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# ON SOME INTERPOLATION RULES FOR LATTICE ORDERED GROUPS

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*Abstract.* Let $\alpha$ be an infinite cardinal. In this paper we define an interpolation rule $\mathrm{IR}(\alpha)$ for lattice ordered groups. We denote by $C(\alpha)$ the class of all lattice ordered groups satisfying $\mathrm{IR}(\alpha)$, and prove that $C(\alpha)$ is a radical class.

*Keywords*: lattice ordered group, interpolation rule, radical class

*MSC 2000*: 06F15

## 1. Introduction

Darnel and Martinez [4] studied the relations between radical classes of lattice ordered groups and classes of compact Hausdorff spaces. In Section 8 of [4] the following condition (called the *σ-interpolation property*) for a lattice ordered group $G$ was considered: for each pair of sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in $G$ with

$$a_1 < a_2 < \ldots < b_2 < b_1$$

there exists $c \in G$ such that $a_m < c < b_n$ for each $m$ and $n$. The authors remark that this condition had been dealt with for Boolean algebras by Walker [10].

In the present paper we consider analogous interpolation rules for $G$ with the distinction that transfinite sequences can be also taken into account. For each infinite cardinal $\alpha$ we define an interpolation rule $\mathrm{IR}(\alpha)$ concerning the lattice ordered group $G$.

We prove that for each infinite cardinal $\alpha$ the class $C(\alpha)$ of all lattice ordered groups satisfying $\mathrm{IR}(\alpha)$ is a radical class. The correspondence $\alpha \to C(\alpha)$ is an

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injective mapping of the class of all infinite cardinals into the collection of all radical classes of lattice ordered groups.

For a lattice ordered group $H$ let $k(H)$ be the class of all infinite cardinals $\alpha$ such that $H$ satisfies the condition $\mathrm{IR}(\alpha)$. We distinguish the following cases:

a) $k(H)=\emptyset$; in this case we put $f(H)=0$.

b) $k(H)$ is the class of all infinite cardinals; then we set $f(H)=\infty$.

c) In the other cases we put $f(H)=\sup k(H)$.

Let $\mathscr{G}$ be the class of all lattice ordered groups and $G\in\mathscr{G}$. We prove that

$$\{G_1\in\mathscr{G}\colon\ f(G_1)\geqslant f(G)\}$$

is a radical class.

We remark that a different type of interpolation rule (denoted as the Riesz interpolation property) for partially ordered groups was studied by Goodearl in the monograph [5]. Further, the term "$\sigma$-interpolation" for lattice ordered groups was used in a different meaning by Darnel [3] and the author [8].

The notion of a radical class of lattice ordered groups was introduced by the author [7]. In what follows, we always write "radical class" meaning a radical class of lattice ordered groups.

From the result of Holland [6] it follows that each variety of lattice ordered groups is a radical class. Conrad [2] dealt with $K$-radical classes; these are defined to be radical classes which can be characterized by using merely the lattice properties of the corresponding lattice ordered groups. A problem on radical classes proposed in [7] was solved by Medvedev [9].

## 2. Preliminaries

For lattice ordered groups we apply the terminology and notation as in Conrad [1].

Let $G$ be a lattice ordered group. The system of all convex $\ell$-subgroups of $G$ will be denoted by $c(G)$. Under the partial order defined by the set-theoretical inclusion, $c(G)$ is a complete lattice.

A nonempty subclass $A$ of $\mathscr{G}$ is a *radical class* if it satisfies the following conditions:

a) $A$ is closed with respect to isomorphisms;

b) if $G\in A$, then $c(G)\subseteq A$;

c) if $G\in\mathscr{G}$ and $\{G_j\}_{j\in J}\subseteq c(G)\cap A$, then $\bigvee_{j\in J} G_j\in A$.

Let $\beta$ be a limit ordinal. We denote by $I(\beta)$ the set of all ordinals less than $\beta$. Let $G\in\mathscr{G}$ and $g_i\in G$ for each $i\in I(\beta)$. Then $(g_i)_{i\in I(\beta)}$ is a *transfinite sequence* (of type $\beta$). This transfinite sequence is *strictly increasing* (*strictly decreasing*) if

$g_{i(1)} < g_{i(2)}$ (or $g_{i(1)} > g_{i(2)}$, respectively) whenever $i(1), i(2) \in I(\beta)$ and $i(1) < i(2)$. Further, $(g_i)_{i\in I(\beta)}$ is *increasing* (*decreasing*) if $g_{i(1)} \leqslant g_{i(2)}$ ($g_{i(1)} \geqslant g_{i(2)}$) for $i(1), i(2) \in I(\beta)$ with $i(1) < i(2)$.

