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BOUNDEDNESS OF RIESZ POTENTIAL GENERATED BY GENERALIZED SHIFT OPERATOR ON Ba SPACES

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Abstract. In this paper, the boundedness of the Riesz potential generated by generalized shift operator $I^{\alpha}_{B_k}$ from the spaces $\mathrm{B}a = (L_{p_m,\nu}(\mathbb{R}^k_n), a_m)$ to the spaces $\mathrm{B}a' = (L_{q_m,\nu}(\mathbb{R}^k_n), a'_m)$ is examined.

Keywords: generalized shift operator, Riesz-Bessel transformations

MSC 2000: 47G10, 45E10, 47B37

1. Introduction

A new class of function spaces, denoted by Ba, was introduced by X. Ding and P. Luo in [5]. This class of spaces is a very natural generalization of the classical L_p spaces and also includes some important Orlicz spaces, Orlicz-Sobolev spaces, etc. In the past few years, many results have been obtained pertaining to Ba spaces and have been used in both classical analysis and other branches of mathematics (see [3]–[7]). The boundedness of the Riesz potential in Ba spaces was investigated by Y. Deng, W. Chang and Y. Li [7]. The Riesz potential generated by generalized shift operator was introduced by I. A. Aliev and A. D. Gadzhiev [8], where weighted L_p estimates were obtained for $I^{\alpha}_{B_k}$.

The aim of this paper is to prove the boundedness of the Riesz potential generated by a generalized shift operator $I^{\alpha}_{B_k}$ from the spaces $\mathrm{B}a = (L_{p_m,\nu}(\mathbb{R}^k_n), a_m)$ to the spaces $\mathrm{B}a' = \big(L_{q_m,\nu}(\mathbb{R}^k_n), a'_m\big)$.

2. Ba spaces and Riesz potential generated by a generalized shift operator

Let $B = \{B_1, \dots, B_m, \dots\}$ be a sequence of Banach function spaces and $a = \{a_1, a_2, \dots, a_m, \dots\}$ be a sequence of non-negative real numbers. Let $\varphi(z) = \sum\limits_{m=1}^{\infty} a_m z^m$ be an entire function. For $f \in \bigcap\limits_{m=1}^{\infty} B_m$, we form a power series as follows

$$I(f,\lambda) = \sum_{m=1}^{\infty} a_m \|f\|_{B_m}^m \lambda^m,$$

where $\|\cdot\|_{B_m}$ is the B_m-norm of f. Let R_f denote the radius of convergence of the series $I(f,\lambda)$ and Ba denote the following function set

$$\mathrm{B}a = \Big\{f\colon\ f \in \bigcap_{m=1}^{\infty} B_m, R_f > 0\Big\}.$$

The set Ba is proved to be a Banach space when we define the norm of an element $f \in \mathrm{B}a$ by

$$\|f\|_{\mathrm{B}a} = \inf_{\lambda>0}\Big\{\frac{1}{\lambda}\colon\ I(f,\lambda) \leqslant 1\Big\},$$

(see, for details, [5]).

In this paper we will confine ourselves to the Banach spaces

$$B_m = L_{p_m,\nu}(\mathbb{R}_n^k) = \bigg\{f\colon\ \|f\|_{L_{p_m,\nu}} \equiv \bigg(\int_{\mathbb{R}_n^k} |f(x)|^{p_m} \prod_{j=1}^{k} x_{n-k+j}^{2\nu_j}\,\mathrm{d}x\bigg)^{1/p_m} < \infty\bigg\},$$

where $\nu_j > 0$, $j = 1, 2, \dots, k$ are fixed parameters, $1 < p_m < \infty$ $(m = 1, 2, \dots)$, and $\mathbb{R}_n^k = \{x\colon\ x = (x_1, x_2, \dots, x_n), x_{n-k+1} \geqslant 0, \dots, x_n \geqslant 0, 1 \leqslant k \leqslant n\}$.

For simplicity, we will denote $\|\cdot\|_{L_{p,\nu}}$ by $\|\cdot\|_{p,\nu}$.

