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ON EQUITORSION HOLOMORPHICALLY PROJECTIVE MAPPINGS OF GENERALIZED KÄHLERIAN SPACES

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Abstract. In this paper we investigate holomorphically projective mappings of generalized Kählerian spaces. In the case of equitorsion holomorphically projective mappings of generalized Kählerian spaces we obtain five invariant geometric objects for these mappings.

Keywords: Generalized Riemannian space, Kählerian space, generalized Kählerian space, holomorphically projective mapping, equitorsion holomorphically projective mapping, holomorphically projective parameter, holomorphically projective tensor

MSC 2000: 53B05

1. Introduction

A generalized Riemannian space GR_N in the sense of Eisenhart's definition [1] is a differentiable N-dimensional manifold, equipped with a nonsymmetric basic tensor g_{ij}. Connection coefficients of this space are generalized Cristoffel's symbols of the second kind. Generally, $\Gamma^i_{jk} \neq \Gamma^i_{kj}$.

In a generalized Riemannian space one can define four kinds of covariant derivatives [3], [4]. For example, for a tensor a^i_j in GR_N we have

$$a^i_{j\underset{1}{|}m} = a^i_{j,m} + \Gamma^i_{pm}a^p_j - \Gamma^p_{jm}a^i_p, \quad a^i_{j\underset{2}{|}m} = a^i_{j,m} + \Gamma^i_{mp}a^p_j - \Gamma^p_{mj}a^i_p,$$
$$a^i_{j\underset{3}{|}m} = a^i_{j,m} + \Gamma^i_{pm}a^p_j - \Gamma^p_{mj}a^i_p, \quad a^i_{j\underset{4}{|}m} = a^i_{j,m} + \Gamma^i_{mp}a^p_j - \Gamma^p_{jm}a^i_p.$$

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In the case of the space GR_N we have five independent curvature tensors [6] (in [6] $\underset{5}{R}$ is denoted by $\underset{2}{\tilde{R}}$):

$$\begin{aligned}
\underset{1}{R}{}^i_{jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^p_{jm}\Gamma^i_{pn} - \Gamma^p_{jn}\Gamma^i_{pm},\\
\underset{2}{R}{}^i_{jmn} &= \Gamma^i_{mj,n} - \Gamma^i_{nj,m} + \Gamma^p_{mj}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{mp},\\
\underset{3}{R}{}^i_{jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm} + \Gamma^p_{nm}(\Gamma^i_{pj} - \Gamma^i_{jp}),\\
\underset{4}{R}{}^i_{jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm} + \Gamma^p_{mn}(\Gamma^i_{pj} - \Gamma^i_{jp}),\\
\underset{5}{R}{}^i_{jmn} &= \frac{1}{2}(\Gamma^i_{jm,n} + \Gamma^i_{mj,n} - \Gamma^i_{jn,m} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{pn} + \Gamma^p_{mj}\Gamma^i_{np}\\
&\qquad -\Gamma^p_{jn}\Gamma^i_{mp} - \Gamma^p_{nj}\Gamma^i_{pm}).
\end{aligned}$$

Kählerian spaces and their mappings were investigated by many authors, for example K. Yano [11], [12], M. Prvanović [9], N. S. Sinyukov [10], J. Mikeš [2] and many others.

In [7] we defined a generalized Kählerian space GK_N as a generalized N-dimensional Riemannian space with a (nonsymmetric) metric tensor g_{ij} and an almost complex structure $F^i_j(x)$ such that

$$F^h_p(x)F^p_i(x) = -\delta^h_i, \tag{1.1}$$

$$g_{\underline{pq}}F^p_iF^q_j = g_{\underline{ij}}, \quad g^{\underline{ij}} = g^{\underline{pq}}F^i_pF^j_q, \tag{1.2}$$

$$F^h_{i\underset{\theta}{|}j} = 0 \quad (\theta = 1, 2), \tag{1.3}$$

where $\underset{\theta}{|}$ denotes the covariant derivative of the kind θ with respect to the metric tensor g_{ij}, and $\underline{ij}$ denotes the symmetrization over i, j.

Generalizing the concept of an analytic planar curve in a Kählerian space [8], [10] we obtained in [7] an analogous notion for a generalized Kählerian space.

A curve

$$l\colon\ x^h = x^h(t) \qquad (h = 1, 2, \ldots, N) \tag{1.4}$$

is said to be analytic planar if the relation

$$\frac{\mathrm{d}\lambda^h}{\mathrm{d}t} + \Gamma^h_{pq}\lambda^p\lambda^q = a(t)\lambda^h + b(t)F^h_p\lambda^p \tag{1.5}$$

is satisfied, where $\lambda^h = \mathrm{d}x^h/\mathrm{d}t$ and $a(t)$, $b(t)$ are functions of the parameter t.

Let us consider two N-dimensional generalized Kählerian spaces GK_N and $G\overline{K}_N$ with almost complex structures F^h_i and $\overline{F}^h_i$, respectively. We introduce a common by the mapping f coordinate system, i.e. if $M \in GK_N$, $\overline{M} = f(M) \in G\overline{K}_N$ and at the local carte $(\mathscr{U}_M, \varphi)$ we have $\varphi(M) = (x^1, x^2, \dots, x^N) \in R^N$ then at the local carte $(\overline{\mathscr{U}}_{\overline{M}}, \overline{\varphi} = \varphi \circ f^{-1})$ we have

$$\overline{\varphi}(\overline{M}) = (\varphi \circ f^{-1})(\overline{M}) = \varphi(M) = (x^1, x^2, \dots, x^N).$$

We suppose that

$$F^h_i = \overline{F}^h_i \tag{1.6}$$

in the common by the mapping $f\colon GK_N \to G\overline{K}_N$ coordinate system.

Diffeomorfism $f\colon GK_N \to G\overline{K}_N$ is *holomorphically projective* or *analytic planar* [7] if it maps analytic planar curves of the space GK_N into analytic planar curves of the space $G\overline{K}_N$. In this case, Cristoffel's symbols of the second kind of these spaces satisfy the relations [7]

$$\overline{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_{(i}\delta^h_{j)} + \sigma_{(i}F^h_{j)} + \xi^h_{ij}, \tag{1.7}$$

where (ij) denotes a symmetrization without division by indices i, j and ξ^h_{ij} is an anti-symmetric tensor. In (1.7) the vector σ_i can be selected such that $\sigma_i = -\psi_p F^p_i$. Then we have

$$\overline{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_{(i}\delta^h_{j)} - \psi_p F^p_{(i}F^h_{j)} + \xi^h_{ij}. \tag{1.8}$$

2. Equitorsion holomorphically projective mappings

Let $f\colon GK_N \to G\overline{K}_N$ be a holomorphically projective mapping, and let the torsion tensor $\Gamma^h_{\underset{\vee}{ij}}$ and $\overline{\Gamma}^h_{\underset{\vee}{ij}}$ of the spaces GK_N and $G\overline{K}_N$ satisfy the condition

$$\overline{\Gamma}^h_{\underset{\vee}{ij}} = \Gamma^h_{\underset{\vee}{ij}}. \tag{2.1}$$