Instead of $(g_i)_{i\in I(\beta)}$ we apply also the notation $(g_i)_{i<\beta}$.

Let $\alpha$ be an infinite cardinal and let $G$ be a lattice ordered group. We define the condition $\mathrm{IR}(\alpha)$ for $G$ as follows.

($\mathrm{IR}(\alpha)$) Assume that

(i) $\beta_1$ and $\beta_2$ are limit ordinals with $\operatorname{card} I(\beta_1) \leqslant \alpha$, $\operatorname{card} I(\beta_2) \leqslant \alpha$;

(ii) $(a_i)_{i<\beta_1}$ is a strictly increasing transfinite sequence of elements of $G$;

(iii) $(b_i)_{i<\beta_2}$ is a strictly decreasing transfinite sequence of elements of $G$;

(iv) $a_{i(1)} < b_{i(2)}$ for each $i(1) \in I(\beta_1)$ and $i(2) \in I(\beta_2)$.

Then there is an element of $c$ of $G$ such that $a_{i(1)} < c < a_{i(2)}$ for each $i(1) \in I(\beta_1)$ and $i(2) \in I(\beta_2)$.

Further, we denote by $\mathrm{IR}_0(\alpha)$ the condition which we obtain from $\mathrm{IR}(\alpha)$ if we perform the following modifications: in (ii), we suppose that $(a_i)_{i<\beta(1)}$ is increasing; in (iii), we suppose that $(b_i)_{i<\beta(2)}$ is decreasing; in (iv) we have $a_{i(1)} \leqslant b_{i(2)}$; and, finally, for the element $c$ we get $a_{i(1)} \leqslant c \leqslant b_{i(2)}$.

**Lemma 2.1.** *Let $G \in \mathscr{G}$. Then the conditions* $\mathrm{IR}(\alpha)$ *and* $\mathrm{IR}_0(\alpha)$ *for $G$ are equivalent.*

Proof. a) Assume that the condition $\mathrm{IR}_0(\alpha)$ is valid for $G$. Further, suppose that the assumptions of $\mathrm{IR}(\alpha)$ (i.e., (i)–(iv)) are fulfilled. Then in view of $\mathrm{IR}_0(\alpha)$, there exists $c \in G$ such that $a_{i(1)} \leqslant c \leqslant a_{i(2)}$ for each $i(1) \leqslant \beta_1$, $i(2) \leqslant \beta_2$.

We have to verify that $a_{i(1)} < c < a_{i(2)}$ for each $i(1) \leqslant \beta_1$ and each $i(2) \leqslant \beta_2$. By way of contradiction, suppose that there exists $i(1) \leqslant \beta_1$ with $a_{i(1)} = c$. There exists $i(3) \leqslant \beta_1$ such that $i(1) < i(3)$. Then $a_{i(3)} > a_{i(1)}$, whence $a_{i(3)} > c$, which is a contradiction. Thus $a_{i(1)} < c$ for each $i(1) < \beta_1$. Analogously, $c < a_i(2)$ for each $i(2) < \beta_2$. Therefore $\mathrm{IR}(\alpha)$ is valid for $G$.

b) Conversely, assume that the condition $\mathrm{IR}(\alpha)$ holds for $G$. We distinguish the following cases.

b1) Suppose that there exists $i(1) < \beta_1$ such that $a_{i(1)} = \max\{a_i \colon i < \beta_1\}$. We put $c = a_{i(1)}$ and we have $a_{i(1)} \leqslant c \leqslant a_{i(2)}$ for each $i(1) < \beta_1$, $i(2) < \beta_2$. Hence $\mathrm{IR}_0(\alpha)$ holds for $G$.

b2) Assume that there exists $i(2) < \beta_2$ with $b_{i(2)} = \min\{b_i \colon i < \beta_2\}$. Then, similarly as in b1), $\mathrm{IR}_0(\alpha)$ is valid for $G$.

b3) Suppose that neither the assumption b1) nor the assumption b2) is satisfied. Then there exists a strictly increasing transfinite sequence $(a'_i)_{i<\beta'_1}$ of elements of $G$ such that

(i) for each $i < \beta_1'$ there exists $i(1) < \beta_1$ with $a_i' = a_{i(1)}$;
(ii) for each $i(1) < \beta_1$ there exists $i < \beta_1'$ such that $a_{i(1)} < a_i'$;
(iii) $\beta_1' \leqslant \beta_1$.