The generalized shift operator is defined by

$$T^y f(x) = c_{\nu_j} \int_0^{\pi}\!\!\dots\!\int_0^{\pi} f\Big[x' - y', \sqrt{x_{n-k+1}^2 + y_{n-k+1}^2 - 2x_{n-k+1}y_{n-k+1}\cos\alpha_1}, \dots, \\ \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos\alpha_k}\Big] \prod_{j=1}^{k} [\sin^{2\nu_j - 1}\alpha_j\,\mathrm{d}\alpha_j],$$

where

$$c_{\nu_j} = \pi^{-k/2} \prod_{j=1}^{k} \frac{\Gamma(\nu_j + \frac{1}{2})}{\Gamma(\nu_j)},$$

$x = (x', x_{n-k+1}, \ldots, x_n)$, $y = (y', y_{n-k+1}, \ldots, y_n)$, and $x', y' \in \mathbb{R}_{n-k}$. We remark that T^y is closely connected with the Bessel differential operator

$$B_r = \frac{\partial^2}{\partial r^2} \frac{2\nu}{r} \frac{\partial}{\partial r}, \qquad r > 0.$$

Let Δ_{B_k} denote the Laplace-Bessel operators,

$$\Delta_{B_k} = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \sum_{j=1}^{k} \frac{2\nu_j}{x_{n-k+j}} \frac{\partial}{\partial x_{n-k+j}}, \quad 1 \leqslant k \leqslant n,\ \nu_j > 0\ (j = 1, \ldots, k).$$

The shift T^y generates the corresponding convolution ("B-convolution")

$$(f_1 * f_2)(y) = \int_{\mathbb{R}_n^k} f_1(x)[T^x f_2(y)]\ x_n^{2\nu}\,\mathrm{d}x.$$

We note that this convolution satisfies the property $f_1 * f_2 = f_2 * f_1$ (see [1], [2], [9]–[11]).

The Riesz potential I^α is defined by

$$(I^\alpha f)(x) = r(\alpha) \int_{\mathbb{R}_n} \frac{f(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y, \quad 0 < \alpha < n,$$

where $f \in L_p(\mathbb{R}_n)$ and $r(\alpha) = \left[\pi^{n/2} 2^\alpha \Gamma(\frac{1}{2}\alpha)/\Gamma(\frac{1}{2}(n-\alpha))\right]^{-1}$ (see [12]). It is well-known that, for $p \in (1, n/\alpha)$, I^α is bounded operator from L_p to L_q, with $1/q = 1/p - \alpha/n$, i.e., there exists a constant $A(p)$ such that $\|I^\alpha f\|_q \leqslant A(p)\|f\|_p$. In [7], the boundedness of Riesz potential I^α in Ba spaces were investigated by Y. Deng, W. Chang and Y. Li

Now let T^y be the generalized shift operator. The Riesz potential generated by generalized shift operator is defined by

$$(I_{B_k}^\alpha f)(x) = c(\alpha) \int_{\mathbb{R}_n^k} f(y) T^y\big(|x|^{\alpha-n-2|\nu|}\big) \prod_{j=1}^{k} y_{n-k+j}^{2\nu_j}\,\mathrm{d}y, \quad 0 < \alpha < n + 2|\nu|, \tag{2.1}$$

where $f(x)$ belongs to the space of test functions, denoted by $\mathcal{Z}_+(\mathbb{R}_n^k) = \mathcal{Z}_+$, and

$$c(\alpha) = 2^{\alpha-k} \pi^{(k-n)/2} \Gamma\Big(|\nu| + \frac{n-\alpha}{2}\Big) \Big[\Gamma\Big(\frac{\alpha}{2}\Big) \prod_{j=1}^{k} \Gamma\Big(\nu_j + \frac{1}{2}\Big)\Big]^{-1}.$$

The Riesz potential generated by generalized shift operator $I_{B_k}^\alpha$ is a bounded operator from $L_{p,\nu}(\mathbb{R}_n^k)$ to $L_{q,\nu}(\mathbb{R}_n^k)$ with $1 < p < q < \infty$, $1/q = 1/p - \alpha/(n + 2|\nu|)$, i.e., there exists some constant $A_\alpha(p, q, \nu)$ such that for all $f \in L_p(\mathbb{R}_n^k)$ (see [8])

$$\|I_{B_k}^\alpha f\|_{q,\nu} \leqslant A_\alpha(p, q, \nu)\|f\|_{p,\nu} \quad . \tag{2.2}$$

It is natural to expect that $I^{\alpha}_{B_k}$ is bounded from $\mathrm{B}a = (L_{p_m,\nu}(\mathbb{R}^k_n), a_m)$ to $\mathrm{B}a' = (L_{q_m,\nu}(\mathbb{R}^k_n), a'_m)$, where $1/q_m = 1/p_m - \alpha/(n+2|\nu|)$, $m = 1, 2, \ldots$. However, this is not true in general. Indeed, we have