In this case the mapping f is called *an equitorsion holomorphically projective mapping* of the spaces GK_N and $G\overline{K}_N$. Then (1.7) implies

$$\xi^h_{ij} = 0. \tag{2.2}$$

2.1. Holomorphically projective parameters of the first kind.

Curvature tensors of the first kind $\underset{1}{R}$ and $\underset{1}{\overline{R}}$ of the spaces GK_N and $G\overline{K}_N$, respectively, are connected by the relation [5]

$$\underset{1}{\overline{R}}{}^i_{jmn} = \underset{1}{R}{}^i_{jmn} + P^i_{jm\underset{1}{|}n} - P^i_{jn\underset{1}{|}m} + P^p_{jm}P^i_{pn} - P^p_{jn}P^i_{pm} + 2\Gamma^p_{\underset{\vee}{mn}}P^i_{jp}, \tag{2.3}$$

where $P^h_{ij} = \overline{\Gamma}^h_{ij} - \Gamma^h_{ij}$ is a deformation tensor. Substituting (1.8) and (2.2) into (2.3) we get

$$
\begin{aligned}
(2.4)\qquad \overline{\underset{1}{R}}{}^i_{jmn} = {} & \underset{1}{R}{}^i_{jmn} + \delta^i_m \underset{1}{\psi}_{jn} + \delta^i_j \underset{1}{\psi}_{[mn]} - \delta^i_j \underset{1}{\psi}_{jm} \\
& + F^p_j (F^i_n \underset{1}{\psi}_{pm} - F^i_m \underset{1}{\psi}_{pn}) + F^i_j (F^p_n \underset{1}{\psi}_{pm} - F^p_m \underset{1}{\psi}_{pn}) \\
& + 2\underset{\vee}{\Gamma}{}^i_{mn}\psi_j + 2\underset{\vee}{\Gamma}{}^p_{mn}\psi_p \delta^i_j - 2\underset{\vee}{\Gamma}{}^p_{mn}\psi_q F^q_j F^i_p - 2\underset{\vee}{\Gamma}{}^p_{mn}\psi_q F^q_p F^i_j,
\end{aligned}
$$

where we denote

$$
(2.5)\qquad \underset{1}{\psi}_{ij} = \psi_{i\underset{1}{|}j} - \psi_i \psi_j + \psi_p F^p_i \psi_q F^q_j .
$$

Contracting with respect to indices i, n in (2.4) we obtain

$$
\begin{aligned}
(2.6)\qquad \overline{\underset{1}{R}}_{jm} = {} & \underset{1}{R}_{jm} + \underset{1}{\psi}_{[mj]} - N\underset{1}{\psi}_{jm} - F^p_j F^q_m \underset{1}{\psi}_{(pq)} \\
& + 2\underset{\vee}{\Gamma}{}^p_{mj}\psi_p - 2\underset{\vee}{\Gamma}{}^p_{mr}\psi_q F^q_j F^r_p - 2\underset{\vee}{\Gamma}{}^p_{mr}\psi_q F^q_p F^r_j .
\end{aligned}
$$

Anti-symmetrization without division in (2.6) with respect to indices j, m yields

$$
\begin{aligned}
(2.7)\qquad (N+2)\underset{1}{\psi}_{[jm]} = {} & \underset{1}{R}_{[jm]} - \overline{\underset{1}{R}}_{[jm]} + 4\underset{\vee}{\Gamma}{}^p_{mj}\psi_p - 2\underset{\vee}{\Gamma}{}^p_{mr}\psi_q F^q_j F^r_p \\
& + 2\underset{\vee}{\Gamma}{}^p_{jr}\psi_q F^q_m F^r_p - 2\underset{\vee}{\Gamma}{}^p_{mr}\psi_q F^q_p F^r_j + 2\underset{\vee}{\Gamma}{}^p_{jr}\psi_q F^q_p F^r_m .
\end{aligned}
$$

Symmetrizing without division with respect to j, m in (2.6) we arrive at

$$
\begin{aligned}
(2.8)\qquad \overline{\underset{1}{R}}_{(jm)} = {} & \underset{1}{R}_{(jm)} - N\underset{1}{\psi}_{(jm)} - 2F^p_j F^q_m \underset{1}{\psi}_{(pq)} - 2\underset{\vee}{\Gamma}{}^p_{mr}\psi_q F^q_j F^r_p \\
& - 2\underset{\vee}{\Gamma}{}^p_{jr}\psi_q F^q_m F^r_p - 2\underset{\vee}{\Gamma}{}^p_{mr}\psi_q F^q_p F^r_j - 2\underset{\vee}{\Gamma}{}^p_{jr}\psi_q F^q_p F^r_m .
\end{aligned}
$$

For the spaces GK_N and $G\overline{K}_N$ the relations [7]

$$
(2.9)\qquad \underset{1}{R}_{(pq)} F^p_i F^q_j = \underset{1}{R}_{(ij)} - 2\underset{\vee}{\Gamma}{}^p_{rq}\underset{\vee}{\Gamma}{}^q_{ps} F^r_j F^s_m + 2\underset{\vee}{\Gamma}{}^p_{jq}\underset{\vee}{\Gamma}{}^q_{pm}
$$

and

$$
(2.10)\qquad \overline{\underset{1}{R}}_{(pq)} F^p_i F^q_j = \overline{\underset{1}{R}}_{(ij)} - 2\underset{\vee}{\Gamma}{}^p_{rq}\underset{\vee}{\Gamma}{}^q_{ps} F^r_j F^s_m + 2\underset{\vee}{\Gamma}{}^p_{jq}\underset{\vee}{\Gamma}{}^q_{pm}
$$

are valid, respectively.

By composition with $F^j_p F^m_q$, contraction with respect to j, m, and using the conditions (2.9), (2.10) we get from (2.8)

$$\overline{\underset{1}{R}}_{(jm)} = \underset{1}{R}_{(jm)} - N\underset{1}{\psi}_{(pq)} F^p_j F^q_m - 2\underset{1}{\psi}_{(jm)} + 2\Gamma^p_{\underset{\vee}{qr}}\psi_j F^r_p F^q_m + 2\Gamma^p_{\underset{\vee}{qr}}\psi_m F^r_p F^q_j + 2\Gamma^p_{\underset{\vee}{rj}}\psi_q F^q_p F^r_m + 2\Gamma^p_{\underset{\vee}{rm}}\psi_m F^q_p F^r_j . \tag{2.11}$$

From (2.8) and (2.11) we have

$$(N-2)F^p_j F^q_m \underset{1}{\psi}_{(pq)} = (N-2)\underset{1}{\psi}_{(jm)} + 2\Gamma^p_{\underset{\vee}{mr}}\psi_q F^q_j F^r_p + 2\Gamma^p_{\underset{\vee}{jr}}\psi_q F^q_m F^r_p + 2\Gamma^p_{\underset{\vee}{qr}}\psi_j F^r_p F^q_m + 2\Gamma^p_{\underset{\vee}{qr}}\psi_m F^r_p F^q_j . \tag{2.12}$$

