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Czechoslovak Mathematical Journal, Vol. 54 (2004), No. 3, 609–621

Persistent URL: <http://dml.cz/dmlcz/127915>

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AN ALGEBRAIC VERSION OF THE CANTOR-BERNSTEIN-SCHRÖDER THEOREM

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(Received October 8, 2001)

Abstract. The Cantor-Bernstein-Schröder theorem of the set theory was generalized by Sikorski and Tarski to σ -complete boolean algebras, and recently by several authors to other algebraic structures. In this paper we expose an abstract version which is applicable to algebras with an underlying lattice structure and such that the central elements of this lattice determine a direct decomposition of the algebra. Necessary and sufficient conditions for the validity of the Cantor-Bernstein-Schröder theorem for these algebras are given. These results are applied to obtain versions of the Cantor-Bernstein-Schröder theorem for σ -complete orthomodular lattices, Stone algebras, BL -algebras, MV -algebras, pseudo MV -algebras, Łukasiewicz and Post algebras of order n .

Keywords: lattices, central elements, factor congruences, varieties

MSC 2000: 06B99, 08B99

0. INTRODUCTION

The famous Cantor-Bernstein-Schröder theorem (CBS theorem, for short) states that, if a set X can be embedded into a set Y and viceversa, then there is a one-to-one function of X onto Y . At the end of the forties, Sikorski [20] (see also Tarski [21]) showed that the CBS theorem is a particular case of a statement on σ -complete boolean algebras. Recently several authors extended Sikorski's result to classes of algebras more general than boolean algebras, like orthomodular lattices [9], MV -algebras [8], pseudo MV -algebras [15].

During the preparation of this paper the author was supported by a Fellowship from the FOMEC Program.

The aim of this paper is to give a general algebraic frame for the validity of the CBS theorem, from which all the versions mentioned above can be derived, as well as versions of the CBS theorem for residuated lattices [18], Stone algebras [1], *BL*-algebras [13], Lukasiewicz and Post algebras of order n [1], [3].

The abstract frame for the CBS theorem is given by the \mathcal{L} -varieties of algebras, introduced in Section 1. In Section 2 we show that there are many examples of \mathcal{L} -varieties. Necessary and sufficient conditions for the validity of the CBS theorem in algebras belonging to an \mathcal{L} -variety are given in Section 3, which is the main section of this paper. In Section 4 we look for some simple global conditions on algebras of an \mathcal{L} -variety that are sufficient for the validity of the CBS theorem. Finally, in Section 5 we give a version of the CBS theorem for partially ordered sets.

1. BASIC NOTIONS

We recall from [2], [19] some notions of the lattice theory that will play an important role in what follows. Let $L = \langle L, \vee, \wedge \rangle$ be a lattice. Given a, b, c in L , we write: $(a, b, c)D$ iff $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$; $(a, b, c)D^*$ iff $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ and $(a, b, c)T$ iff $(a, b, c)D, (a, b, c)D^*$ hold for all permutations of a, b, c . In this case we say that $\{a, b, c\}$ is a *distributive triple*. A lattice L is called *bounded* provided it has a smallest element 0 and a greatest element 1. An element z of a lattice L is called a *neutral element* iff for all elements $a, b \in L$ we have $(a, b, z)T$. An element z of a bounded lattice is called a *central element* iff z is a neutral element having a complement, which we shall denote by $\neg z$. The set of all central elements of L is called the *center* of L and is denoted by $Z(L)$. An interval $[a, b]$ of a lattice A is defined as the set $\{x \in A : a \leq x \leq b\}$. A sequence $(a_n)_{n \in \omega}$ of elements of a lattice L with 0 is called *orthogonal* iff $a_n \wedge a_m = 0$ whenever m, n are distinct elements. In particular, L is called *orthogonally σ -complete* iff, for all orthogonal sequences $(a_n)_{n \in \omega}$, $\bigvee_{n \in \omega} a_n$ exists. A subset S of L is called a *σ -sublattice* of L when it contains with any countable subset X also $\bigwedge X$ and $\bigvee X$.

Proposition 1.1. *For each bounded lattice L , its center $Z(L)$ is a boolean sublattice of L .*

Notation. The supremum (infimum) in $Z(L)$ of a family $(a_i)_{i \in I}$ of $Z(L)$, if it exists, will be denoted by $\bigsqcup_{i \in I} a_i$ ($\bigsqcap_{i \in I} a_i$), to distinguish it from the supremum $\bigvee_{i \in I} a_i$ (infimum $\bigwedge_{i \in I} a_i$) in L , which need not belong to $Z(L)$.

Definition 1.2. A variety \mathcal{V} of algebras is an \mathcal{L} -variety iff

- (1) there are terms of the language of \mathcal{V} defining on each $A \in \mathcal{V}$ operations $\vee, \wedge, 0, 1$ such that $L(A) = \langle A, \vee, \wedge, 0, 1 \rangle$ is a bounded lattice;
- (2) for all $A \in \mathcal{V}$ and for all $z \in Z(L(A))$, the binary relation Θ_z on A defined by $a\Theta_z b$ iff $a \wedge z = b \wedge z$ is a congruence on A , such that $A \cong A/\Theta_z \times A/\Theta_{\neg z}$.