Theorem 2.1. *Let $0 < \alpha < n + 2|\nu|$ and let $1 < p_m < (n+2|\nu|)/\alpha$. Then the Riesz potential generated by generalized shift operator $I^{\alpha}_{B_k}$ is bounded from $\mathrm{B}a = (L_{p_m,\nu}(\mathbb{R}^k_n), a_m)$ to $\mathrm{B}a' = (L_{q_m,\nu}(\mathbb{R}^k_n), a'_m)$ if and only if there exist two positive constant β and γ such that*

$$1 < \beta < p_m < \gamma < \frac{n+2|\nu|}{\alpha} \quad \textit{for all} \quad a_m \neq 0. \tag{2.3}$$

To prove of this theorem we first give the following two lemmas.

Lemma 2.2.

$$\int\limits_{\{u\in\mathbb{R}^k_n:\ |u|>\varepsilon\}} |u|^{\alpha-n-2|\nu|}[T^u f(x)] \prod_{j=1}^{k} u^{2\nu_j}_{n-k+j}\,\mathrm{d}u = c_\nu \int\limits_{|\tilde{x}-\tilde{y}|>\varepsilon} |\tilde{x}-\tilde{y}|^{\alpha-n-2|\nu|}$$
$$\times f\Big(y', \sqrt{y^2_{n-k+1}+y^2_{n-k+2}}, \ldots, \sqrt{y^2_{n+k-1}+y^2_{n+k}}\Big) \prod_{j=1}^{k} |y_{n-k+2j}|^{2\nu_j-1}\,\mathrm{d}\tilde{y},$$

where $c_\nu = \pi^{-k/2}2^k \prod\limits_{j=1}^{k} \Gamma(\nu_j+\frac{1}{2})/\Gamma(\nu_j)$ *and* $\tilde{x} = (x', x_{n-k+1}, \ldots, x_n, \underbrace{0, \ldots, 0}_{k\text{-terms}})$, $x' = (x_1, \ldots, x_{n-k})$, $\tilde{y} = (y', y_{n-k+1}, \ldots, y_n, y_{n+1}, \ldots, y_{n+k})$.

Proof. We denote the first part by I,

$$I = \pi^{-\frac{k}{2}}2^k \prod_{j=1}^{k} \frac{\Gamma(\nu_j+\frac{1}{2})}{\Gamma(\nu_j)} \int_{|u|>\varepsilon} |u|^{\alpha-n-2|\nu|}$$
$$\int_0^{\pi}\!\!\ldots\!\int_0^{\pi} f\Big(x'-u', \sqrt{x^2_{n-k+1}+u^2_{n-k+1}-2x_{n-k+1}u_{n-k+1}\cos\alpha_1}, \ldots,$$
$$\sqrt{x^2_n+u^2_n-2x_nu_n\cos\alpha_n}\Big) \prod_{j=1}^{k} [u^{2\nu_j}_{n-k+j}\sin^{2\nu_j-1}\alpha_j\,\mathrm{d}\alpha_j]\,\mathrm{d}u$$
$$= \pi^{-\frac{k}{2}}2^k \prod_{j=1}^{k} \frac{\Gamma(\nu_j+\frac{1}{2})}{\Gamma(\nu_j)} \int_{|u|>\varepsilon} |u|^{\alpha-n-2|\nu|} \int_0^{\pi}\!\!\ldots\!\int_0^{\pi} f\Big(x'-u',$$
$$\sqrt{x^2_{n-k+1}-2x_{n-k+1}u_{n-k+1}\cos\alpha_1+(u_{n-k+1}\cos\alpha_1)^2+(u_{n-k+1}\sin\alpha_1)^2}, \ldots,$$
$$\sqrt{x^2_n-2x_nu_n\cos\alpha_k+(u_n\cos\alpha_k)^2+(u_n\sin\alpha_k)^2}\Big) \prod_{j=1}^{k} [u^{2\nu_j}_{n-k+j}\sin^{2\nu_j-1}\alpha_j\,\mathrm{d}\alpha_j]\,\mathrm{d}u$$

$$= c_\nu \int_{|s|>\varepsilon} |u|^{\alpha-n-2|\nu|} \int_0^\pi \ldots \int_0^\pi f\Big(x'-u',$$
$$\sqrt{(x_{n-k+1}-u_{n-k+1}\cos\alpha_1)^2+(u_{n-k+1}\sin\alpha_1)^2}, \ldots,$$
$$\sqrt{(x_n-u_n\cos\alpha_k)^2+(u_n\sin\alpha_k)^2}\Big) \prod_{j=1}^k [u_{n-k+j}^{2\nu_j} \sin^{2\nu_j-1}\alpha_j \,\mathrm{d}\alpha_j]\,\mathrm{d}u.$$