Substituting (2.12) into (2.9) we obtain

$$\begin{aligned}(N+2)\underset{1}{\psi}_{(jm)} = {}& \underset{1}{R}_{(jm)} - \overline{\underset{1}{R}}_{(jm)} - \frac{2}{N-2}\big(N\Gamma^p_{\underset{\vee}{mr}}\psi_q F^q_j F^r_p \\ & + N\Gamma^p_{\underset{\vee}{jr}}\psi_q F^q_m F^r_p + 2\Gamma^p_{\underset{\vee}{qr}}\psi_j F^r_p F^q_m + 2\Gamma^p_{\underset{\vee}{qr}}\psi_m F^r_p F^q_j\big) \\ & - 2\Gamma^p_{\underset{\vee}{mr}}\psi_q F^q_p F^r_j - 2\Gamma^p_{\underset{\vee}{jr}}\psi_q F^q_p F^r_m .\end{aligned} \tag{2.13}$$

Using (2.7) and (2.13) we get

$$\begin{aligned}(N+2)\underset{1}{\psi}_{jm} = {}& \underset{1}{R}_{jm} - \overline{\underset{1}{R}}_{jm} + 2\Gamma^p_{\underset{\vee}{mj}}\psi_p - \frac{2N-2}{N-2}\Gamma^p_{\underset{\vee}{mr}}\psi_q F^q_j F^r_p \\ & - \frac{2}{N-2}\Gamma^p_{\underset{\vee}{jr}}\psi_q F^q_m F^r_p - \frac{2}{N-2}\Gamma^p_{\underset{\vee}{qr}}\psi_j F^r_p F^q_m \\ & - \frac{2}{N-2}\Gamma^p_{\underset{\vee}{qr}}\psi_m F^r_p F^q_j - 2\Gamma^p_{\underset{\vee}{mr}}\psi_q F^q_p F^r_j .\end{aligned} \tag{2.14}$$

Eliminating ψ_i by using the condition

$$\overline{\Gamma}^p_{pj} - \Gamma^p_{pj} = (N+2)\psi_j \tag{2.15}$$

we reduce the equation (2.14) to the form

$$(N+2)\underset{1}{\psi}_{jm} = \underset{1}{R}_{jm} - \overline{\underset{1}{R}}_{jm} + \overline{\underset{1}{P}}_{jm} - \underset{1}{P}_{jm}, \tag{2.16}$$

where we denote

$$\begin{aligned}\underset{1}{P}_{jm} = \frac{2}{N+2}\Big(&\Gamma^p_{\underset{\vee}{mj}}\Gamma^q_{qp} - \frac{N-1}{N-2}\Gamma^p_{\underset{\vee}{mr}}\Gamma^s_{sq}F^q_j F^r_p - \frac{1}{N-2}\Gamma^p_{\underset{\vee}{jr}}\Gamma^s_{sq}F^q_m F^r_p \\ & - \frac{1}{N-2}\Gamma^p_{\underset{\vee}{qr}}\Gamma^s_{sj}F^r_p F^q_m - \frac{1}{N-2}\Gamma^p_{\underset{\vee}{qr}}\Gamma^s_{sm}F^r_p F^q_j - \Gamma^p_{\underset{\vee}{mr}}\Gamma^s_{sq}F^q_p F^r_j\Big).\end{aligned} \tag{2.17}$$

In the same manner the geometric objects $\overline{P}_{\underset{1}{}jm}$ of the space $G\overline{K}_N$ is defined. Eliminating $\underset{1}{\psi}_{jm}$ from (2.4) we obtain

$$HP\underset{1}{\overline{W}}{}^i{}_{jmn} = HP\underset{1}{W}{}^i{}_{jmn}, \tag{2.18}$$

where the magnitude

$$\begin{aligned} HP\underset{1}{W}{}^i{}_{jmn} = \underset{1}{R}{}^i_{jmn} + \frac{1}{N+2}[\delta^i_m(\underset{1}{R}_{jn} - \underset{1}{P}_{jn}) + \delta^i_j(\underset{1}{R}_{[mn]} - \underset{1}{P}_{[mn]}) \\ - \delta^i_n(\underset{1}{R}_{jm} - \underset{1}{P}_{jm}) + F^p_j F^i_n(\underset{1}{R}_{pm} - \underset{1}{P}_{pm}) - F^p_j F^i_m(\underset{1}{R}_{pn} - \underset{1}{P}_{pn}) \\ + F^i_j F^p_n(\underset{1}{R}_{pm} - \underset{1}{P}_{pm}) - F^i_j F^p_m(\underset{1}{R}_{pn} - \underset{1}{P}_{pn}) - 2\Gamma^i_{\underset{\vee}{mn}}\Gamma^q_{qj} - 2\delta^i_j\Gamma^p_{\underset{\vee}{mn}}\Gamma^q_{qp} \\ + 2\Gamma^p_{\underset{\vee}{mn}}\Gamma^s_{sq}F^q_j F^i_p + 2\Gamma^p_{\underset{\vee}{mn}}\Gamma^s_{sq}F^q_p F^i_j] \end{aligned} \tag{2.19}$$

is expressed by geometric objects of the space GK_N. In the same manner the magnitude $HP\underset{1}{\overline{W}}{}^i{}_{jmn}$ is expressed by geometric objects of the space $G\overline{K}_N$. The magnitude $HP\underset{1}{W}{}^i{}_{jmn}$ is not a tensor, and we call it *the equitorsion holomorphically projective parameter of the first kind* of the space GK_N. From the facts given above, we have

Theorem 2.1. *The equitorsion holomorphically projective parameter* (2.19) *is an invariant of equitorsion holomorphically projective mappings* $f\colon GK_N \to G\overline{K}_N$.

2.2. Holomorphically projective parameters of the second kind.

For the curvature tensors $\underset{2}{R}$ and $\underset{2}{\overline{R}}$ of the spaces GK_N and $G\overline{K}_N$ the relation

$$\underset{2}{\overline{R}}{}^i_{jmn} = \underset{2}{R}{}^i_{jmn} + P^i_{mj\underset{2}{|}n} - P^i_{nj\underset{2}{|}m} + P^p_{mj}P^i_{np} - P^p_{nj}P^i_{mp} + 2\Gamma^p_{\underset{\vee}{nm}}P^i_{pj} \tag{2.20}$$

is valid [5], where P^i_{jm} is a deformation tensor. Substituting (1.8) and (2.2) into (2.20) we get

$$\begin{aligned} \underset{2}{\overline{R}}{}^i_{jmn} = \underset{2}{R}{}^i_{jmn} + \delta^i_m\underset{2}{\psi}_{jn} + \delta^i_j\underset{2}{\psi}_{[mn]} - \delta^i_n\underset{2}{\psi}_{jm} \\ + F^p_j(F^i_n\underset{2}{\psi}_{pm} - F^i_m\underset{2}{\psi}_{pn}) + F^i_j(F^p_n\underset{2}{\psi}_{pm} - F^p_m\underset{2}{\psi}_{pn}) \\ + 2\Gamma^p_{\underset{\vee}{nm}}\psi_p\delta^i_j + 2\Gamma^i_{\underset{\vee}{nm}}\psi_j - 2\Gamma^p_{\underset{\vee}{nm}}\psi_q F^q_p F^i_j - 2\Gamma^p_{\underset{\vee}{nm}}\psi_q F^q_j F^i_p, \end{aligned} \tag{2.21}$$

where we denote

$$\underset{2}{\psi}_{ij} = \psi_{i\underset{2}{|}j} - \psi_i\psi_j + \psi_p F^p_i \psi_q F^q_j. \tag{2.22}$$

