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ON THE AFFINE COMPLETENESS OF LATTICE ORDERED GROUPS

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Abstract. In the paper it is proved that a nontrivial direct product of lattice ordered groups is never affine complete.

Keywords: lattice ordered group, affine completeness, direct product

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1. Introduction

Polynomial completeness and affine completeness of various algebraic structures have been investigated in a rather large series of papers and systematically studied in the monograph [3].

The problem of the existence of a nontrivial affine complete lattice ordered group remains open (cf. [3], p. 331, Problem 5.6.19).

The following negative results have been proved.

(A_1) Let G be a complete lattice ordered group. Then G is affine complete if and only if $G = \{0\}$. (Cf. [1].)

More generally, we have

(A_2) Let G be an abelian projectable lattice ordered group. Then G is affine complete if and only if $G = \{0\}$. (Cf. [2].)

(A_3) Let G be an abelian lattice ordered group, $G = A \times B$, $A \neq \{0\} \neq B$. Then G is not affine complete. (Cf. [1].)

(A_4) A direct product of a nonzero subdirectly irreducible lattice ordered group and any lattice ordered group is never affine complete (cf. [3], Section 3.6.4).

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In the present paper we prove that (A_2) and (A_3) remain valid without assuming that G is abelian.

2. Preliminaries

We apply the terminology as in [3]. An algebra is *affine complete* if every congruence compatible function is induced by a polynomial.

Let $G \neq \{0\}$ be a lattice ordered group. We denote by $P(G)$ the set of all polynomials over G and by $\operatorname{Con} G$ the set of all congruence relations on G.

Let $p(x) \in P(G)$. From the basic properties of lattice ordered groups we easily obtain that $p(x)$ can be represented in the form

$$p(x) = \bigwedge_{i \in I} \bigvee_{j \in J(i)} \left(a_{ij}^1 + a_{ij}^2 + \ldots + a_{ij}^{n(i,j)}\right), \tag{1}$$

where I, $J(i)$ are nonempty finite sets and for each $i \in I$, $j \in J(i)$, $k \in \{1, 2, \ldots, n(i,j)\}$ we have either

a) $a_{ij}^k \in G$,

or

b) $a_{ij}^k \in \{x, -x\}$.

We denote by $[a]$ the set of all triples (i, j, k) (under the notation as above) such that the condition a) is valid.

In this section we assume that $[a] \neq \emptyset$. Let m_0 be the number of elements of the set $[a]$.

There exists $s \in G^+$ such that

$$s \geqslant \bigvee_{(i,j,k) \in [a]} |a_{ij}^k|.$$

This condition is satisfied if and only if

$$s \geqslant a_{ij}^k \quad \text{and} \quad s \geqslant -a_{ij}^k \qquad \text{for each } (i,j,k) \in [a]. \tag{α}$$

Put

$$x_1 = 3m_0 s.$$

In the present section we deal with the properties of the element $p(x_1)$.

Let i, j be fixed and let $1 \leqslant k < n(i,j)$. Suppose that

$$a_{ij}^k = x, \quad a_{ij}^{k+1} \in G.$$

Then in the corresponding expression for $p(x_1)$ (cf. (1)) we have

$$x_1 + a_{ij}^k = (x_1 + a_{ij}^k - x_1) + x_1.$$

Since

$$-s \leqslant a_{ij}^k \leqslant s,$$

we obtain

$$-s \leqslant x_1 + a_{ij}^k - x_1 \leqslant s.$$

In a similar way we can proceed if $a_{ij}^k = -x$.

We put

$$p_{ij}(x) = a_{ij}^1 + a_{ij}^2 + \ldots + a_{ij}^{n(i,j)}.$$

Applying the above mentioned steps and using the obvious induction we conclude that $p_{ij}(x_1)$ can be written in the form

$$p_{ij}(x_1) = \bar{a}_{ij}^1 + \bar{a}_{ij}^2 + \ldots + \bar{a}_{ij}^{\ell(i,j)} + k_{ij}x_1, \tag{2}$$

where $0 \leqslant \ell(i,j) \leqslant n(i,j)$, k_{ij} is an integer and for each $k \in \{1, 2, \ldots, \ell(i,j)\}$ we have

$$\bar{a}_{ij}^k \in [-s, s].$$

Denote

$$\bar{\bar{a}}_{ij} = \bar{a}_{ij}^1 + \ldots + \bar{a}_{ij}^{\ell(i,j)}.$$

Keeping the element i fixed we put

$$\bar{j} = \{j(1) \in J(i)\colon\ k_{i,j(1)} = k_{ij}\}$$
$$p_{i\bar{j}}(x) = \bigvee_{i(1)\in\bar{j}} \big(a_{ij(1)}^1 + a_{ij(1)}^2 + \ldots + a_{ij(1)}^{n(i,j(1))}\big).$$