Now, we pass to the new variables $\tilde{x} = (x', x_{n-k+1}, \ldots, x_n, 0, \ldots, 0)$, $\tilde{y} = (y', y_{n-k+1}, \ldots, y_n, y_{n+1}, \ldots, y_{n+k})$: $x' - u' = y'$, $y_{n-k+(2j-1)} = x_{n-k+j} - u_{n-k+j} \times \cos\alpha_j$, $|y_{n-k+2j}| = u_{n-k+j}\sin\alpha_j$, $0 \leqslant \alpha_j < \pi$ and $u_{n-k+j} > 0$, $j = 1, 2, \ldots, k$. Since the Jacobian of the transformation is equal to $(u_{n-k+1} \cdot u_{n-k+2} \ldots u_n)^{-1}$ we have

$$I = c_\nu \int_{|\tilde{x}-\tilde{y}|>\varepsilon} |\tilde{x}-\tilde{y}|^{\alpha-n-2|\nu|} f\Big(y', \sqrt{y_{n-k+1}^2 + y_{n-k+2}^2}, \ldots, \sqrt{y_{n+k-1}^2+y_{n+k}^2}\Big)$$
$$\times \prod_{j=1}^k |y_{n-k+2j}|^{2\nu_j-1}\,\mathrm{d}\tilde{y}.$$

□

Lemma 2.3.

$$\int_{\mathbb{R}_n^k} f(u) \prod_{j=1}^k u_{n-k+j}^{2\nu_j}\,\mathrm{d}u = c_\nu \int_{\mathbb{R}_{n+k}^k} f\Big(y', \sqrt{y_{n-k+1}^2+y_{n-k+2}^2}, \ldots, \sqrt{y_{n+k-1}^2+y_{n+k}^2}\Big)$$
$$\times \prod_{j=1}^k |y_{n-k+2j}|^{2\nu_j-1}\,\mathrm{d}y\,\mathrm{d}y_{n+1}\ldots \mathrm{d}y_{n+k},$$

where $c_\nu = \pi^{-k/2} 2^k \prod_{j=1}^k \Gamma(\nu_j + \frac{1}{2})/\Gamma(\nu_j)$.

The proof is straightforward by substituting $y' = u'$, $y_{n-k+(2j-1)} = u_{n-k+j}\cos\alpha_j$, $y_{n-k+2j} = u_{n-k+j}\sin\alpha_j$, $0 \leqslant \alpha_j < \pi$ and $u_{n-k+j} > 0$, $j = 1, 2, \ldots, k$.

Proof of Theorem 2.1. If $\beta < p < \gamma$ then there exists a $K > 0$ such that $A_\alpha(p, q, \nu) \leqslant K$ by the continuity. Now suppose that (2.3) holds, then we have $A_\alpha(p_m, q_m, \nu) \leqslant K$, $m = 1, 2, \ldots$. By the definition of the Ba-norm, for all $f \in \mathrm{B}a = (L_{p_m,\nu}(\mathbb{R}_n^k), a_m)$, we have

$$I\Big(f, \frac{1}{\|f\|_{\mathrm{B}a}}\Big) = \sum_{m=1}^\infty a_m \frac{\|f\|_{p_m,\nu}^m}{\|f\|_{\mathrm{B}a}^m} \leqslant 1,$$

so

$$I\Big(I_{B_k}^{\alpha}f,\frac{1}{K\|f\|_{\mathrm{Ba}}}\Big)=\sum_{m=1}^{\infty}a_m\frac{\|I_{B_k}^{\alpha}\|_{q_m,\nu}^m}{(K\|f\|_{\mathrm{Ba}})^m}$$
$$\leqslant\sum_{m=1}^{\infty}a_mA_{\alpha}^m(p_m,q_m,\nu)\frac{\|f\|_{p_m,\nu}^m}{(K\|f\|_{\mathrm{Ba}})^m}\leqslant 1.$$

This implies

$$\|I_{B_k}^{\alpha}f\|_{\mathrm{Ba}'}=\inf_{\lambda>0}\{1/\lambda\colon\ I(I_{B_k}^{\alpha}f,\lambda)\leqslant 1\}\leqslant K\|f\|_{\mathrm{Ba}},$$

and the sufficiency is thus proved.