For an algebra A in an \mathcal{L} -variety, we will write simply $Z(A)$ instead of $Z(L(A))$.

Observe that each subvariety of an \mathcal{L} -variety is an \mathcal{L} -variety.

Definition 1.3. Let \mathcal{V} be an \mathcal{L} -variety of algebras of similarity type τ . For all $A \in \mathcal{V}$, all $z \in Z(A)$ and all operation symbols $f \in \tau$, we define $f_z(x_1, \dots, x_n) = z \wedge f(x_1, \dots, x_n)$, where n is the arity of f . Moreover, we define $[0, z]_A = \langle [0, z], (f_z)_{f \in \tau} \rangle$.

Taking into account that for each $f \in \tau$ of arity n and elements x_1, \dots, x_n in A , $x_i\Theta_z(x_i \wedge z)$ for $i = 1 \dots n$; we have $f(x_1, \dots, x_n)\Theta_z f(x_1 \wedge z, \dots, x_n \wedge z)$, i.e., $f(x_1 \wedge z, \dots, x_n \wedge z) \wedge z = f(x_1, \dots, x_n) \wedge z$. Now it is easy to prove the following result:

Proposition 1.4. *The correspondence $x/\Theta_z \mapsto x \wedge z$ defines an isomorphism from A/Θ_z onto $[0, z]_A$. Moreover, the correspondence $x \mapsto (x \wedge z, x \wedge \neg z)$ defines an isomorphism from A onto $A/\theta_z \times A/\theta_{\neg z}$.*

2. EXAMPLES OF \mathcal{L} -VARIETIES

Example 2.1. *The variety \mathcal{L}_{01} of bounded lattices and its subvarieties.* In particular, the subvarieties of modular and of distributive lattices.

Example 2.2. *A lattice with involution [16] is an algebra $\langle L, \vee, \wedge, \sim \rangle$ such that $\langle L, \vee, \wedge \rangle$ is a lattice and \sim is a unary operation on L that fulfils the following conditions:*

- (i) $\sim \sim x = x$ and
- (ii) $\sim (x \vee y) = \sim x \wedge \sim y$.

The variety \mathcal{L}_i of bounded lattices with involution which satisfy the *Kleene equation* (iii) $x \wedge \sim x = (x \wedge \sim x) \wedge (y \vee \sim y)$ is an \mathcal{L} -variety. Indeed, suppose $L \in \mathcal{L}_i$ and let $z \in Z(L)$. It is clear that Θ_z is a lattice congruence. To see that Θ_z also preserves the operation \sim , observe first that $\sim z = \neg z$. Indeed, we have

$$\begin{aligned} \neg z &= \neg z \wedge 1 = \neg z \wedge (\sim z \vee \sim \neg z) = (\neg z \wedge \sim z) \vee (\neg z \wedge \sim \neg z) \\ &\leq (\neg z \wedge \sim z) \vee (z \vee \sim z) = z \vee \sim z. \end{aligned}$$

Hence $\neg z = \neg z \wedge (z \vee \sim z) = \neg z \wedge \sim z$, and then $z \vee \sim z \geq z \vee \neg z = 1$. Consequently, taking into account properties (i) and (ii), we can conclude that $\sim z$ is the complement of z , i.e., $\sim z = \neg z$. Suppose now that $x \wedge z = y \wedge z$. Then $\sim x \vee \neg z = \sim y \vee \neg z$, which implies $z \wedge x = z \wedge y$. This shows that \sim is preserved by Θ_z .

Subvarieties of \mathcal{L}_i are the variety \mathcal{OL} of *ortholattices* [2], [19], characterized by the equation $x \wedge \sim x = 0$, and the variety \mathcal{K} of *Kleene algebras* [1], characterized by the distributive law. The intersection $\mathcal{OL} \cap \mathcal{K}$ is the variety \mathcal{B} of boolean algebras. An important subvariety of \mathcal{OL} is the variety \mathcal{OML} of *orthomodular lattices* [2], [19].

Example 2.3. *The variety \mathcal{B}_ω of pseudocomplemented distributive lattices* [1]. We prove that the pseudo complement $*$ has Θ_z -compatibility. Indeed, let $B \in \mathcal{B}_\omega$, $z \in Z(B)$, and $a, b \in B$. If $a \wedge z = b \wedge z$, then $(a \wedge z) \vee \neg z = (b \wedge z) \vee \neg z$. Hence $a \vee \neg z = b \vee \neg z$ because $z \in Z(A)$. Consequently, $(a \vee \neg z)^* = (b \vee \neg z)^*$ and $a^* \wedge z = b^* \wedge z$.

The variety of Stone algebras \mathcal{ST} is the subvariety of \mathcal{B}_ω characterized by the equation $(x \wedge y)^* = x^* \vee y^*$ [1].

Example 2.4. *The variety \mathcal{RL} of residuated lattices.* A residuated lattice is an algebra $\langle A, \vee, \wedge, \odot, \rightarrow, 0, 1 \rangle$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$ satisfying the following axioms:

1. $\langle A, \odot, 1 \rangle$ is an abelian monoid,
2. $L(A) = \langle A, \vee, \wedge, 0, 1 \rangle$ is a bounded lattice,
3. $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$,
4. $((x \rightarrow y) \odot x) \wedge y = (x \rightarrow y) \odot x$,
5. $(x \wedge y) \rightarrow y = 1$.