Then we get

$$p_{i\bar{j}}(x_1) = \bigvee_{j(1)\in\bar{j}} (\bar{\bar{a}}_{ij(1)} + k_{ij}x_1) = \Bigg(\bigvee_{j(1)\in\bar{j}} \bar{\bar{a}}_{ij(1)}\Bigg) + k_{ij}x_1.$$

We set

$$\bigvee_{j(1)\in\bar{j}} \bar{\bar{a}}_{ij(1)} = a_{ij}^*.$$

For each $j \in J(i)$ we have

$$\bar{\bar{a}}_{ij} \in [-m_0 s, m_0 s],$$

whence

$$a_{ij}^* \in [-m_0 s, m_0 s]. \tag{3}$$

Now let j and j' be elements of $J(i)$ such that $\bar{j} \neq \overline{j'}$. Hence we have $k_{ij} \neq k_{ij'}$.

2.1. Lemma. *Assume that* $k_{ij} < k_{ij'}$. *Then* $p_{i\bar{j}}(x_1) < p_{i\overline{j'}}(x_1)$.

Proof. We have

$$p_{i\bar{j}}(x_1) = a^*_{ij} + k_{ij}x_1, \quad p_{i\overline{j'}}(x_1) = a^*_{ij'} + k_{ij'}x_1.$$

We want to show that

$$(\alpha_1) \qquad a^*_{ij} + k_{ij}x_1 < a^*_{ij'} + k_{ij'}x_1.$$

The relation (α_1) is equivalent to

$$(\alpha_2) \qquad -a^*_{ij'} + a^*_{ij} < (k_{ij'} - k_{ij})x_1.$$

In view of (3) we get

$$-a^*_{ij'} \in [-m_0 s, m_0 s],$$

whence

$$-a^*_{ij'} + a^*_{ij} \in [-2m_0 s, 2m_0 s],$$

thus according to the definition of x_1 we obtain

$$-a^*_{ij'} + a^*_{ij} < x_1 \leqslant (k_{ij'} - k_{ij})x_1,$$

which completes the proof. □

For $i \in I$ we put

$$p_i(x) = \bigvee_{j \in J(i)} \left(a^1_{ij} + a^2_{ij} + \ldots + a^{n(i,j)}_{ij}\right) = \bigvee_{j \in J(i)} p_{ij}(x).$$

Hence we obtain

$$p_i(x_1) = \bigvee_{j \in J(i)} p_{ij}(x_1) = \bigvee_{j \in J(i)} p_{i\bar{j}}(x_1).$$

There exists a pair $(i, j(0))$ such that

$$k_{i,j(0)} = \max\{k_{ij}\}_{j \in J(i)}.$$

Then in view of 2.1 we conclude

2.2. Lemma. $p_i(x_1) = a^*_{ij(0)} + k_{ij(0)}x_1$.

Let us now write $j(i)$ instead of $j(0)$. Since

$$p(x) = \bigwedge_{i\in I} p_i(x)$$

we get

$$p(x_1) = \bigwedge_{i\in I} p_i(x_1) = \bigwedge_{i\in I} (a^*_{i,j(i)} + k_{ij(i)}x_1).$$

For the indices belonging to I we proceed analogously as we did above for the indices belonging to $J(i)$.

Let $i \in I$. We put

$$\bar{i} = \{i(1) \in I\colon\ k_{i(1),j(i(1))} = k_{i,j(i)}\},$$

$$p_{\bar{i}}(x) = \bigwedge_{i(1)\in\bar{i}} p_{i(1)}(x),$$

$$a^{**}_{\bar{i}} = \bigwedge_{i(1)\in\bar{i}} a^*_{i(1),j(i(1))}.$$

Then we have

$$\begin{aligned} p_{\bar{i}}(x_1) &= \bigwedge_{i(1)\in\bar{i}} p_{i(1)}(x_1) = \bigwedge_{i(1)\in\bar{i}} (a^*_{i(1),j(i(1))} + k_{i,j(i)}x_1) \\ &= \Bigg(\bigwedge_{i(1)\in\bar{i}} a^*_{i(1),j(i(1))}\Bigg) + k_{i,j(1)}x_1 = a^{**}_i + k_{i,j(i)}x_1. \end{aligned}$$

From (3) we conclude that

$$a^{**}_{\bar{i}} \in [-m_0 s, m_0 s] \tag{4}$$

for each $i \in I$.