Contracting by indices i, n in (2.21) we arrive at

$$\overline{\underset{2}{R}}_{jm} = \underset{2}{R}_{jm} + \underset{2}{\psi}_{[mj]} - N\underset{2}{\psi}_{jm} - F^p_j F^q_m \underset{2}{\psi}_{(pq)} + 2\Gamma^p_{\underset{\vee}{jm}}\psi_p - 2\Gamma^p_{\underset{\vee}{rm}}\psi_q F^q_p F^r_j - 2\Gamma^p_{\underset{\vee}{rm}}\psi_q F^q_j F^r_p. \tag{2.23}$$

Anti-symmetrization without division in (2.23) with respect to indices j, m yields

$$(N+2)\underset{2}{\psi}_{[jm]} = \underset{2}{R}_{[jm]} - \overline{\underset{2}{R}}_{[jm]} + 4\Gamma^p_{\underset{\vee}{jm}}\psi_p - 2\Gamma^p_{\underset{\vee}{rm}}\psi_q F^q_p F^r_j + 2\Gamma^p_{\underset{\vee}{rj}}\psi_q F^q_p F^r_m - 2\Gamma^p_{\underset{\vee}{rm}}\psi_q F^q_j F^r_p + 2\Gamma^p_{\underset{\vee}{rj}}\psi_q F^q_m F^r_p. \tag{2.24}$$

Symmetrizing with respect to indices j, m in (2.23) we get

$$\overline{\underset{2}{R}}_{(jm)} = \underset{2}{R}_{(jm)} - N\underset{2}{\psi}_{(jm)} - 2F^p_j F^q_m \underset{2}{\psi}_{(pq)} - 2\Gamma^p_{\underset{\vee}{rm}}\psi_q F^q_p F^r_j - 2\Gamma^p_{\underset{\vee}{rj}}\psi_q F^q_p F^r_m - 2\Gamma^p_{\underset{\vee}{rm}}\psi_q F^q_j F^r_p - 2\Gamma^p_{\underset{\vee}{rj}}\psi_q F^q_m F^r_p. \tag{2.24$'$}$$

For the spaces GK_N and $G\overline{K}_N$the following relations are valid [7]:

$$\underset{2}{R}_{(pq)} F^p_i F^q_j = \underset{2}{R}_{(ij)} - 2\Gamma^p_{\underset{\vee}{rq}}\Gamma^q_{\underset{\vee}{ps}} F^r_j F^s_m + 2\Gamma^p_{\underset{\vee}{jq}}\Gamma^q_{\underset{\vee}{pm}} \tag{2.25}$$

and

$$\overline{\underset{2}{R}}_{(pq)} F^p_i F^q_j = \overline{\underset{2}{R}}_{(ij)} - 2\Gamma^p_{\underset{\vee}{rq}}\Gamma^q_{\underset{\vee}{ps}} F^r_j F^s_m + 2\Gamma^p_{\underset{\vee}{jq}}\Gamma^q_{\underset{\vee}{pm}}. \tag{2.26}$$

By composition with $F^j_p F^m_q$, contracting with respect to indices j, m, and using (2.25), (2.26) we obtain from (2.24$'$) the relation

$$\overline{\underset{2}{R}}_{(jm)} = \underset{2}{R}_{(jm)} - N\underset{2}{\psi}_{(pq)} F^p_j F^q_m - 2\underset{2}{\psi}_{(jm)} + 2\Gamma^p_{\underset{\vee}{jr}}\psi_q F^q_p F^r_m + 2\Gamma^p_{\underset{\vee}{mr}}\psi_q F^q_p F^r_j + 2\Gamma^p_{\underset{\vee}{rq}}\psi_j F^r_p F^q_m + 2\Gamma^p_{\underset{\vee}{rq}}\psi_m F^r_p F^q_j. \tag{2.27}$$

From (2.24$'$) and (2.27) we get

$$(N-2)F^p_j F^q_m \underset{2}{\psi}_{(pq)} = (N-2)\underset{2}{\psi}_{(jm)} + 2\Gamma^p_{\underset{\vee}{rm}}\psi_q F^q_j F^r_p + 2\Gamma^p_{\underset{\vee}{rj}}\psi_q F^q_m F^r_p + 2\Gamma^p_{\underset{\vee}{rq}}\psi_j F^r_p F^q_m + 2\Gamma^p_{\underset{\vee}{rq}}\psi_m F^r_p F^q_j. \tag{2.28}$$

Substituting (2.28) in (2.27) we conclude that

$$\begin{aligned}(N+2)\underset{2}{\psi}_{(jm)} = \underset{2}{R}_{(jm)} - \overline{\underset{2}{R}}_{(jm)} &- \frac{2}{N-2}\big(N\Gamma^p_{\underset{\vee}{rm}}\psi_q F^q_j F^r_p \\ &+ N\Gamma^p_{\underset{\vee}{rj}}\psi_q F^q_m F^r_p + 2\Gamma^p_{\underset{\vee}{rq}}\psi_j F^r_p F^q_m + 2\Gamma^p_{\underset{\vee}{rq}}\psi_m F^r_p F^q_j\big) \\ &- 2\Gamma^p_{\underset{\vee}{rm}}\psi_q F^q_p F^r_j - 2\Gamma^p_{\underset{\vee}{rj}}\psi_q F^q_p F^r_m.\end{aligned} \tag{2.29}$$

From (2.24) and (2.29) we get

$$(2.30)\qquad \begin{aligned}(N+2)\underset{2}{\psi}_{jm} &= \underset{2}{R}_{jm} - \underset{2}{\overline{R}}_{jm} + 2\Gamma^p_{\underset{\vee}{jm}}\psi_p - 2\Gamma^p_{\underset{\vee}{rm}}\psi_q F^q_p F^r_j \\ &- \frac{2N-2}{N-2}\Gamma^p_{\underset{\vee}{rm}}\psi_q F^q_j F^r_p - \frac{2}{N-2}\Gamma^p_{\underset{\vee}{rj}}\psi_q F^q_m F^r_p \\ &- \frac{2}{N-2}\Gamma^p_{\underset{\vee}{rq}}\psi_j F^r_p F^q_m - \frac{2}{N-2}\Gamma^p_{\underset{\vee}{rq}}\psi_m F^r_p F^q_j.\end{aligned}$$