Similarly, there exists a strictly decreasing transfinite sequence $(b_i')_{i<\beta_2'}$ such that

($\mathrm{i}_1$) for each $i < \beta_2'$ there exists $i(2) < \beta_2$ with $b_i' = b_{i(2)}$;
($\mathrm{ii}_1$) for each $i(2) < \beta_2$ there exists $i < \beta_2'$ such that $b_{i(2)} > b_i'$;
($\mathrm{iii}_1$) $\beta_2' \leqslant \beta_2$.

Then we have $a'_{i(3)} < b'_{i(4)}$ for each $i(3) < \beta_1'$ and each $i(4) < \beta_2'$. Thus in view of $\mathrm{IR}(\alpha)$ there exists $c \in G$ such that $a'_{i(3)} < c < b'_{i(4)}$ for each $i(3) < \beta_1'$ and each $i(4) < \beta_2'$. According to (ii) and ($\mathrm{ii}_1$) we obtain $a_{i(1)} < c < b_{i(2)}$ for each $i(1) < \beta_1$ and each $i(2) < \beta_2$. Hence the condition $\mathrm{IR}_0(\alpha)$ is satisfied for $G$. □

## 3. The class $C(\alpha)$

Let $\alpha$ be an infinite cardinal and $a, b \in G \in \mathscr{G}$, $a \leqslant b$. We say that the interval $[a, b]$ of $G$ satisfies the condition $\mathrm{IR}_0(\alpha)$ if $\mathrm{IR}_0(\alpha)$ holds whenever the elements $a_i$, $b_i$ under consideration belong to the interval $[a, b]$.

**Lemma 3.1.** *Let $a, b \in G^+$. Assume that both the intervals $[0, a]$ and $[0, b]$ satisfy the condition $\mathrm{IR}_0(\alpha)$. Then the interval $[0, a + b]$ also satisfies this condition.*

Proof. Since the interval $[a, a + b]$ is isomorphic to $[0, b]$ we conclude that $[a, a + b]$ satisfies the condition $\mathrm{IR}_0(\alpha)$.

Let $\beta_1$ and $\beta_2$ be as above. Assume that $(a_i)_{i<\beta_1}$, $(b_i)_{i<\beta_2}$ satisfy the conditions from $\mathrm{IR}_0(\alpha)$ and that these elements belong to the interval $[0, a + b]$.

For each $x \in [0, a + b]$ we put $x^1 = x \wedge a$, $x^2 = x \vee b$. Consider the transfinite sequences

$$(a_i^1)_{i<\beta_1}, \quad (b_i^1)_{i<\beta_1}.$$

From the assumptions concerning $a_i$ and $b_i$, and from the fact that $[0, a]$ satisfies $\mathrm{IR}_0(\alpha)$ we infer that there exists $y \in [0, a]$ such that

$$a^1_{i(1)} \leqslant y \leqslant b^1_{i(2)} \tag{1}$$

for each $i(1) < \beta_1$ and each $i(2) < \beta_2$.

Analogously, there exists $z \in [a, a + b]$ such that

$$a^2_{i(1)} \leqslant z \leqslant b^2_{i(2)} \tag{2}$$

for each $i(1) < \beta_1$ and each $i(2) < \beta_2$.

For each $x \in [0, a+b]$ we have

$$x = x^1 + (-x^1 + x) = x^1 + (-a + x^2).$$

(cf. Fig. 1). Put $c = y + (-a + z)$. Then $0 \leqslant c \leqslant a + b$. In view of (1) and (2) we obtain

$$a_{i(1)} = a^1_{i(1)} + (-a^1_{i(1)} + a_{i(1)}) = a^1_{i(1)} + (-a + a^2_{i(1)}) \leqslant y + (-a + z) = c$$

for each $i(1) < \beta_1$.