From [18, Lemma 1.4] it follows that \mathcal{RL} is an \mathcal{L} -variety. Subvarieties of \mathcal{RL} are the variety \mathcal{H} of Heyting algebras [1] characterized by the equation $x \odot y = x \wedge y$, and the variety \mathcal{BL} of *BL-algebras* [13] characterized by the equations

$$(i) \quad x \wedge y = x \odot (x \rightarrow y) \quad \text{and} \quad (ii) \quad (x \rightarrow y) \vee (y \rightarrow x) = 1.$$

Important subvarieties of \mathcal{BL} are the variety \mathcal{MV} of *MV-algebras* [6], characterized by the equation $\neg\neg x = x$ (see [13]), the variety \mathcal{PL} of *PL-algebras*, characterized by the equations $(\neg\neg z \odot ((x \odot z) \rightarrow (y \odot z))) \rightarrow (x \rightarrow y) = 1$ and $x \wedge \neg x = 0$ [13], [7], and the variety \mathcal{HL} of linear Heyting algebras, i.e., Heyting algebras satisfying the equation $(x \rightarrow y) \vee (y \rightarrow x) = 1$ (also known as Gödel algebras [13]).

Example 2.5. \mathcal{L}_n , *the varieties of Łukasiewicz and of Post algebras of order $n \geq 2$* [1], as well as the various types of Łukasiewicz-Moisil algebras which are considered in [3].

Example 2.6. \mathcal{PMV} , *the variety of pseudo MV-algebras.* A pseudo *MV-algebra* [15] is an algebra $\langle A, \oplus, ^-, \sim, 0, 1 \rangle$ of type $\langle 2, 1, 1, 0, 0 \rangle$ such that when defining the derived operations by $y \odot x := (x^- \oplus y^-)^\sim$, $x \vee y = x \oplus (x^\sim \odot y)$, $x \wedge y := x \odot (x^- \oplus y)$ the following axioms are satisfied:

1. $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
2. $x \oplus 0 = 0 \oplus x = x$ and $x \oplus 1 = 1 \oplus x = 1$,

3. $1^{\sim} = 0$ and $1^{-} = 0$,
4. $(x^{-} \oplus y^{-})^{\sim} = (x^{\sim} \oplus y^{\sim})^{-}$,
5. $x \oplus (x^{\sim} \odot y) = y \oplus (y^{\sim} \odot x) = (x \odot y^{-}) \oplus y = (y \odot x^{-}) \oplus x$,
6. $x \odot (x^{-} \oplus y) = (x \oplus y^{\sim}) \odot y$,
7. $(x^{-})^{\sim} = x$.

\mathcal{PMV} is categorically equivalent to lattice ordered (not necessarily abelian) groups with a strong unit [10]. From this result it is not very hard to prove that \mathcal{PMV} is an \mathcal{L} -variety.

3. THE CBS PROPERTY

The aim of this section is to give a formulation of the CBS theorem for algebras in \mathcal{L} -varieties. We begin by proving some technical results.

Proposition 3.1. *Let L be a bounded lattice. Then following assertions hold for all $z \in Z(L)$:*

1. $Z([0, z]) = Z(L) \cap [0, z]$.
2. *If $x \in Z([0, z])$ then the complement of x relative to $[0, z]$ is $\neg_z x = z \wedge \neg x$.*

Proof. Let $x \in Z([0, z])$. We first prove that, if x is a neutral element in $[0, z]$, then x is a neutral element in L . Let $a, b \in L$.

$$\text{a } (a, b, x)D: x \wedge (a \vee b) = (x \wedge (a \vee b)) \wedge (z \vee \neg z) = (x \wedge (a \vee b) \wedge z) \vee (x \wedge (a \vee b) \wedge \neg z) = (x \wedge (a \vee b) \wedge z) \vee 0 = x \wedge ((a \wedge z) \vee (b \wedge z)) = (x \wedge (a \wedge z)) \vee (x \wedge (b \wedge z)) = (x \wedge a) \vee (x \wedge b).$$

By the same argument it is possible to check $(b, a, x)D$.

$$\text{b } (x, b, a)D: a \wedge (x \vee b) = (a \wedge (x \vee b)) \wedge (z \vee \neg z) = (a \wedge (x \vee b) \wedge z) \vee (a \wedge (x \vee b) \wedge \neg z) = ((a \wedge z) \wedge ((x \vee b) \wedge z)) \vee (a \wedge ((x \wedge \neg z) \vee (b \wedge \neg z))) = ((a \wedge z) \wedge ((x \wedge z) \vee (b \wedge z))) \vee (a \wedge (0 \vee (b \wedge \neg z))) = ((a \wedge z) \wedge (x \vee (b \wedge z))) \vee (a \wedge b \wedge \neg z) = ((a \wedge z \wedge x) \vee (a \wedge b \wedge z)) \vee (a \wedge b \wedge \neg z) = (a \wedge x) \vee ((a \wedge b) \vee (z \vee \neg z)) = (a \wedge x) \vee (a \wedge b). \text{ By the same argument it is possible to check } (b, x, a)D, (x, a, b)D \text{ and } (a, x, b)D.$$