Eliminating ψ_i by condition (2.15) we reduce the last equation to the form

$$(2.31)\qquad (N+2)\underset{2}{\psi}_{jm} = \underset{2}{R}_{jm} - \underset{2}{\overline{R}}_{jm} + \underset{2}{\overline{P}}_{jm} - \underset{2}{P}_{jm},$$

where we denote

$$(2.32)\quad \begin{aligned}\underset{2}{P}_{jm} = \frac{2}{N+2}\Big(&\Gamma^p_{\underset{\vee}{jm}}\Gamma^q_{qp} - \Gamma^p_{\underset{\vee}{rm}}\Gamma^s_{sq}F^q_p F^r_j - \frac{N-1}{N-2}\Gamma^p_{\underset{\vee}{rm}}\Gamma^s_{sq}F^q_j F^r_p \\ &- \frac{1}{N-2}\Gamma^p_{\underset{\vee}{rj}}\Gamma^s_{sq}F^q_m F^r_p - \frac{1}{N-2}\Gamma^p_{\underset{\vee}{rq}}\Gamma^s_{sj}F^r_p F^q_m - \frac{1}{N-2}\Gamma^p_{\underset{\vee}{rq}}\Gamma^s_{sm}q F^r_p F^q_j\Big).\end{aligned}$$

Eliminating $\underset{2}{\psi}_{jm}$ from (2.21) we get

$$(2.33)\qquad HP\underset{2}{\overline{W}}{}^i{}_{jmn} = HP\underset{2}{W}{}^i{}_{jmn},$$

where

$$(2.34)\quad \begin{aligned}HP\underset{2}{W}{}^i{}_{jmn} &= \underset{2}{R}{}^i_{jmn} + \frac{1}{N+2}[\delta^i_m(\underset{2}{R}_{jn} - \underset{2}{P}_{jn}) + \delta^i_j(\underset{2}{R}_{[mn]} - \underset{2}{P}_{[mn]}) \\ &- \delta^i_n(\underset{2}{R}_{jm} - \underset{2}{P}_{jm}) + F^p_j F^i_n(\underset{2}{R}_{pm} - \underset{2}{P}_{pm}) - F^p_j F^i_m(\underset{2}{R}_{pn} - \underset{2}{P}_{pn}) \\ &+ F^i_j F^p_n(\underset{2}{R}_{pm} - \underset{2}{P}_{pm}) - F^i_j F^p_m(\underset{2}{R}_{pn} - \underset{2}{P}_{pn}) - 2\delta^i_j\Gamma^p_{\underset{\vee}{nm}}\Gamma^q_{qp} - 2\Gamma^i_{\underset{\vee}{nm}}\Gamma^q_{qj} \\ &+ 2\Gamma^p_{\underset{\vee}{nm}}\Gamma^s_{sq}F^q_p F^i_j + 2\Gamma^p_{\underset{\vee}{nm}}\Gamma^s_{sq}F^q_j F^i_p].\end{aligned}$$

The magnitude $HP\underset{2}{W}{}^i{}_{jmn}$ is not a tensor, and we call it *the equitorsion holomorphically projective parameter of the second kind* of the space GK_N. From the facts given above, we have

Theorem 2.2. *The equitorsion holomorphically projective parameter of the second kind is an invariant of equitorsion holomorphically projective mappings of the spaces GK_N and $G\overline{K}_N$.*

2.3. Holomorphically projective parameters of the third kind.

The curvature tensors $\underset{3}{R}$ and $\underset{3}{\overline{R}}$ of the space GK_N and $G\overline{K}_N$ satisfy the relation [5]

$$\underset{3}{\overline{R}}{}^i_{jmn} = \underset{3}{R}{}^i_{jmn} + P^i_{jm\underset{2}{|}n} - P^i_{nj\underset{1}{|}m} + P^p_{jm}P^i_{np} - P^p_{nj}P^i_{pm} + 2P^p_{nm}\Gamma^i_{\underset{\vee}{pj}} + 2\Gamma^p_{\underset{\vee}{nm}}P^i_{\underset{\vee}{pj}}. \tag{2.35}$$

Substituting (1.8) and (2.2) in (2.35) we get

$$\begin{aligned}\underset{3}{\overline{R}}{}^i_{jmn} = {} & \underset{3}{R}{}^i_{jmn} + \delta^i_m \underset{2}{\psi}_{jn} + \delta^i_j(\underset{2}{\psi}_{mn} - \underset{1}{\psi}_{nm}) - \delta^i_n \underset{1}{\psi}_{jm} \\ & + F^p_j(F^i_n \underset{1}{\psi}_{pm} - F^i_m \underset{2}{\psi}_{pn}) + F^i_j(F^p_n \underset{1}{\psi}_{pm} - F^p_m \underset{2}{\psi}_{pn}) \\ & + 2\Gamma^i_{\underset{\vee}{mj}}\psi_n + 2\Gamma^i_{\underset{\vee}{nj}}\psi_m - 2\Gamma^i_{\underset{\vee}{pj}}\psi_q F^q_n F^p_m - 2\Gamma^i_{\underset{\vee}{pj}}\psi_q F^q_m F^p_n,\end{aligned} \tag{2.36}$$

where we denote

$$\underset{\theta}{\psi}_{ij} = \psi_{i\underset{\theta}{|}j} - \psi_i\psi_j + \psi_p F^p_i \psi_q F^q_j, \quad (\theta = 1, 2). \tag{2.37}$$

The following equality is also valid [5]:

$$\underset{2}{\psi}_{[mn]} = \underset{1}{\psi}_{[mn]} + 2\Gamma^p_{\underset{\vee}{mn}}\psi_p.$$

From the equation (2.36) we get

$$\begin{aligned}\underset{3}{\overline{R}}{}^i_{jmn} = {} & \underset{3}{R}{}^i_{jmn} + \delta^i_m \underset{1}{\psi}_{jn} + \delta^i_j(\underset{1}{\psi}_{mn} - \underset{1}{\psi}_{nm}) - \delta^i_n \underset{1}{\psi}_{jm} \\ & + F^p_j(F^i_n \underset{1}{\psi}_{pm} - F^i_m \underset{1}{\psi}_{pn}) + F^i_j(F^p_n \underset{1}{\psi}_{pm} - F^p_m \underset{1}{\psi}_{pn}) \\ & + 2\Gamma^p_{\underset{\vee}{jn}}\psi_p\delta^i_m + 2\Gamma^p_{\underset{\vee}{mn}}\psi_p\delta^i_j - 2\Gamma^q_{\underset{\vee}{pn}}\psi_q F^p_j F^i_m - 2\Gamma^q_{\underset{\vee}{pn}}\psi_q F^i_j F^p_m \\ & + 2\Gamma^i_{\underset{\vee}{mj}}\psi_n + 2\Gamma^i_{\underset{\vee}{nj}}\psi_m - 2\Gamma^i_{\underset{\vee}{pj}}\psi_q F^q_n F^p_m - 2\Gamma^i_{\underset{\vee}{pj}}\psi_q F^q_m F^p_n.\end{aligned} \tag{2.38}$$

