{n} \widetilde{a}_k \widetilde{F}_{k,p}\Big(\frac{1}{z'}\Big) &= (\varphi_1'(z'))^{\frac{1}{p}}(\varphi_1(z'))^{-\frac{2}{p}} \sum_{k=1}^{n} \widetilde{a}_k \varphi_1^k(z') \\ &\quad - \frac{1}{2\pi \mathrm{i}} \int_L \frac{(\varphi_1'(\zeta))^{\frac{1}{p}}(\varphi_1(\zeta))^{-\frac{2}{p}} \sum\limits_{k=1}^{n} \widetilde{a}_k \varphi_1^k(\zeta)}{\zeta - z'}\, \mathrm{d}\zeta\end{aligned}$$

$$
\begin{aligned}
&= (\varphi_1'(z'))^{\frac{1}{p}}(\varphi_1(z'))^{-\frac{2}{p}}\sum_{k=1}^{n}\widetilde{a}_k\varphi_1^k(z') \\
&\quad - \frac{1}{2\pi i}\int_L \frac{(\varphi_1')(\zeta))^{\frac{1}{p}}(\varphi_1(z'))^{-\frac{2}{p}}\Big[\sum\limits_{k=1}^{n}\widetilde{a}_k\varphi_1^k(\zeta) - f_1^+(\varphi_1(\zeta))\Big]}{\zeta - z'}\,\mathrm{d}\zeta \\
&\quad - \frac{1}{2\pi i}\int_L \frac{(\varphi_1'(\zeta))^{\frac{1}{p}}(\varphi_1(z'))^{-\frac{2}{p}}f_1^-(\varphi(\zeta))}{\zeta - z'}\,\mathrm{d}\zeta - \frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta - z'}\,\mathrm{d}\zeta.
\end{aligned}
$$

Since $(\varphi_1'(\zeta))^{\frac{1}{p}}(\varphi_1(z'))^{-\frac{2}{p}}f_1^-(\varphi_1(\zeta)) \in E^p(G,\omega)$ we get

$$
\frac{1}{2\pi i}\int_L \frac{(\varphi_1'(\zeta))^{\frac{1}{p}}(\varphi_1(z'))^{-\frac{2}{p}}f_1^-(\varphi(\zeta))}{\zeta - z'}\,\mathrm{d}\zeta = (\varphi_1'(z'))^{\frac{1}{p}}(\varphi_1(z'))^{-\frac{2}{p}}f_1^-(\varphi_1(z')).
$$

Then

$$
\begin{aligned}
\sum_{k=1}^{n}\widetilde{a}_k\widetilde{F}_{k,p}\Big(\frac{1}{z'}\Big) &= (\varphi_1'(z'))^{\frac{1}{p}}(\varphi_1(z'))^{-\frac{2}{p}}\sum_{k=1}^{n}\widetilde{a}_k\varphi_1^k(z') \\
&\quad - \frac{1}{2\pi i}\int_L \frac{(\varphi_1'(\zeta))^{\frac{1}{p}}(\varphi_1(z'))^{-\frac{2}{p}}\Big[\sum\limits_{k=1}^{n}\widetilde{a}_k\varphi_1^k(\zeta) - f_1^+(\varphi_1(\zeta))\Big]}{\zeta - z'}\,\mathrm{d}\zeta \\
&\quad - (\varphi_1'(z'))^{\frac{1}{p}}(\varphi_1(z'))^{-\frac{2}{p}}f_1^-(\varphi_1(z')) - f^+(z').
\end{aligned}
$$

Taking limit as $z' \to z$ along all nontangential paths inside of L, it appears that

$$
\begin{aligned}
\sum_{k=1}^{n}\widetilde{a}_k\widetilde{F}_{k,p}(\frac{1}{z}) &= (\varphi_1'(z))^{\frac{1}{p}}(\varphi_1(z))^{-\frac{2}{p}}\sum_{k=1}^{n}\widetilde{a}_k\varphi_1^k(z) \\
&\quad - \frac{1}{2}(\varphi_1'(\zeta))^{\frac{1}{p}}(\varphi_1(z))^{-\frac{2}{p}}\Big(\sum_{k=1}^{n}\widetilde{a}_k\varphi_1^k(z) - f_1^+(\varphi_1(z))\Big) \\
&\quad - S_L\Big[(\varphi_1')^{\frac{1}{p}}\varphi_1^{-\frac{2}{p}}\Big(\sum_{k=1}^{n}\widetilde{a}_k\varphi_1^k - f_1^+\circ\varphi_1\Big)\Big](z) \\
&\quad - (\varphi_1'(z))^{\frac{1}{p}}(\varphi_1(z))^{-\frac{2}{p}}f_1^-(\varphi_1(z)) - f^+(z)
\end{aligned}
$$

a.e. on L. Using the relations (6) and (20), from the last equality we obtain

$$
\begin{aligned}
f^-(z) + \sum_{k=1}^{n}\widetilde{a}_k\widetilde{F}_{k,p}\Big(\frac{1}{z}\Big) &= \frac{1}{2}(\varphi_1'(z))^{\frac{1}{p}}(\varphi_1(z))^{-\frac{2}{p}}\Big(\sum_{k=1}^{n}\widetilde{a}_k\varphi_1^k(z) - f_1^+(\varphi_1(z))\Big) \\
&\quad - S_L\Big[(\varphi_1')^{\frac{1}{p}}\varphi_1^{-\frac{2}{p}}\Big(\sum_{k=1}^{n}\widetilde{a}_k\varphi_1^k - f_1^+\circ\varphi_1\Big)\Big](z).
\end{aligned}
$$