$$\text{c } (a, b, x)D^*: x \vee (a \wedge b) = (x \vee (a \wedge b)) \wedge (z \vee \neg z) = ((x \vee (a \wedge b)) \wedge z) \vee ((x \vee (a \wedge b)) \wedge \neg z) = ((x \wedge z) \vee (a \wedge b \wedge z)) \vee ((x \wedge \neg z) \vee (a \wedge b \wedge \neg z)) = (x \vee (a \wedge b \wedge z)) \vee (0 \vee (a \wedge b \wedge \neg z)) = (x \vee ((a \wedge z) \wedge (b \wedge z))) \vee (a \wedge b \wedge \neg z) = ((x \vee (a \wedge z)) \wedge (x \vee (b \wedge z))) \vee (a \wedge b \wedge \neg z) = ((x \vee a) \wedge (x \vee z)) \wedge (x \vee b) \wedge (x \vee z) \vee (a \wedge b \wedge \neg z) = ((x \vee a) \wedge (x \vee b) \wedge z) \vee (a \wedge b \wedge \neg z) = ((x \vee a) \wedge (x \vee b)) \vee (a \wedge b \wedge \neg z) \wedge (z \vee (a \wedge b \wedge \neg z)) = (x \vee a) \wedge (x \vee b) \wedge (z \vee (a \wedge b)) = (x \vee a) \wedge (x \vee b) \wedge (z \vee a) \wedge (z \vee b) = (x \vee a) \wedge (x \vee b). \text{ By the same argument it is possible to check that } (b, a, x)D^*.$$

$$\text{d } (x, b, a)D^*: a \vee (x \wedge b) = (a \vee (x \wedge b)) \wedge (z \vee \neg z) = ((a \vee (x \wedge b)) \wedge z) \vee ((a \vee (x \wedge b)) \wedge \neg z) = ((a \wedge z) \vee (x \wedge b \wedge z)) \vee ((a \wedge \neg z) \vee (x \wedge b \wedge \neg z)) = ((a \wedge z) \vee x) \wedge ((a \wedge z) \vee (b \wedge z)) \vee ((a \wedge \neg z) \vee 0) = ((a \vee x) \wedge (a \vee b) \wedge z) \vee (a \wedge \neg z) = (((a \vee x) \wedge (a \vee b)) \vee$$

$(a \wedge \neg z) \wedge (z \vee (a \wedge \neg z)) = (a \vee x) \wedge (a \vee b) \wedge (z \vee a) = (a \vee x) \wedge (a \vee b)$. By the same argument it is possible to check $(b, x, a)D^*$, $(x, a, b)D^*$ and $(a, x, b)D^*$.

Thus x is neutral in L . We proceed now to prove that if x is complemented in $[0, z]$ then x is also complemented in L . In fact, let $\neg_z x$ be the complement of x in $[0, z]$ and define x_1 by $x_1 = \neg_z x \vee \neg z$. Hence $x \vee x_1 = x \vee (\neg_z x \vee \neg z) = z \vee \neg z = 1$ and since x is a neutral element, $x \wedge x_1 = 0$. Thus x_1 is the complement of x in L . From the two preceding results, it follows that $x \in Z(L)$.

On the other hand, it is easy to verify that if x is a neutral element in L then x is a neutral element in $[0, z]$. Moreover, if x has a complement $\neg x$ in L , then $\neg_z x = \neg x \wedge z$ is the complement of x in $[0, z]$. Therefore if $x \in [0, z]$ is a central element in the lattice L , then x is a central element in the lattice $[0, z]$. \square

Proposition 3.2. *Let \mathcal{V} be an \mathcal{L} -variety, $A, B \in \mathcal{V}$, $\alpha: A \rightarrow B$ an isomorphism. Then*

- (1) *for all $z \in Z(A)$, $\alpha(z) \in Z(B)$, and the restriction of α to $Z(A)$ is a boolean algebra isomorphism from $Z(A)$ onto $Z(B)$;*
- (2) *for all $z \in Z(A)$, the restriction of α to $[0, z]_A$ is an isomorphism from $[0, z]_A$ onto $[0, \alpha(z)]_B$.*

Definition 3.3. Let \mathcal{V} be an \mathcal{L} -variety. We say that $A \in \mathcal{V}$ possesses the CBS property iff the following holds: Given $B \in \mathcal{V}$ and $b \in Z(B)$ such that there is $a \in Z(A)$ with $A \cong [0, b]_B$ and $B \cong [0, a]_A$, it follows that $A \cong B$.