Contracting with respect to i, n in (2.38) we arrive at

$$\begin{aligned}\underset{3}{\overline{R}}_{jm} = {} & \underset{3}{R}_{jm} + \underset{1}{\psi}_{[mj]} - N\underset{1}{\psi}_{jm} - F^p_j F^q_m \underset{1}{\psi}_{(pq)} \\ & + 2\Gamma^p_{\underset{\vee}{mj}}\psi_p - 2\Gamma^r_{\underset{\vee}{pm}}\psi_q F^q_r F^p_m - 2\Gamma^r_{\underset{\vee}{pj}}\psi_q F^q_m F^p_r.\end{aligned} \tag{2.39}$$

By anti-symmetrization without division (2.39) with respect to j, m we get

$$\begin{aligned}(N+2)\underset{1}{\psi}_{[jm]} = {} & \underset{3}{R}_{[jm]} - \underset{3}{\overline{R}}_{[jm]} + 4\Gamma^p_{\underset{\vee}{mj}}\psi_p - 2\Gamma^r_{\underset{\vee}{pj}}\psi_q F^q_r F^p_m \\ & + 2\Gamma^q_{\underset{\vee}{pm}}\psi_q F^q_r F^p_j - 2\Gamma^r_{\underset{\vee}{pj}}\psi_q F^q_m F^p_r + 2\Gamma^q_{\underset{\vee}{pm}}\psi_q F^q_j F^p_r.\end{aligned} \tag{2.40}$$

Symmetrizing with respect to j, m in (2.39) we obtain

$$
\begin{aligned}
(2.41)\qquad \underset{3}{\overline{R}}_{(jm)} = \underset{3}{R}_{(jm)} - N\underset{1}{\psi}_{(jm)} - 2F^p_j F^q_m \underset{1}{\psi}_{(pq)} - 2\Gamma^q_{\underset{\vee}{pj}}\psi_q F^q_r F^p_m \\
- 2\Gamma^q_{\underset{\vee}{pm}}\psi_q F^q_r F^p_j - 2\Gamma^q_{\underset{\vee}{pj}}\psi_q F^q_m F^p_r - 2\Gamma^r_{\underset{\vee}{pm}}\psi_q F^q_j F^p_r.
\end{aligned}
$$

By composition with $F^j_p F^m_q$, contraction with respect to j, m, and using the conditions [7], one obtains

$$
(2.42)\qquad \underset{3}{R}_{(pq)} F^p_i F^q_j = \underset{3}{R}_{(ij)} - 2\Gamma^p_{\underset{\vee}{rq}}\Gamma^q_{\underset{\vee}{ps}} F^r_j F^s_m + 2\Gamma^p_{\underset{\vee}{jq}}\Gamma^q_{\underset{\vee}{pm}}
$$

and

$$
(2.43)\qquad \underset{3}{\overline{R}}_{(pq)} F^p_i F^q_j = \underset{3}{\overline{R}}_{(ij)} - 2\Gamma^p_{\underset{\vee}{rq}}\Gamma^q_{\underset{\vee}{ps}} F^r_j F^s_m + 2\Gamma^p_{\underset{\vee}{jq}}\Gamma^q_{\underset{\vee}{pm}}.
$$

From (2.41) we have

$$
\begin{aligned}
(2.44)\qquad \underset{3}{\overline{R}}_{(jm)} = \underset{3}{R}_{(jm)} - N\underset{1}{\psi}_{(pq)} F^p_j F^q_m - 2\underset{1}{\psi}_{(jm)} + 2\Gamma^r_{\underset{\vee}{mp}}\psi_q F^q_r F^p_j \\
+ 2\Gamma^r_{\underset{\vee}{jp}}\psi_q F^q_r F^r_m + 2\Gamma^r_{\underset{\vee}{pq}}\psi_m F^p_r F^q_j + 2\Gamma^r_{\underset{\vee}{pq}}\psi_j F^p_r F^q_m
\end{aligned}
$$

and (2.41) and (2.44) imply

$$
\begin{aligned}
(2.45)\qquad (N-2)F^p_j F^q_m \underset{1}{\psi}_{(pq)} = (N-2)\underset{1}{\psi}_{(jm)} + 2\Gamma^r_{\underset{\vee}{pq}}\psi_m F^p_r F^q_j \\
+ 2\Gamma^r_{\underset{\vee}{pq}}\psi_j F^p_r F^q_m + 2\Gamma^r_{\underset{\vee}{pj}}\psi_q F^q_m F^p_r + 2\Gamma^r_{\underset{\vee}{pm}}\psi_q F^q_j F^p_r.
\end{aligned}
$$

Substituting (2.45) into (2.44) we have

$$
\begin{aligned}
(2.46)\qquad (N+2)\underset{1}{\psi}_{(jm)} = \underset{3}{R}_{(jm)} - \underset{3}{\overline{R}}_{(jm)} - \frac{2}{N-2}(N\Gamma^q_{\underset{\vee}{pj}}\psi_q F^q_m F^p_r \\
+ N\Gamma^q_{\underset{\vee}{pm}}\psi_q F^q_j F^p_r + 2\Gamma^r_{\underset{\vee}{pq}}\psi_m F^p_r F^q_j + 2\Gamma^r_{\underset{\vee}{pq}}\psi_j F^p_r F^q_m) \\
- 2\Gamma^r_{\underset{\vee}{pj}}\psi_q F^q_r F^p_m - 2\Gamma^r_{\underset{\vee}{pm}}\psi_q F^q_r F^p_j.
\end{aligned}
$$

From (2.40) and (2.46) we get

$$
\begin{aligned}
(2.47)\qquad (N+2)\underset{1}{\psi}_{jm} = \underset{3}{R}_{jm} - \underset{3}{\overline{R}}_{jm} + 2\Gamma^p_{\underset{\vee}{mj}}\psi_p - \frac{2N-2}{N-2}\Gamma^r_{\underset{\vee}{pj}}\psi_q F^q_m F^p_r \\
- \frac{2}{N-2}\Gamma^r_{\underset{\vee}{pm}}\psi_q F^q_j F^p_r - \frac{2}{N-2}\Gamma^r_{\underset{\vee}{pq}}\psi_m F^p_r F^q_j \\
- \frac{2}{N-2}\Gamma^r_{\underset{\vee}{pq}}\psi_j F^p_r F^q_m - \Gamma^q_{\underset{\vee}{pj}}\psi_q F^q_r F^p_m.
\end{aligned}
$$

Eliminating ψ_i from the last equation we have

(2.48) $$(N+2)\underset{1}{\psi}_{jm} = \underset{3}{R}_{jm} - \underset{3}{\overline{R}}_{jm} + \underset{3}{\overline{P}}_{jm} - \underset{3}{P}_{jm},$$

where we denote

(2.49) $$\underset{3}{P}_{jm} = \frac{2}{N+2}\Big(\Gamma^p_{\underset{\vee}{mj}}\Gamma^q_{qp} - \frac{N-1}{N-2}\Gamma^q_{\underset{\vee}{pj}}\Gamma^s_{sq}F^q_m F^p_r - \frac{1}{N-2}\Gamma^r_{\underset{\vee}{pm}}\Gamma^s_{sq}F^q_j F^p_r \\ - \frac{1}{N-2}\Gamma^r_{\underset{\vee}{pq}}\Gamma^s_{sm}F^p_r F^q_j - \frac{1}{N-2}\Gamma^r_{\underset{\vee}{pq}}\Gamma^s_{sj}F^p_r F^s_m - \Gamma^r_{\underset{\vee}{pj}}\Gamma^s_{sq}F^q_r F^p_m\Big).$$