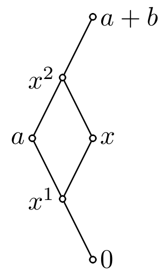


Fig. 1

Analogously, we get $b_{i(2)} \geqslant c$ for each $i(2) < \beta_2$. Hence $\mathrm{IR}_0(\alpha)$ is valid for the interval $[0, a+b]$. □

The following assertion is an immediate consequence of the definition of $C(\alpha)$.

**Lemma 3.2.** *Let $\alpha$ be an infinite cardinal. The class $C(\alpha)$ is closed with respect to isomorphisms. If $G \in C(\alpha)$, then $c(G) \subseteq C(\alpha)$. If $G, G' \in \mathscr{G}$, $G \in C(\alpha)$ and if the underlying lattices of $G$ and $G'$ are isomorphic, then $G' \in C(\alpha)$.*

The following result is well-known.

**Lemma 3.3.** *Let $G \in \mathscr{G}$, $\emptyset \neq \{G_j\}_{j \in J} \subseteq c(G)$, $\bigvee_{j \in J} G_j = G^0$, $0 \leqslant x \in G^0$. Then there exist $j(1), j(2), \ldots, j(n) \in J$ and elements $a_1 \in G_{j(1)}, \ldots, a_n \in G_{j(n)}$ such that $x = a_1 + a_2 + \ldots + a_n$.*

**Lemma 3.4.** *Let $G \in \mathscr{G}$, $\emptyset \neq \{G_j\}_{j \in J} \subseteq c(G) \cap C(\alpha)$. Then $\bigvee_{j \in J} G_j \in C(\alpha)$.*

P r o o f. Let $G^0$ be as in 3.3. It is obvious that $G^0$ satisfies the condition $\mathrm{IR}_0(\alpha)$ if and only if, for each $a, b \in G^0$ with $a \leqslant b$, the interval $[a, b]$ satisfies this condition. Since the interval $[a, b]$ is isomorphic to the interval $[0, b - a]$, we can restrict ourselves to the intervals of the form $[0, x]$ for $0 \leqslant x$. Let $a_1, a_2, \ldots, a_n$ be as in 3.3. By using 3.1 and the obvious induction on $n$ we obtain that $[0, x]$ satisfies $\mathrm{IR}_0(\alpha)$; hence $G^0$ satisfies this condition. Then according to 2.1, $G^0$ belongs to $C(\alpha)$. □

**Theorem 3.5.** *Let $\alpha$ be an infinite cardinal. Then $C(\alpha)$ is a $K$-radical class.*

Proof. This is a consequence of 3.2 and 3.4. □

**Example 3.6.** Let $\alpha$ be an infinite cardinal and let $G$ be a complete lattice ordered group. Further, let $(a_i)_{i<\beta_1}$ and $(b_i)_{i<\beta_2}$ be as in the definition of IR$(\alpha)$. Then $\bigvee_{i<\beta_1} a_i$ exists in $G$; we denote this element by $c$. Then, with this element $c$ under consideration, IR$(\alpha)$ is satisfied for $G$. Hence $G \in C(\alpha)$. Therefore for each infinite cardinal $\alpha$, $C(\alpha)$ is a proper class.

It is obvious that if $\alpha_1$ and $\alpha_2$ are infinite cardinals with $\alpha_1 < \alpha_2$, then $C(\alpha_2) \subseteq C(\alpha_1)$. Let $Q$ be the additive group of all rationals with the natural linear order. It is easy to verify that $Z$ does not belong to $C(\aleph_0)$. Hence, for each infinite cardinal $\alpha$, $Z \notin C(\alpha)$; thus $C(\alpha) \neq \mathscr{G}$.

**Example 3.7.** Let $\alpha$ be an infinite cardinal. Let $T$ be a linearly ordered set which is isomorphic to the first ordinal whose cardinality is equal to $\alpha$. Hence for each $t \in T$ we have card$\{t_1 \in T\colon\ t_1 \leqslant t\} < \alpha$. Let $G(\alpha)$ be the set of all real functions $x$ defined on $T$ having the property that there exists $t_x \in T$ with $x(t) = x(t_x)$ for each $t > t_x$.

Let $\alpha_1$ be an infinite cardinal; assume that $\alpha_1 < \alpha$. Further, let $\beta_1$ and $\beta_2$ be limit ordinals with card $\beta_i \leqslant \alpha_1$ $(i = 1, 2)$. Suppose that $(a_i)_{i<\beta_1}$ and $(b_i)_{i<\beta_2}$ are transfinite sequences of elements of $G(\alpha)$ such that the conditions (ii), (iii) and (iv) from the definition of IR$(\alpha)$ are satisfied.