We now proceed to prove the necessity of condition (2.3). We need some estimates concerning the functions f_l and g_l defined by

$$f_l(x)=\begin{cases}1, & x\in I=\{x\colon\ |x|\leqslant l,\ x_{n-k+j}\geqslant 0\ (j=1,2,\dots,k)\},\\ 0, & \text{otherwise.}\end{cases}$$

and

$$g_l(x)=\begin{cases}|x|^{-\alpha} & \log(1/|x|)^{-\alpha(n+2|\nu|)^{-1}(1+\varepsilon)},\ x\in I,\\ 0, & \text{otherwise.}\end{cases}$$

First, we claim that there exists some constant $B_{\alpha}(p,q,\nu)$, which depends only on p and $B_{\alpha}(p,q,\nu)\to\infty$ as $p\to 1^+$, such that

$$\frac{\|I_{B_k}^{\alpha}f_l\|_{q,\nu}}{\|f_l\|_{p,\nu}}\geqslant B_{\alpha}(p,q,\nu)\tag{2.4}$$

holds for all p near 1, where $1/q=1/p-\alpha(n+2|\nu|)^{-1}$. For any $y\in I$ and $x\notin I$, if we use Lemma 2.2 we have

$$(I_{B_k}^{\alpha}f_l)(x)=c(\alpha)\int_{\mathbb{R}_n^k}f_l(y)T^y\big(|x|^{\alpha-n-2|\nu|}\big)\prod_{j=1}^{k}y_{n-k+j}^{2\nu_j}\,\mathrm{d}y$$
$$=c(\alpha)\int_{\mathbb{R}_n^k}|y|^{\alpha-n-2\nu}T^yf_l(x)\prod_{j=1}^{k}y_{n-k+j}^{2\nu_j}\,\mathrm{d}y$$
$$=c(\alpha)\int_{\mathbb{R}_{n+k}^k(\tilde y)}|\tilde x-\tilde y|^{\alpha-n-2|\nu|}f_l\Big(y',\sqrt{y_{n-k+1}^2+y_{n-k+2}^2},\dots,\sqrt{y_{n+k-1}^2+y_{n+k}^2}\Big)$$
$$\times\prod_{j=1}^{k}|y_{n-k+2j}|^{2\nu_j-1}\,\mathrm{d}\tilde y.$$

Since $|x| > l$, $|\tilde{y}| \leqslant l$ and, $|\tilde{x}| = x$, it follows that $|\tilde{x} - \tilde{y}| \leqslant |\tilde{x}| + |\tilde{y}| \leqslant |x| + l \leqslant 2|x|$. Thus, for any $x \notin I$,

$$\begin{aligned}
&(I^{\alpha}_{B_k} f_l)(x) \\
&\quad \geqslant c(\alpha) \int\limits_{\mathbb{R}^k_{n+k}(\tilde{y})} (2|x|)^{\alpha-n-2|\nu|} f_l\Big(y', \sqrt{y^2_{n-k+1} + y^2_{n-k+2}}, \ldots, \sqrt{y^2_{n+k-1} + y^2_{n+k}}\Big) \\
&\qquad\qquad \times \prod_{j=1}^{k} y^{2\nu_j - 1}_{n-k+2j} \, \mathrm{d}\tilde{y}.
\end{aligned}$$

By Lemma 2.3

$$\begin{aligned}
I^{\alpha}_{B_k} f(x) &\geqslant c(\alpha) \int_{\mathbb{R}^k_n} (2|x|)^{\alpha-n-2|\nu|} f_l(u) \prod_{j=1}^{k} u^{2\nu_j}_{n-k+j} \, \mathrm{d}u \\
&= c(\alpha)(2|x|)^{\alpha-n-2|\nu|} \int_{|u|\leqslant l} f_l(u) \prod_{j=1}^{k} u^{2\nu_j}_{n-k+j} \, \mathrm{d}u \\
&= c(\alpha) 2^{\alpha-n-2|\nu|} |x|^{\alpha-n-2|\nu|} c \, \frac{l^{n+2|\nu|}}{n+2|\nu|}.
\end{aligned}$$