Eliminating $\underset{1}{\psi}_{jm}$ from (2.36) we obtain

(20.50) $$HP\underset{3}{\overline{W}}{}^i{}_{jmn} = HP\underset{3}{W}{}^i{}_{jmn},$$

where

(2.51) $$HP\underset{3}{W}{}^i{}_{jmn} = \underset{3}{R}{}^i_{jmn} + \frac{1}{N+2}[\delta^i_m(\underset{3}{R}_{jn} - \underset{3}{P}_{jn}) + \delta^i_j(\underset{3}{R}_{[mn]} - \underset{3}{P}_{[mn]}) \\ - \delta^i_n(\underset{3}{R}_{jm} - \underset{3}{P}_{jm}) + F^p_j F^i_n(\underset{3}{R}_{pm} - \underset{3}{P}_{pm}) - F^p_j F^i_m(\underset{3}{R}_{pn} - \underset{3}{P}_{pn}) \\ + F^i_j F^p_n(\underset{3}{R}_{pm} - \underset{3}{P}_{pm}) - F^i_j F^p_m(\underset{3}{R}_{pn} - \underset{3}{P}_{pn}) - 2\Gamma^p_{\underset{\vee}{jn}}\Gamma^q_{qp}\delta^i_m \\ - 2\Gamma^p_{\underset{\vee}{mn}}\Gamma^q_{qp}\delta^i_j + 2\Gamma^q_{\underset{\vee}{pn}}\Gamma^s_{sq}F^p_j F^i_m + 2\Gamma^q_{\underset{\vee}{pn}}\Gamma^s_{sq}F^i_j F^p_m - 2\Gamma^i_{\underset{\vee}{mj}}\Gamma^p_{pn} \\ - 2\Gamma^i_{\underset{\vee}{nj}}\Gamma^p_{pm} + 2\Gamma^i_{\underset{\vee}{pj}}\Gamma^s_{sq}F^q_n F^p_m + 2\Gamma^i_{\underset{\vee}{pj}}\Gamma^s_{sq}F^q_m F^p_n].$$

The magnitude $HP\underset{3}{W}{}^i{}_{jmn}$ is not a tensor and we call it *the equitorsion holomorphically projective parameter of the third kind* of the space GK_N. By virtue of the facts given above we have the following result:

Theorem 2.3. *The equitorsion holomorphically projective parameter of the third kind is an invariant of equitorsion holomorphically projective mappings of generalized Kählerian spaces.*

2.4. Holomorphically projective parameters of the fourth kind. For curvature tensors $\underset{4}{R}$ and $\underset{4}{\overline{R}}$ of spaces GK_N and $G\overline{K}_N$ the following relation is valid [5]:

(2.52) $$\underset{4}{\overline{R}}{}^i_{jmn} = \underset{4}{R}{}^i_{jmn} + P^i_{jm\underset{2}{|}n} - P^i_{nj\underset{1}{|}m} + P^p_{jm}P^i_{np} - P^p_{nj}P^i_{pm} \\ + 2P^p_{mn}\Gamma^i_{\underset{\vee}{pj}} + 2\Gamma^p_{\underset{\vee}{mn}}P^i_{\underset{\vee}{pj}}.$$

where P^i_{jm} is the deformation tensor. Substituting (1.8) and (2.2) in (2.52) we have

$$
\begin{aligned}
(2.53)\qquad \overline{\underset{4}{R}}{}^i_{jmn} = {} & \underset{4}{R}{}^i_{jmn} + \delta^i_m \underset{2}{\psi}_{jn} + \delta^i_j(\underset{2}{\psi}_{mn} - \underset{1}{\psi}_{nm}) - \delta^i_n \underset{1}{\psi}_{jm} \\
& + F^p_j(F^i_n \underset{1}{\psi}_{pm} - F^i_m \underset{2}{\psi}_{pn}) + F^i_j(F^p_n \underset{1}{\psi}_{pm} - F^p_m \underset{2}{\psi}_{pn}) \\
& + 2\Gamma^i_{\underset{\vee}{mj}}\psi_n + 2\Gamma^i_{\underset{\vee}{nj}}\psi_m - 2\Gamma^i_{\underset{\vee}{pj}}\psi_q F^q_n F^p_m - 2\Gamma^i_{\underset{\vee}{pj}}\psi_q F^q_m F^p_n .
\end{aligned}
$$

In the same manner as in the previous cases we get

$$
(2.54)\qquad HP\overline{\underset{4}{W}}{}^i{}_{jmn} = HP\underset{4}{W}{}^i{}_{jmn},
$$

where we denote

$$
\begin{aligned}
(2.55)\ HP\underset{4}{W}{}^i{}_{jmn} = {} & \underset{4}{R}{}^i_{jmn} + \frac{1}{N+2}\Big[\delta^i_m(\underset{4}{R}_{jn} - \underset{3}{P}_{jn}) + \delta^i_j(\underset{4}{R}_{[mn]} - \underset{3}{P}_{[mn]}) \\
& - \delta^i_n(\underset{4}{R}_{jm} - \underset{3}{P}_{jm}) + F^p_j F^i_n(\underset{4}{R}_{pm} - \underset{3}{P}_{pm}) - F^p_j F^i_m(\underset{4}{R}_{pn} - \underset{3}{P}_{pn}) \\
& + F^i_j F^p_n(\underset{4}{R}_{pm} - \underset{3}{P}_{pm}) - F^i_j F^p_m(\underset{4}{R}_{pn} - \underset{3}{P}_{pn}) - 2\Gamma^p_{\underset{\vee}{jn}}\Gamma^q_{qp}\delta^i_m \\
& - 2\Gamma^p_{\underset{\vee}{mn}}\Gamma^q_{qp}\delta^i_j + 2\Gamma^q_{\underset{\vee}{pn}}\Gamma^s_{sq}F^p_j F^i_m + 2\Gamma^q_{\underset{\vee}{pn}}\Gamma^s_{sq}F^i_j F^p_m - 2\Gamma^i_{\underset{\vee}{mj}}\Gamma^p_{pn} \\
& - 2\Gamma^i_{\underset{\vee}{nj}}\Gamma^p_{pm} + 2\Gamma^i_{\underset{\vee}{pj}}\Gamma^s_{sq}F^q_n F^p_m + 2\Gamma^i_{\underset{\vee}{pj}}\Gamma^s_{sq}F^q_m F^p_n\Big].
\end{aligned}
$$

The magnitude $HP\underset{4}{W}{}^i{}_{jmn}$ is not a tensor either, and we call it *the equitorsion projective parameter of the fourth kind* of the space GK_N. In this case we have