Proposition 3.4. *Let \mathcal{V} be an \mathcal{L} -variety. The following conditions are equivalent for each $A \in \mathcal{V}$:*

- (1) *A possesses the CBS property.*
- (2) *For all $b \in Z(A)$, if $A \cong [0, b]_A$, then for all $z \in Z(A)$ such that $z \geq b$ we have $A \cong [0, z]_A$.*

Proof. We suppose that A possesses the CBS property. Let $z, b \in Z(A)$ be such that $z \geq b$ and $A \cong [0, b]_A$. We denote by B the \mathcal{V} -algebra $[0, z]_A$. By Proposition 3.1, $b \in Z(B)$. Now we have $A \cong [0, b]_B$ and $B \cong [0, z]_A$ (for this we use the identity $\text{id}_{[0, z]}$), and we conclude that $A \cong [0, z]_A$. For the converse, suppose that $B \in \mathcal{V}$, $a \in Z(A)$, $b \in Z(B)$ and that there are morphisms $\alpha: A \rightarrow [0, b]_B$ and $\beta: B \rightarrow [0, a]_A$. If $z = \beta(b)$, then $A \cong [0, z]_A$ and $a \geq z$. Now by the hypothesis $A \cong [0, a]_A$. This proves that $A \cong B$. \square

Let \mathcal{V} be an \mathcal{L} -variety, $A \in \mathcal{V}$, $b \in Z(A)$ and let $\alpha: A \rightarrow [0, b]_A$ be an isomorphism. If we consider $z \in Z(A)$ such that $z \geq b$ and the \mathcal{V} -algebra $B = [0, z]_A$, then there is an isomorphism $\beta: B \rightarrow [0, a]_A$ (for instance we can take $\beta = \text{id}_{[0, z]}$). We

define recursively two sequences, $(a_n)_{n \in \omega}$ in A , $(b_n)_{n \in \omega}$ in B , called respectively the A -sequence and the B -sequence as follows:

$$\begin{aligned} a_0 &= 1_A, & b_0 &= 1_B = z, \\ a_1 &= \beta(z) = a, & b_1 &= \alpha(a_0) = b, \\ a_{n+1} &= \beta(b_n), & b_{n+1} &= \alpha(a_n). \end{aligned}$$

Then the sequence

$$(a_2 \wedge \neg a_3, a_4 \wedge \neg a_5, \dots) = (a_{2n} \wedge \neg a_{2n+1})_{n \in \omega, n \geq 1}$$

is called a *CBS sequence*. Fixing b, z as above, then for each pair of isomorphisms $\alpha: A \rightarrow [0, b]_B$, $\beta: B \rightarrow [0, a]_A$ we have a CBS sequence, which we will denote by $\langle b, z, \alpha, \beta \rangle$.

Proposition 3.5. *Let \mathcal{V} be an \mathcal{L} -variety, $A \in \mathcal{V}$, and let $\langle b, z, \alpha, \beta \rangle$ be a CBS sequence. Then*

- (1) *the A -, B -sequences are strictly decreasing in $Z(A)$,*
- (2) *$\langle b, z, \alpha, \beta \rangle$ is an orthogonal sequence in $Z(A)$, and $\beta\alpha(a_{2n} \wedge \neg a_{2n+1}) = a_{2n+2} \wedge \neg a_{2n+3}$ for $n \geq 0$.*

Proof. By Proposition 3.2 it is easy to see that $a_1 = a$, $b_1 = b$, and that all a_n, b_n are central elements. Hence $\langle b, z, \alpha, \beta \rangle$ is in $Z(A)$. By the injectivity of α and β , $(a_n)_{n \in \omega}, (b_n)_{n \in \omega}$ are strictly decreasing. Let $m, n \in \omega$ such that $m < n$. Since $(a_n)_{n \in \omega}$ is strictly decreasing, $(a_{2m} \wedge \neg a_{2m+1}) \wedge (a_{2n} \wedge \neg a_{2n+1}) \leq (a_{2m} \wedge \neg a_{2m+1}) \wedge (a_{2m+1} \wedge \neg a_{2n+1}) = 0$. Finally, $\beta\alpha(a_{2n} \wedge \neg a_{2n+1}) = \beta(\alpha(a_{2n}) \wedge \alpha(\neg a_{2n+1})) = \beta(\alpha(a_{2n}) \wedge b \wedge \neg \alpha(a_{2n+1})) = \beta(b_{2n+1} \wedge \neg b_{2n+2}) = \beta(b_{2n+1}) \wedge a \wedge \neg \beta(b_{2n+2}) = a_{2n+2} \wedge \neg a_{2n+3}$. \square

Definition 3.6. Let \mathcal{V} be an \mathcal{L} -variety and $A \in \mathcal{V}$. Then A is called *CBS complete* iff for all $b \in Z(A)$ such that $A \cong_{\mathcal{V}} [0, b]_A$ and for all $z \in Z(A)$ such that $z \geq b$ there exists a CBS sequence $\langle b, z, \alpha, \beta \rangle$ which has the (boolean) supremum $\bigsqcup_{n \geq 1} (a_{2n} \wedge \neg a_{2n+1})$.