There exists $t_0 \in T$ such that $t_0 > t_{a_i}$ for each $i < \beta_1$ and $t_0 > t_{b_i}$ for each $i > \beta_2$. Hence $a_i$ $(i < \beta_1)$ and $b_i$ $(b_i < \beta_2)$ are constants for $t \geqslant t_0$.

Let $t \in T$. The set $\{a_i(t)\colon\ i < \beta_1\}$ is upper bounded, hence there exists a real

$$c_t = \sup\{a_i(t)\colon\ i < \beta_1\}.$$

Consider a real function $y$ on $T$ such that $y(t) = c_t$ for each $t \in T$. Then $y$ is a constant for $t \geqslant t_0$, whence $y \in G(\alpha)$. Further, we have

$$y = \bigvee_{i<\beta_1} a_i.$$

Hence $a_{i(1)} < y$ for each $i(1) < \beta_1$ and $y < b_{i(2)}$ for each $i(2) < \beta_2$.

We have verified that $G(\alpha)$ satisfies the condition IR$(\alpha_1)$. Now we want to show that $G(\alpha)$ does not satisfy the condition IR$(\alpha)$.

Put $\beta_1 = \beta_2 = T$. We define elements $a_i$ and $b_i$ $(i < \beta_1)$ as follows.

Let $t \in T$. Then $t$ can be uniquely expressed in the form $t = t_1 + t_2$, where

(i) $t_1 = 0$ or $t_0$ is a limit ordinal,

(ii) $t_2$ is a non-negative integer.

We have to define $a_i(t)$ and $b_i(t)$. Recall that both $i$ and $t$ are elements of $T$; let $i = t_1 + t_2$ (under the notation as above).

a) If $t > i$, then we put

$$a_i(t) = 0, \quad b_i(t) = 3.$$

b) Let $t \leqslant i$. We set

$$a_i(t) = b_i(t) = \begin{cases} 1 & \text{if } t \text{ is even,} \\ 2 & \text{if } t \text{ is odd.} \end{cases}$$

Then the conditions (ii), (iii) and (iv) from the definition of $\mathrm{IR}(\alpha)$ are satisfied. But there is no $c \in C(\alpha)$ such that $a_i < c < b_i$ for each $i < \beta_1$. Hence $G(\alpha)$ does not satisfy $\mathrm{IR}(\alpha)$.

As a corollary we obtain:

**Proposition 3.8.** *Let $\alpha_1$ and $\alpha_2$ be infinite cardinals, $\alpha_1 < \alpha_2$. Then $C(\alpha_2) \subset C(\alpha_1)$.*

**Corollary 3.9.** *The correspondence $\alpha \to C(\alpha)$ is an injective mapping of the class of all infinite cardinals into the collection of all radical classes.*

**Proposition 3.10.** *Let $\alpha$ be an infinite cardinal. Then the class $C(\alpha)$ is closed with respect to direct products.*

Proof. Let $\{G_j\}_{j\in J} \subseteq \mathscr{G}$, $G = \prod_{j\in J} G_j$. Assume that all $G_j$ belong to $C(\alpha)$. Then in view of 2.1, all $G_j$ satisfy the condition $\mathrm{IR}_0(\alpha)$. This yields that $G$ satisfies this condition as well. By using 2.1 again we conclude that $G$ satisfies the condition $\mathrm{IR}(\alpha)$. Therefore $G$ belongs to $C(\alpha)$. □

We conclude this section by remarking that if we replace, in the above construction of $C(\alpha)$, the infinite cardinal $\alpha$ by a limit ordinal $\beta$, then by applying the same method we obtain again a radical class; let us denote it by $C_1(\beta)$. But there exist distinct limit ordinals $\beta_1$ and $\beta_2$ such that $C_1(\beta_1) = C_1(\beta_2)$; hence the result analogous to 3.8 does not hold in this case.

## 4. The mapping $f$

Let $f$ be as in Section 1. We start by giving some examples.

**Example 4.1.** Let $G$ be a complete lattice ordered group. Then in view of 3.6, $G \in C(\alpha)$ for each infinite cardinal $\alpha$. Hence $f(G) = \infty$.