By a simple computation we see that

$$\begin{aligned}
&\frac{\|I^{\alpha}_{B_k} f_l\|_{q,\nu}}{\|f_l\|_{p,\nu}} \geqslant \frac{\Big(\int_{|x|>l} \big|I^{\alpha}_{B_k} f_l(x)\big|^q \prod_{j=1}^{k} x^{2\nu_j}_{n-k+j} \, \mathrm{d}x\Big)^{1/q}}{\Big(c \, \frac{l^{n+2|\nu|}}{n+2|\nu|}\Big)^{1/p}} \\
&\geqslant \frac{c(\alpha) 2^{\alpha-n-2|\nu|} c \, \frac{l^{n+2|\nu|}}{n+2|\nu|}}{c^{1/p} \frac{l^{(n+2|\nu|)/p}}{(n+2|\nu|)^{1/p}}} \Big(\int_{|x|>l} |x|^{(\alpha-n-2|\nu|)q} \prod_{j=1}^{k} x^{2\nu_j}_{n-k+j} \, \mathrm{d}x\Big)^{1/q} \\
&= c(\alpha) 2^{\alpha-n-2|\nu|} \Big(\frac{c}{n+2|\nu|}\Big)^{1-1/p} l^{n+2|\nu| - (n+2|\nu|)/p} \\
&\quad \times \Big(c' \int_{l}^{\infty} r^{(\alpha-n-2|\nu|)q + n + 2|\nu| - 1} \, \mathrm{d}r\Big)^{1/q} \\
&= c(\alpha) 2^{\alpha-n-2|\nu|} \Big(\frac{c}{n+2|\nu|}\Big)^{1-1/p} l^{n+2|\nu| - (n+2|\nu|)/p} (c')^{1/q} \\
&\quad \times \Big[\frac{l^{(\alpha-n-2|\nu|)q + n + 2|\nu|}}{q(n+2|\nu|-\alpha) - (n+2|\nu|)}\Big]^{\frac{1}{q}} = \frac{c(\alpha) 2^{\alpha-n-2|\nu|} \Big(\frac{c}{n+2|\nu|}\Big)^{1-1/p} (c')^{1/q}}{\big[q(n+2|\nu|-\alpha) - (n+2|\nu|)\big]^{1/q}}.
\end{aligned}$$

Thus we obtain (2.4) by taking

$$B_\alpha(p,q,\nu)=\frac{c(\alpha)2^{\alpha-n-2|\nu|}\Bigl(\dfrac{c}{n+2|\nu|}\Bigr)^{1-1/p}(c')^{1/q}}{[q(n+2|\nu|-\alpha)-(n+2|\nu|)]^{1/q}},$$

where $B_\alpha(p,q,\nu)$ is independent of l and, $B_\alpha(p,q,\nu)\to\infty$ as $p\to1^+$ as desired.

Next, we assert that if $l<\frac{1}{2}$, then

$$\frac{\|I^\alpha_{B_k}g_l\|_{q,\nu}}{\|g_l\|_{p,\nu}}\geqslant C_\alpha(p,q,\nu) \tag{2.5}$$

holds for all p sufficiently near $(n+2|\nu|)/\alpha$, where $C_\alpha(p,q,\nu)$ is independent of l and $C_\alpha(p,q,\nu)\to\infty$ as $p\to\bigl((n+2\nu)/\alpha\bigr)^-$. In fact, if it is not the case, then there exists some K, which is independent of p, such that

$$\Bigl(\int_{|x|\leqslant l}\bigl|I^\alpha_{B_k}g_l(x)\bigr|^q\prod_{j=1}^k x_{n-k+j}^{2\nu_j}\,\mathrm{d}x\Bigr)^{\frac{1}{q}}\leqslant\|I^\alpha_{B_k}g_l\|_{q,\nu}\leqslant K\|g_l\|_{p,\nu}. \tag{2.6}$$

Now $q\to\infty$ as $p\to\bigl((n+2\nu)/\alpha\bigr)^-$, and by a similar argument as in [7], [12], it is easy to see that

$$\begin{aligned}\|g_l\|_{p,\nu}&=\Bigl(\int_{|x|\leqslant l}\Bigl\{|x|^{-\alpha}\Bigl|\log\frac{1}{|x|}\Bigr|^{-\frac{\alpha}{n+2|\nu|}(1+\varepsilon)}\Bigr\}^p\prod_{j=1}^k x_{n-k+j}^{2\nu_j}\,\mathrm{d}x\Bigr)^{\frac{1}{p}}\\&\leqslant\Bigl(\int_{|x|\leqslant l}\Bigl\{|x|^{-\alpha}\Bigl|\log\frac{1}{|x|}\Bigr|^{-(1+\varepsilon)}\Bigr\}^p\prod_{j=1}^k x_{n-k+j}^{2\nu_j}\,\mathrm{d}x\Bigr)^{\frac{1}{p}}<\infty.\end{aligned}$$