Theorem 3.7. *Let \mathcal{V} be an \mathcal{L} -variety. Then the following conditions are equivalent for each $A \in \mathcal{V}$:*

- (1) *A is CBS complete.*
- (2) *A possesses the CBS property.*

Proof. Suppose that A is CBS complete. Let $z, b \in Z(A)$ be such that $z \geq b$, $A \cong [0, b]_A$ and $B = [0, z]_A$. We want to prove that $A \cong [0, z]_A = B$. By the

hypothesis there are isomorphisms, $\alpha: A \rightarrow [0, b]_B$, $\beta: B \rightarrow [0, a]_A$ defining A -, B -sequences

$$\begin{aligned} a_0 &= 1_A, & b_0 &= 1_B = z, \\ a_1 &= a, & b_1 &= b, \\ a_{n+1} &= \beta(b_n), & b_{n+1} &= \alpha(a_n) \end{aligned}$$

and the CBS sequence $\langle b, z, \alpha, \beta \rangle = (a_{2n} \wedge \neg a_{2n+1})_{n \in \omega, n \geq 1}$ with $y = \bigsqcup_{n \geq 1} (a_{2n} \wedge \neg a_{2n+1})$. Let $x = y \vee \neg a$. By Proposition 1.4 we have

$$(1) \quad A \cong [0, \neg x] \times [0, x].$$

Since $y \in Z([0, a])$ by Proposition 3.1, we have

$$(2) \quad [0, a]_A \cong [0, \neg a y] \times [0, y] = [0, a \wedge \neg y] \times [0, y].$$

But $\neg x = a \wedge \neg y$, hence

$$(3) \quad [0, \neg x] = [0, a \wedge \neg y].$$

By Proposition 3.2,

$$[0, x] \cong [0, \beta\alpha(x)] = \left[0, \beta\alpha \left(\bigsqcup_{n \in \omega} a_{2n} \wedge \neg a_{2n+1} \right) \right] = \left[0, \bigsqcup_{n \in \omega} \beta\alpha(a_{2n} \wedge \neg a_{2n+1}) \right],$$

and by Proposition 3.5, $\beta\alpha(a_{2n} \wedge \neg a_{2n+1}) = (a_{2n+2} \wedge \neg a_{2n+3})$. Thus we have

$$(4) \quad [0, x] \cong \left[0, \bigsqcup_{n \geq 1} (a_{2n} \wedge \neg a_{2n+1}) \right] = [0, y].$$

From (1), (2), (3) and (4) we obtain that $A \cong [0, a]$, hence $A \cong_{\mathcal{V}} B$.

Suppose now that A possesses the CBS property. Let $b \in Z(A)$ be such that we can find an isomorphism $\alpha: A \rightarrow [0, b]_A$ and a $z \in Z(A)$ such that $z \geq b$. By hypothesis there is an isomorphism $\beta: [0, z]_A \rightarrow A$. The corresponding A , $[0, z]_A$ -sequences have the form

$$\begin{aligned} a_0 &= 1_A, & b_0 &= z, \\ a_1 &= \beta(b_0) = 1, & b_1 &= \alpha(a_0) = z, \\ a_2 &= \beta(b_1) = \beta(z), & b_2 &= \alpha(a_1) = z, \\ a_3 &= \beta(b_2) = \beta(z), & b_3 &= \alpha(a_2) = \alpha\beta(z), \\ & \vdots & & \vdots \end{aligned}$$

It is easy to show (by induction) that $a_{2n} = a_{2n+1}$ for all $n \geq 1$. Thus we have $\langle b, z, \alpha, \beta \rangle = (0, 0, 0, \dots)$ and the boolean supremum is 0. Therefore there exists at least one CBS sequence associated with $z \geq b$ admitting the boolean supremum. Therefore A is CBS complete. \square

Corollary 3.8. *Let \mathcal{V} be an \mathcal{L} -variety and $A \in \mathcal{V}$. If $Z(A)$ is an orthogonally σ -complete lattice, then A possesses the CBS property.*

Corollary 3.9 (Sikorski). *The σ -complete Boolean algebras possesses the CBS property.*

Corollary 3.10. *Let A be a CBS complete algebra in an \mathcal{L} -variety \mathcal{V} . Then $A \cong A^2$ iff $A \cong A^n$ for all $n \geq 2$.*

Proof. It is an easy adaptation of the proof of Proposition 3.2 in [9]. \square

Remark 3.11. It is worth noting that the σ -completeness condition for Boolean algebras is not necessary for the CBS property, as is shown by the Boolean algebra $B_{\mathbb{N}}$ of finite and cofinite subsets of \mathbb{N} . $B_{\mathbb{N}}$ is not even orthogonally σ -complete. Indeed, $\{2n\}_{n \in \mathbb{N}}$ is an orthogonal sequence in $B_{\mathbb{N}}$, but $\bigvee_{n \in \mathbb{N}} \{2n\}$ is not in $B_{\mathbb{N}}$. By cardinality arguments it is very easy to see that $B_{\mathbb{N}} \cong [\emptyset, X]_{B_{\mathbb{N}}}$ iff X is a cofinite set. Thus $B_{\mathbb{N}}$ possesses the CBS property. On the other hand, there are Boolean algebras which do not possess the CBS property. For instance, Hanf constructed a Boolean algebra B such that $B \cong B^3$ but $B \not\cong B^2$ [17, §6.2]. This means that $B \cong [(0, 0, 0), (0, 0, 1)]_{B^3}$ but $B \not\cong [(0, 0, 0), (0, 1, 1)]_{B^3}$.