Theorem 2.4. *The equitorsion holomorphically projective parameter of the fourth kind is an invariant of equitorsion holomorphically projective mappings of generalized Kählerian spaces GK_N and $G\overline{K}_N$.*

2.5. Holomorphically projective tensor.

For curvature tensors of the fifth kind $\underset{5}{R}$ and $\overline{\underset{5}{R}}$ of the spaces GK_N and $G\overline{K}_N$ the following relation is valid:

$$
\begin{aligned}
(2.56)\qquad \overline{\underset{5}{R}}{}^i_{jmn} = {} & \underset{5}{R}{}^i_{jmn} + \frac{1}{2}(P^i_{jm\underset{1}{|}n} - P^i_{jn\underset{2}{|}m} + P^i_{mj\underset{2}{|}n} - P^i_{nj\underset{1}{|}m} + P^p_{jm}P^i_{pn} \\
& - P^p_{jn}P^i_{mp} + P^p_{mj}P^i_{np} - P^p_{nj}P^i_{pm} + 4\Gamma^p_{\underset{\vee}{jn}}P^i_{\underset{\vee}{pm}} + 4\Gamma^p_{\underset{\vee}{jm}}P^i_{\underset{\vee}{pn}}).
\end{aligned}
$$

Substituting (1.8) and (2.2) in (2.56) we have

$$
\begin{aligned}
(2.57)\qquad \overline{\underset{5}{R}}{}^i_{jmn} = {} & \underset{5}{R}{}^i_{jmn} + \delta^i_m \underset{12}{\psi}_{jn} + \delta^i_j \underset{12}{\psi}_{[mn]} - \delta^i_n \underset{12}{\psi}_{jm} \\
& + F^p_j(F^i_n \underset{12}{\psi}_{pm} - F^i_m \underset{12}{\psi}_{pn}) + F^i_j(F^p_n \underset{12}{\psi}_{pm} - F^p_m \underset{12}{\psi}_{pn}),
\end{aligned}
$$

where we denote

$$(2.58) \qquad \underset{12}{\psi}_{jm} = \frac{1}{2}(\psi_{j\underset{1}{|}m} + \psi_{j\underset{2}{|}m}) - \psi_j\psi_m + \psi_p F^p_j \psi_q F^q_m.$$

Contracting with respect to i, n in (2.57) we get

$$(2.59) \qquad \underset{5}{\overline{R}}_{jm} = \underset{5}{R}_{jm} - \underset{12}{\psi}_{[jm]} - N\underset{12}{\psi}_{jm} - F^p_j F^q_m \underset{12}{\psi}_{(pq)}.$$

Using anti-symmetrization without division in (2.59) with respect to j, m we get

$$(2.60) \qquad (N+2)\underset{1}{\psi}_{[jm]} = \underset{5}{R}_{[jm]} - \underset{5}{\overline{R}}_{[jm]}.$$

Symmetrizing without division with respect to j, m in (2.59) we have

$$(2.61) \qquad \underset{5}{\overline{R}}_{(jm)} = \underset{5}{R}_{(jm)} - N\underset{12}{\psi}_{(jm)} - 2F^p_j F^q_m \underset{12}{\psi}_{(pq)}.$$

By composition with $F^j_p F^m_q$, contracting with respect to j, m, and using the relations

$$(2.62) \qquad \underset{5}{R}_{(pq)} F^p_i F^q_j = \underset{5}{R}_{(ij)} + 2\Gamma^p_{\underset{\vee}{rq}}\Gamma^q_{\underset{\vee}{ps}} F^r_j F^s_m - 2\Gamma^p_{\underset{\vee}{jq}}\Gamma^q_{\underset{\vee}{pm}}$$

and

$$(2.63) \qquad \underset{5}{\overline{R}}_{(pq)} F^p_i F^q_j = \underset{5}{\overline{R}}_{(ij)} + 2\Gamma^p_{\underset{\vee}{rq}}\Gamma^q_{\underset{\vee}{ps}} F^r_j F^s_m - 2\Gamma^p_{\underset{\vee}{jq}}\Gamma^q_{\underset{\vee}{pm}},$$

one obtains from (2.61) the equality

$$(2.64) \qquad \underset{5}{\overline{R}}_{(jm)} = \underset{5}{R}_{(jm)} - N\underset{12}{\psi}_{(pq)} F^p_j F^q_m - 2\underset{12}{\psi}_{(jm)}.$$

From (2.61) and (2.64) we obtain

$$(2.65) \qquad F^p_j F^q_m \underset{12}{\psi}_{(pq)} = \underset{12}{\psi}_{(jm)}.$$

Substituting (2.65) in (2.64) we have

$$(2.66) \qquad (N+2)\underset{12}{\psi}_{(jm)} = \underset{5}{R}_{(jm)} - \underset{5}{\overline{R}}_{(jm)}.$$

From (2.61) and (2.66) one gets

$$(2.67) \qquad (N+2)\underset{12}{\psi}_{jm} = \underset{5}{R}_{jm} - \underset{5}{\overline{R}}_{jm}.$$

Eliminating $\underset{12}{\psi}_{jm}$ from (2.57) we have

$$HP\overline{\underset{5}{W}}{}^{i}{}_{jmn} = HP\underset{5}{W}{}^{i}{}_{jmn}, \tag{2.68}$$

where

$$HP\underset{5}{W}{}^{i}{}_{jmn} = \underset{5}{R}{}^{i}_{jmn} + \frac{1}{N+2}[\delta^{i}_{m}\underset{5}{R}_{jn} + \delta^{i}_{j}\underset{5}{R}_{[mn]} - \delta^{i}_{n}\underset{5}{R}_{jm} + F^{p}_{j}(F^{i}_{n}\underset{5}{R}_{pm} - F^{i}_{m}\underset{5}{R}_{pn}) + F^{i}_{j}(F^{p}_{n}\underset{5}{R}_{pm} - F^{p}_{m}\underset{5}{R}_{pn})]. \tag{2.69}$$

Contrary to the previous cases the magnitude $HP\underset{5}{W}{}^{i}{}_{jmn}$ is a tensor and we call it *the equitorsion holomorphically projective tensor* of the space GK_N.

Based on the facts given above, we have the

Theorem 2.5. *The equitorsion holomorphically projective tensor is an invariant of equitorsion holomorphically projective mappings of generalized Kählerian spaces.*

2.6. The case of Kählerian spaces.

In the case of holomorphically projective mappings of Kählerian spaces the magnitudes $HP\underset{\theta}{W}{}^{i}{}_{jmn}$, $(\theta = 1, \dots, 5)$, given by (2.19, 34, 51, 55, 69) reduce to the holomorphically projective curvature tensor [10]

$$HPW^{i}{}_{jmn} = R^{i}_{jmn} + \frac{1}{N+2}(R_{j[n}\delta^{i}_{m]} + F^{p}_{j}R_{p[m}F^{i}_{n]} + 2F^{i}_{j}F^{p}_{n}R_{pm}).$$

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