Applying here Minkowski's inequality and Theorem 5, we conclude that

$$\Big\| f^{-}(z) + \sum_{k=1}^{n} \widetilde{a}_k \widetilde{F}_{k,p}\Big(\frac{1}{z}\Big) \Big\|_{L^p(L,\omega)} \leqslant c \Big\| f_1^{+}(w) - \sum_{k=1}^{n} \widetilde{\alpha}_k w^k \Big\|_{L^p(T,\omega_1)}.$$

Now, by virtue of Lemmas 5 and 4 we have

$$\Big\| f^{-}(z) + \sum_{k=1}^{n} \widetilde{a}_k \widetilde{F}_{k,p}\Big(\frac{1}{z}\Big) \Big\|_{L^p(L,\omega)} \leqslant c\,\Omega_{p,\omega_1,k}\Big(f_1, \frac{1}{n}\Big).$$

The proof of the relation (18) proceeds similarly to that of (17), using the relations (9) and (19) instead of the relations (10) and (20), respectively, and limiting along all nontangential path outside of L.

Now the relation (6) and the estimate (17) and (18) complete the proof. □

P r o o f of Theorem 3. If $f \in E^p(G^-, \omega)$ we apply Theorem 1 to the function $f_* := f - f(\infty)$. The approximate rational function $R_n(z, f)$ is constructed as

$$f(\infty) + \sum_{k=1}^{n} \widetilde{a}_k \widetilde{F}_{k,p}\Big(\frac{1}{z}\Big).$$

□

P r o o f of Theorem 4. It can be shown easily that, under the condition (1), the relations

$$\omega \in A_p(L), \quad \omega_0 \in A_p(T), \quad \omega_1 \in A_p(T),$$

are equivalent. Therefore, the proof of Theorem 4 proceeds similarly to that of Theorem 1. □

References

[1] *S. Y. Alper*: Approximation in the mean of analytic functions of class E^p. In: Investigations on the Modern Problems of the Function Theory of a Complex Variable. Gos. Izdat. Fiz.-Mat. Lit., Moscow, 1960, pp. 272–286. (In Russian.)

[2] *J. E. Andersson*: On the degree of polynomial approximation in $E^p(D)$. J. Approx. Theory *19* (1977), 61–68.

[3] *A. Çavuş and D. M. Israfilov*: Approximation by Faber-Laurent rational functions in the mean of functions of the class $L_p(\Gamma)$ with $1 < p < \infty$. Approximation Theory App. *11* (1995), 105–118.

[4] *G. David*: Operateurs integraux singulers sur certaines courbes du plan complexe. Ann. Sci. Ecol. Norm. Super. *4* (1984), 157–189.

[5] *P. L. Duren*: Theory of H^p-Spaces. Academic Press, 1970.

[6] *E. M. Dyn'kin and B. P. Osilenker*: Weighted estimates for singular integrals and their applications. In: Mathematical analysis, Vol. 21. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1983, pp. 42–129. (In Russian.)
[7] *D. Gaier*: Lectures on Complex Approximation. Birkhäuser-Verlag, Boston-Stuttgart, 1987.
[8] *G. M. Golusin*: Geometric Theory of Functions of a Complex Variable. Translation of Mathematical Monographs, Vol. 26, AMS, 1969.
[9] *E. A. Haciyeva*: Investigation of the properties of functions with quasimonotone Fourier coefficients in generalized Nikolsky-Besov spaces. Author's summary of candidates dissertation. Tbilisi. (In Russian.)
[10] *I. I. Ibragimov and D. I. Mamedhanov*: A constructive characterization of a certain class of functions. Dokl. Akad. Nauk SSSR *223* (1975), 35–37; Soviet Math. Dokl. *4* (1976), 820–823.
[11] *D. M. Israfilov*: Approximate properties of the generalized Faber series in an integral metric. Izv. Akad. Nauk Az. SSR, Ser. Fiz.-Tekh. Math. Nauk *2* (1987), 10–14. (In Russian.)
[12] *D. M. Israfilov*: Approximation by p-Faber polynomials in the weighted Smirnov class $E^p(G,\omega)$ and the Bieberbach polynomials. Constr. Approx. *17* (2001), 335–351.
[13] *V. M. Kokilashvili*: A direct theorem on mean approximation of analytic functions by polynomials. Soviet Math. Dokl. *10* (1969), 411–414.
[14] *A. I. Markushevich*: Theory of Analytic Functions, Vol. 2. Izdatelstvo Nauka, Moscow, 1968.
[15] *B. Muckenhoupt*: Weighted norm inequalites for Hardy maximal functions. Trans. Amer. Math. Soc. *165* (1972), 207–226.
[16] *P. K. Suetin*: Series of Faber Polynomials. Nauka, Moscow, 1984; Cordon and Breach Publishers, 1998.
[17] *J. L. Walsh and H. G. Russel*: Integrated continuity conditions and degree of approximation by polynomials or by bounded analytic functions. Trans. Amer. Math. Soc. *92* (1959), 355–370.
[18] *M. Wehrens*: Best approximation on the unit sphere in R^n. Funct. Anal. and Approx. Proc. Conf. Oberwolfach. Aug. 9-16, 1980, Basel. 1981, pp. 233–245.

Author's address: Department of Mathematics, Faculty of Arts and Sciences, Balikesir University, 10100, Balikesir, Turkey, e-mail: `mdaniyal@mail.balikesir.edu.tr`.