4. CENTERS AND σ -COMPLETENESS

In general, the σ -completeness of an algebra A in an \mathcal{L} -variety does not imply that $Z(A)$ is an orthogonally σ -complete lattice, as the following example shows:

Example 4.1. Let $B_{\mathbb{N}}$ be as in Remark 3.11 and let $H_{\mathbb{N}}$ be the Heyting algebra of all ideals of $B_{\mathbb{N}}$. We observe that $H_{\mathbb{N}}$ is a complete Heyting algebra such that $Z(H_{\mathbb{N}})$, which is formed by the principal ideals generated by the elements of $B_{\mathbb{N}}$, is not orthogonally σ -complete. Indeed, the principal ideals $(\langle 2n \rangle)_{n \in \mathbb{N}}$ form an orthogonal sequence in $Z(H_{\mathbb{N}})$, but obviously this sequence does not have a central supremum. It is worth noting that $H_{\mathbb{N}}$ possesses the CBS property, as can be shown by cardinality arguments similar to those used in Remark 3.11.

In what follows we give examples of \mathcal{L} -varieties \mathcal{V} with the property that σ -completeness conditions on the algebras in \mathcal{V} guarantee the corresponding σ -completeness of their centers, and then, in the light of Corollary 3.8, the CBS property of these algebras.

4.1. Orthomodular lattices.

Proposition 4.2. *Let L be a σ -complete orthomodular lattice and $(a_n)_{n \in \omega}$ a sequence in $Z(L)$. Then $\bigvee_{n \in \omega} a_n \in Z(L)$, i.e., $\bigsqcup_{n \in \omega} a_n = \bigvee_{n \in \omega} a_n$.*

Proof. The proof is an easy adaptation of the proof of (5.14) and (29.16) in [19]. □

4.2. Stone algebras.

Proposition 4.3. *Let S be a Stone algebra and $(a_i)_{i \in I}$ a family of central elements such that there exist $\bigwedge_{i \in I} a_i$ and $\bigvee_{i \in I} a_i$. Then $\prod_{i \in I} a_i = \bigwedge_{i \in I} a_i$ (i.e. $\bigwedge_{i \in I} a_i \in Z(S)$) and $\bigsqcup_{i \in I} a_i = \neg\neg \bigvee_{i \in I} a_i$. Thus if S is a σ -complete, (orthogonally σ -complete) Stone algebra then $Z(S)$ is a σ -complete (orthogonally σ -complete) lattice.*

Proof. It is well known that $Z(S) = \{x \in S : \neg\neg x = x\}$ (see [1]). Let $a = \bigwedge_{i \in I} a_i$. For all $i \in I$, if $a \leq a_i$, then $\neg\neg a \leq \neg\neg a_i = a_i$. Thus $\neg\neg a \leq \bigwedge_{i \in I} a_i = a$, and since $a \leq \neg\neg a$, we have $a \in Z(S)$. From the basic properties of the pseudocomplement it follows that $\neg\neg \bigvee_{i \in I} a_i \in Z(S)$ and it is easy to see that $\neg\neg \bigvee_{i \in I} a_i$ is the least boolean upper bound of $(a_i)_{i \in I}$. □

4.3. BL-algebras.

Lemma 4.4 [5]. *For each $A \in \mathcal{BL}$, let $\text{Idp}(A) = \{x \in A : x \odot x = x\}$ be the set of all idempotent elements of A . $\text{Idp}(A)$ is a Heyting algebra, $Z(A)$ is a subalgebra of $\text{Idp}(A)$ and $z \in \text{Idp}(A)$ iff $z \odot a = z \wedge a$ for all $a \in A$.*

Lemma 4.5. *Let B be a BL-algebra and $(a_i)_{i \in I}$ a sequence in B such that $\bigvee_{i \in I} a_i$ exists. Then we have*

1. $a \odot \bigvee_{i \in I} a_i = \bigvee_{i \in I} (a \odot a_i)$, $\left(\bigvee_{i \in I} a_i\right) \rightarrow b = \bigwedge_{i \in I} (a_i \rightarrow b)$, $a \wedge \bigvee_{i \in I} a_i = \bigvee_{i \in I} (a \wedge a_i)$ and $\neg\left(\bigvee_{i \in I} a_i\right) = \bigwedge_{i \in I} \neg a_i$;
2. if $(a_i)_{i \in I}$ is a family in $\text{Idp}(B)$ then $\bigvee_{i \in I} a_i \in \text{Idp}(B)$.

Proof. Item 1) follows from basic the properties of residuated lattices [14]. To prove 2), let $a = \bigvee_{i \in I} a_i$. By item 1), we have $a \odot a = a \odot \bigvee_{i \in I} a_i = \bigvee_{i \in I} (a \odot a_i) = \bigvee_{i \in I} (a \wedge a_i) = \bigvee_{i \in I} a_i = a$. \square

Lemma 4.6 [5]. *Let B be a BL-algebra. The following conditions are equivalent:*

1. $z \in Z(B)$,
2. $z \vee \neg z = 1$,
3. *there is $v \in \text{Idp}(b)$ such that $z = \neg v$.*

Proposition 4.7. *Let B be a BL-algebra and $(a_i)_{i \in I}$ a sequence in $Z(B)$ such that there exist $\bigvee_{i \in I} a_i$ and $\bigwedge_{i \in I} a_i$. Then $\bigsqcup_{i \in I} a_i = \neg \neg \bigvee_{i \in I} a_i$ and $\prod_{i \in I} a_i = \bigwedge_{i \in I} a_i$.*