However, $(I^\alpha_{B_k}g_l)(x)$ is essentially unbounded near the origin since

$$\begin{aligned}(I^\alpha_{B_k}g_l)(x)=c(\alpha)&\int_{|y|\leqslant l}\frac{1}{|y|^{-\alpha}}\Bigl(\log\frac{1}{|y|}\Bigr)^{-\frac{\alpha}{n+2|\nu|}(1+\varepsilon)}\\&\times T^y\bigl(|x|^{-\alpha-n-2|\nu|}\bigr)\prod_{j=1}^k y_{n-k+j}^{2\nu_j}\,\mathrm{d}y\end{aligned}$$

is infinity at the origin as long as $\alpha(n+2|\nu|)^{-1}(1+\varepsilon)\leqslant1$.

Now let us suppose that there exists a constant A, independent of f, such that for all $f\in\mathrm{B}a$,

$$\|I^\alpha_{B_k}f_l\|_{\mathrm{B}a'}\leqslant A\|f\|_{\mathrm{B}a}. \tag{2.7}$$

It follows from (2.4) and (2.7) that

$$\sum_{m=1}^{\infty} \frac{a_m[B_\alpha(p_m, q_m, \nu)\|f_l\|_{p_m,\nu}]^m}{(A\|f_l\|_{\mathrm{Ba}})^m} \leqslant \sum_{m=1}^{\infty} \frac{a_m\|I^\alpha_{B_k} f_l\|^m_{q_m,\nu}}{\|I^\alpha_{B_k} f_l\|_{\mathrm{Ba}'}} = 1$$

In particular,

$$\frac{a_m^{1/m} B_\alpha(p_m, q_m, \nu)\|f_l\|_{p_m,\nu}}{A\|f_l\|_{\mathrm{Ba}}} \leqslant 1, \quad \text{or} \quad \frac{a_m^{1/m}\|f_l\|_{p_m,\nu}}{\|f_l\|_{\mathrm{Ba}}} \leqslant \frac{A}{B_\alpha(p_m, q_m, \nu)}, \tag{2.8}$$

where $B_\alpha(p_m, q_m, \nu) \to \infty$ as $p_m \to 0^+$. So if β in (2.3) does not exist, then there is a $p_{m'} > 1$ such that

$$a_m^{1/m} \frac{\|f_l\|_{p_m,\nu}}{\|f_l\|_{\mathrm{Ba}}} < \frac{1}{2} \quad \text{for } p_m \in (1, p_{m'}] \text{ and } l \in (0, \infty). \tag{2.9}$$

Without loss of generality, we assume there is $a_{m''}$ such that $a_{m''} \neq 0$, $p_{m''} < p_{m'}$ and

$$0 < a_{m''}^{1/m''} \frac{\|f_l\|_{p_{m''}}}{\|f_l\|_{\mathrm{Ba}}} < \frac{1}{2}. \tag{2.10}$$

Now let us choose l_0 large enough such that $cl_0^{n+2|\nu|}(n+2|\nu|)^{-1} > 1$ and

$$M\left(c\frac{l_0^{n+2|\nu|}}{n+2|\nu|}\right)^{1/p_{m'}} < a_{m''}^{1/m''}\left(c\frac{l_0^{n+2|\nu|}}{n+2|\nu|}\right)^{1/p_{m''}},$$

where $M = \sup(a_{m''}^{1/m''} : m = 1, 2, \ldots) < \infty$. Then we have for any $p_m > p_{m'}$

$$a_m^{1/m}\left(c\frac{l_0^{n+2|\nu|}}{n+2|\nu|}\right)^{1/p_m} \leqslant M\left(c\frac{l_0^{n+2|\nu|}}{n+2|\nu|}\right)^{1/p_{m'}} \leqslant a_{m''}^{1/m''}\left(c\frac{l_0^{n+2|\nu|}}{n+2|\nu|}\right)^{1/p_{m''}}.$$