**Example 4.2.** Let $Q$ be the additive group of all rationals with the natural linear order. Then $Q$ does not satisfy the condition $\mathrm{IR}(\aleph_0)$, whence $Q \notin C(\aleph_0)$. Thus according to 3.8 we have $Q \notin C(\alpha)$ for each infinite cardinal $\alpha$. Hence $f(Q) = 0$.

**Example 4.3.** Let $\alpha$ be an infinite cardinal, $\alpha > \aleph_0$. Consider the lattice ordered group $G(\alpha)$ from 3.7. Then we have $\alpha \notin k(G(\alpha))$ and $\alpha_1 \in k(G(\alpha))$ for each infinite cardinal $\alpha_1$ with $\alpha_1 < \alpha$. We distinguish two cases.

a) $\alpha$ is a limit cardinal. Then $f(G(\alpha)) = \alpha$.

b) $\alpha$ is a non-limit cardinal. Hence the set of all cardinals less than $\alpha$ has a greatest element $\alpha_0$. Then $f(G(\alpha)) = \alpha_0$.

For each $H \in \mathscr{G}$ we put

$$C(H) = \{G \in \mathscr{G} \colon\ f(G) \geqslant f(H)\}.$$

It is obvious that $C(H)$ is closed with respect to isomorphisms.

**Lemma 4.4.** *Let $H \in \mathscr{G}$, $G \in C(H)$ and $G_1 \in c(G)$. Then $G_1 \in C(H)$.*

P r o o f. If $\alpha$ is an infinite cardinal and $G \in C(\alpha)$, then $G_1$ belongs to $C(\alpha)$ as well. Hence $k(G_1) \supseteq k(G)$ and thus $f(G_1) \geqslant f(G)$. Therefore $G_1$ belongs to $C(H)$. □

**Lemma 4.5.** *Let $H, G \in \mathscr{G}$, $\{G_j\}_{j\in J} \subseteq c(G) \cap C(H)$. Put $G^0 = \bigvee_{j\in J} G_j$. Then $G^0 \in C(H)$.*

P r o o f. Let $\alpha \in k(H)$ and $j \in J$. In view of the assumption we have $f(G_j) \geqslant f(H)$. Our aim is to show that the relation

$$f(G^0) \geqslant f(H) \tag{1}$$

holds. We distinguish the following cases.

a) $f(H) = 0$. We clearly have $f(G^0) \geqslant 0$, whence (1) is valid.

b) $f(H) = \infty$. Then $f(G^0) = \infty$ for each $j \in J$. Thus for each infinite cardinal $\alpha$ and each $j \in J$ we get $G_j \in C(\alpha)$, yielding $G^0 \in C(\alpha)$. Thus $f(G^0) = \infty$.

c) There are infinite cardinals $\alpha_1$ and $\alpha_0$ such that $f(H) = \alpha_0$ and $\alpha_1$ is the greatest cardinal which is less than $\alpha_0$. In this case we necessarily have $H \in C(\alpha_0)$. Let $j \in J$. If $G_j \notin C(\alpha_0)$, then $f(G_j) \leqslant \alpha_1$, which is a contradiction. Thus all $G_j$ belong to $C(\alpha_0)$ and hence $G^0$ belongs to $C(\alpha_0)$ as well. Then $f(G^0) \geqslant \alpha_0$.

d) $f(H) = \alpha_0$ is a limit cardinal, $\alpha_0 \neq \aleph_0$. If $\alpha < \alpha_0$, $j \in J$, then since $f(G_j) \geqslant f(H)$, we obtain $G_j \in C(\alpha)$. This implies that $G^0 \in C(\alpha)$ and thus $f(G^0) \geqslant \alpha_0$.

e) In the remaining case we have $f(H) = \aleph_0$. If $H$ does not belong to $C(\aleph_0)$, then $H$ does not satisfy $\mathrm{IR}(\aleph_0)$, thus $k(H) = \emptyset$. Then $f(H) = 0$, which is a contradiction. Let $j \in J$. If $G_j \notin C(\aleph_0)$, then we have $f(G_j) = 0$, which is impossible in view of $f(G_j) \geqslant f(H)$. Thus $G_j \in C(\aleph_0)$ and then $G^0$ belongs to $C(\aleph_0)$ as well. Hence $f(G^0) \geqslant \aleph_0$. □

Summarizing, we obtain

**Theorem 4.6.** *Let $H \in \mathscr{G}$. Then $C(H)$ is a radical class.*

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