Proof. If $(a_i)_{i \in I}$ is a sequence in $Z(B)$ with $a = \bigwedge_{i \in I} a_i$, by Lemma 4.6 it suffices to show that $a \vee \neg a = 1$. According to Lemma 4.5 we have $a \vee \neg a = \left(\bigwedge_{i \in I} a_i \right) \vee \neg a = \neg \left(\bigvee_{n \in I} \neg a_i \right) \vee \neg a = \neg \left(\left(\bigvee_{i \in I} \neg a_i \right) \wedge a \right) = \neg \bigvee_{i \in I} (\neg a_i \wedge a) = 1$, therefore $\prod_{i \in I} a_i = \bigwedge_{i \in I} a_i$. According to Lemmas 4.6.2, 4.5.3, we have $\neg \neg \bigvee_{i \in I} a_i \in Z(B)$ and $\neg \neg \bigvee_{i \in I} a_i$ is a boolean upper bound of $(a_i)_{i \in I}$. Moreover, if b is a boolean upper bound of $(a_i)_{i \in I}$ then $\bigvee_{i \in I} a_i \leq b$ hence, $\neg \neg \bigvee_{i \in I} a_i \leq b$, thus $\bigsqcup_{i \in I} a_i = \neg \neg \bigvee_{i \in I} a_i$. \square

Corollary 4.8. *If B is a σ -complete (orthogonally σ -complete) BL-algebra then $Z(B)$ is a σ -complete (orthogonally σ -complete) lattice.*

Proposition 4.9. *If B is a σ -complete (orthogonally σ -complete) PL-algebra or MV-algebra then $Z(B)$ is a σ -sublattice (orthogonal σ -sublattice) of $L(B)$.*

Proof. If B is a PL-algebra then according to Proposition 3.1 in [7], $\text{Idp}(B) = Z(B)$. Thus by Lemma 4.5.2, $\bigsqcup_{n \in \omega} a_n = \bigvee_{n \in \omega} a_n$ for $(a_n)_{n \in \omega}$ in $Z(B)$. If B is an MV-algebra then using Lemma 4.5.2 and $\neg \neg x = x$ we have the same result. \square

4.4. Łukasiewicz and Post algebras of order n .

Proposition 4.10 [4, Lemma 3.1]. *Let A be a Łukasiewicz algebra of order $n \geq 2$. If A is σ -complete, then $Z(A)$ is a σ -sublattice of $L(A)$.*

4.5. Pseudo MV-algebra.

Let A be a pseudo MV-algebra. If A is σ -complete, then A is an MV-algebra (see [10, Theorem 4.2] and [11, Proposition 2.8]). Thus by Proposition 4.7, $Z(A)$ is a

σ -sublattice of $L(A)$ and if A is orthogonally σ -complete then $Z(A)$ is an orthogonally σ -complete lattice (see Proposition 3.4 in [15]).

5. CBS-TYPE THEOREM FOR POSETS

The category of posets and monotonic functions will be denoted by Pos . Let A be a poset and $X \subseteq A$. X is *decreasing* (*increasing*) iff for all $x \in X$, if $a \leq x$ ($a \geq x$), then $a \in X$. The set of all decreasing sets in A is denoted by $O(A)$, and it is well known that $O(A)$ has the structure of a complete Heyting algebra. For every $a \in A$ we denote by $\downarrow a$ the decreasing set $\{x \in A: x \leq a\}$, and it is easy to show that $\downarrow a$ is compact in $O(A)$. Moreover, $X \in O(A)$ is compact iff there exist a_1, \dots, a_n in A such that $X = (\downarrow a_1) \cup \dots \cup (\downarrow a_n)$. It is easy to show that $Z(O(A)) = \{B \in O(A): B \text{ is an increasing set}\}$ and that $Z(O(A))$ is a complete lattice.

Lemma 5.1. *Let A, B be posets. If $O(A)$ and $O(B)$ are isomorphic then A and B are isomorphic.*

Theorem 5.2. *Let A, B be posets and let $X \subseteq A$ and $Y \subseteq B$ be simultaneously increasing and decreasing sets. If there are isomorphisms $\alpha: A \rightarrow Y$ and $\beta: B \rightarrow X$, then $A \cong_{\text{Pos}} B$.*

Proof. We first prove that $O(A) \cong_{\text{Pos}} [\emptyset, Y]$. For all $S \in O(A)$ we have $S = \bigcup_{a \in S} (\downarrow a)$ and $\downarrow \alpha(a) \subseteq Y$, since Y is decreasing. Consequently, if $\psi: O(A) \rightarrow [\emptyset, Y]$ is such that $S = \bigcup_{a \in S} (\downarrow a) \mapsto \bigcup_{a \in S} (\downarrow \alpha(a))$, then it is easy to show that ψ is an order isomorphism under \subseteq . Analogously, we can obtain that $O(B) \cong_{\text{Pos}} [\emptyset, X]$. But these Pos -isomorphisms are also Heyting isomorphisms. Then by Theorem 3.7, $O(A) \cong O(B)$ as Heyting algebras. Finally, in view of Lemma 5.1 we have $A \cong_{\text{Pos}} B$. □

Acknowledgement. The author expresses his gratitude to Roberto Cignoli for his advice during the preparation of this paper, and to Daniele Mundici and Mirko Navara for their comments which improved its final version.

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