Now let i and i' be elements of I such that $\bar{i} \neq \overline{i'}$, i.e., $k_{i,j(i)} \neq k_{i'j(i')}$. By an argument similar to that in the proof of 2.1 we obtain

2.3. Lemma. *Assume that* $k_{i,j(i)} < k_{i'j(i')}$. *Then* $p_{\bar{i}}(x_1) < p_{\overline{i'}}(x_1)$.

There exists $i(0) \in I$ such that

$$k_{i(0),j(i(0))} = \min_{i\in I}\{k_{i,j(i)}\}.$$

Then in view of 2.3 we have

2.4. Lemma. $p(x_1) = a^{**}_{\overline{i(0)}} + k_{i(0),j(i(0))}x_1$.

3. Direct products

If a lattice ordered group G is a direct product,

$$G = A \times B \tag{1}$$

and if $g \in G$, then the component of g in A or in B will be denoted by $g(A)$ or by $g(B)$, respectively.

3.1. Theorem. *Let* (1) *be valid. Assume that* $A \neq \{0\} \neq B$. *Then* G *is not affine complete.*

Proof. Consider the mapping $f\colon G \to G$ such that $f(g) = g(A)$ for each $g \in G$. Then in view of 1.4 in [1], f is compatible with all elements of $\operatorname{Con} G$.

By way of contradiction, suppose that G is affine complete. Thus there exists $p(x) \in P(G)$ such that $p(x) = f(x)$.

For $p(x)$ we apply the notation as in Section 2. First let us assume that the set $[a]$ is empty. Hence (cf. (1) in Section 2) we have

$$a^k_{ij} \in \{x, -x\}$$

for each $i \in I$, $j \in J(i)$ and $k \in \{1, 2, \ldots, n(i,j)\}$.

There exist $0 < a \in A$, $0 < b \in B$. Put $g = a + b$. In view of (1) in Section 1 we easily verify that there exists an integer k_0 with

$$p(g) = k_0 g.$$

Thus $g(A) = a \neq k_0 g$, whence

$$f(g) = a \neq k_0 g = p(g),$$

which is a contradiction.

Therefore we must have $[a] \neq \emptyset$. Thus we can apply Lemma 2.4. We will use the simpler notation $k_{i(0)}$ instead of $k_{i(0),j(i(0))}$. Then we have

$$p(x_1) = a^{**}_{\overline{i(0)}} + k_{i(0)}x_1. \tag{$*$}$$

Since the element s from Section 2 is subjected only to the condition (α), we can suppose without loss of generality that

$$s(A) > 0, \quad s(B) > 0.$$

Thus we get

$$x_1(A) > 0, \quad x_1(B) > 0.$$

We put $x_1(A) = a$, $x_1(B) = b$.

Further, according to (4) in Section 2, we obtain

$$a_{\bar{i}}^{**}(A) \in [-m_0 s(A), m_0 s(A)], \tag{4.1}$$

$$a_{\bar{i}}^{**}(B) \in [-m_0 s(B), m_0 s(B)] \tag{4.2}$$

for each $i \in I$.

From $(*)$ and from the assumption we get

$$a = a_{\overline{i(0)}}^{**}(A) + k_{i(0)} a.$$

If $k_{i(0)} \neq 1$, then (4.1) and the relation $a = x_1(A) = 3m_0 s(A)$ imply a contradiction. Hence $k_{i(0)} = 1$. Then

$$p(x_1) = a_{\overline{i(0)}}^{**} + x_1.$$

By considering the components in B, we obtain

$$0 = a_{\overline{i(0)}}^{**}(B) + b.$$

Since $b = x_1(B) = 3m_0 s(B)$ we have arrived at a contradiction with 4.2. □

3.2. Theorem. *Let $G \neq \{0\}$ be a projectable lattice ordered group. Then G is not affine complete.*

P r o o f. If G is linearly ordered, then it is subdirectly irreducible and hence in view of (A_4), G is not affine complete. Suppose that G is not linearly ordered. Then, being projectable, it can be expressed in the form $G = A \times B$, $A \neq \{0\} \neq B$. Thus according to 3.1, G is not affine complete. □

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