Thus by using (2.10) and the fact $\|f_{l_0}\|_{p,\nu} = \left(cl_0^{n+2|\nu|}(n+2|\nu|)^{-1}\right)^{1/p}$, we have for any $p_m > p_{m'}$,

$$a_m^{1/m}\frac{\|f_{l_0}\|_{p_m,\nu}}{\|f_{l_0}\|_{\mathrm{Ba}}} \leqslant a_{m''}^{1/m''}\frac{\|f_{l_0}\|_{p_{m''},\nu}}{\|f_{l_0}\|_{\mathrm{Ba}}} < \frac{1}{2}. \tag{2.11}$$

This together with (2.9) gives

$$\sum_{m=1}^{\infty} a_m \frac{\|f_{l_0}\|^m_{p_m}}{\|f_{l_0}\|^m_{\mathrm{Ba}}} < \sum_{m=1}^{\infty}\left(\frac{1}{2}\right)^m = 1$$

which contradicts the definition of the Ba-norm since $I(f_{l_0}, 1/\|f_{l_0}\|\}_{\mathrm{B}a}) = 1$. We have thus proved that $\beta > 1$. Next we prove $\gamma < (n+2|\nu|)/\alpha$. Using (2.5) and (2.7), we obtain

$$\sum_{m=1}^{\infty} a_m \frac{\big(C_\alpha(p_m, q_m, \nu)\|g_l\|_{p_m,\nu}\big)^m}{\big(A\|g_l\|_{\mathrm{B}a}\big)^m} \leqslant 1.$$

So $a_m^{1/m}\|g_l\|_{p_m,\nu}/\|g_l\|_{\mathrm{B}a} \leqslant A/C_\alpha(p_m, q_m, \nu)$. Note that $C_\alpha(p_m, q_m, \nu) \to \infty$ as $p_m \to \big((n+2|\nu|)/\alpha\big)^-$. Thus if γ does not exist, we can find $p_{m'}$ large enough such that

$$\text{(2.12)} \qquad a_m^{1/m}\frac{\|g_l\|_{p_m,\nu}}{\|g_l\|_{\mathrm{B}a}} \leqslant \frac{1}{2} \quad \text{for } p_m \in \Big[p_{m'}, \frac{n+2|\nu|}{\alpha}\Big) \text{ and } l \in (0,\infty).$$

Similarly we may assume there exists a m'' such that $p_{m''} > p_{m'}$ and

$$\text{(2.13)} \qquad 0 < a_{m''}^{1/m''}\frac{\|g_l\|_{p_{m''},\nu}}{\|g_l\|_{\mathrm{B}a}} < \frac{1}{2} \quad \text{for } l \in (0,\infty).$$

Now choose l_1 small enough such that $1/\varepsilon(-\log l_1)^\varepsilon < 1$ and

$$M\left[\frac{C_1}{\varepsilon(-\log l_1)^\varepsilon}\right]^{1/p_{m'}} < a_{m''}^{1/m''}\left[\frac{C_1}{\varepsilon(-\log l_1)^\varepsilon}\right]^{1/p_{m''}};$$

then for any $p_m < p_{m'}$,

$$a_m^{1/m}\left[\frac{C_1}{\varepsilon(-\log l_1)^\varepsilon}\right]^{1/p_m} \leqslant M\left[\frac{C_1}{\varepsilon(-\log l_1)^\varepsilon}\right]^{1/p_{m'}} < a_{m''}^{1/m''}\left[\frac{C_1}{\varepsilon(-\log l_1)^\varepsilon}\right]^{1/p_{m''}},$$

thus by using (2.12) and the fact that $\|g_{l_1}\|_{p,\nu} \leqslant [C_1/\varepsilon(-\log l_1)^\varepsilon]^{1/p}$, we have

$$\text{(2.14)} \qquad a_m^{1/m}\frac{\|g_{l_1}\|_{p_m,\nu}}{\|g_{l_1}\|_{\mathrm{B}a}} \leqslant a_{m''}^{1/m''}\frac{\|g_{l_1}\|_{p_{m''},\nu}}{\|g_{l_1}\|_{\mathrm{B}a}} \leqslant \frac{1}{2} \quad \text{for } p_m < p_{m'}.$$

From the equations (2.12) and (2.14) we see that

$$\sum_{m=1}^{\infty} a_m^{1/m}\frac{\|g_{l_1}\|_{p_m,\nu}}{\|g_{l_1}\|_{\mathrm{B}a}} \leqslant 1.$$

So we again have a contradiction to the definition of the Ba-norm, and the theorem is proved.

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