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LATTICES WITH COMPLEMENTED TOLERANCE LATTICE

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Abstract. We characterize lattices with a complemented tolerance lattice. As an application of our results we give a characterization of bounded weakly atomic modular lattices with a Boolean tolerance lattice.

Keywords: tolerance simple and tolerance-trivial lattices, locally order-polynomially complete lattices

MSC 2000: 06B05, 06C05

1. Introduction

A lattice with 0 and 1 is called bounded. A *tolerance* T of a lattice L is a binary relation $T \subseteq L^2$ which is reflexive, symmetric and compatible with the lattice operations $\wedge$ and $\vee$. The tolerances of a lattice L form an algebraic lattice denoted by $\operatorname{Tol} L$. As usual, $\operatorname{Con} L$ denotes the congruence lattice of L. Clearly, $\operatorname{Con} L \subseteq \operatorname{Tol} L$. If $\operatorname{Tol} L = \operatorname{Con} L$, then the lattice L is called *tolerance-trivial* [2]. L is called *tolerance simple* if it has only the trivial tolerances, namely the identity relation Δ and the all relation ∇. L is called *tolerance-boolean* if $\operatorname{Tol} L$ is a Boolean lattice.

It is known [21] that a lattice L has a Boolean congruence lattice if and only if it is a *discrete* subdirect product of simple lattices (i.e. the components of arbitrary two elements of L are identical except for a finite number of components). In this paper we prove the following

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Main Theorem. (i) $\operatorname{Tol} L$ *is complemented if and only if the lattice L is tolerance-trivial and it is a discrete subdirect product of tolerance simple lattices.*

(ii) *The tolerance lattice of a bounded lattice L is complemented if and only if L is a finite direct product of tolerance simple lattices.*

The proof of Main Theorem is contained in Section 2. As an application of this theorem, in Section 3 we also show that a bounded weakly atomic modular lattice L is tolerance-boolean if and only if L is a complemented lattice of finite height.

2. Proof of the main results

Let L be an arbitrary lattice. According to H.-J. Bandelt's result [1], $\operatorname{Tol} L$ is a pseudocomplemented and 0-modular lattice. A lattice L with 0 is called *pseudocomplemented* if for every element $x \in L$, there exists an $x^* \in L$ such that for any $y \in L$, $y \wedge x = 0 \Leftrightarrow y \leqslant x^*$. L is said to be 0-*modular* if for any $a, b \in L$, the relations $a \leqslant c$ and $b \wedge c = 0$ imply $(a \vee b) \wedge c = a$ (see [19]). It is well-known that for any $T \in \operatorname{Tol} L$ its transitive closure $\widetilde{T}$ is a congruence. In addition we prove

Proposition 2.1. *For any $T \in \operatorname{Tol} L$ its pseudocomplement T^* is a congruence.*

Proof. We claim that $T^* = \widetilde{T^*}$. Since $T^* \leqslant \widetilde{T^*}$, we have to prove only $\widetilde{T^*} \leqslant T^*$, that is $\widetilde{T^*} \wedge T = \Delta$.

On the contrary, assume that $\widetilde{T^*} \wedge T \neq \Delta$. Then there exist $a, b \in L$ such that $a < b$ and $(a, b) \in \widetilde{T^*} \wedge T$. Thus we have $(a, b) \in T$ and, by the definition of the transitive closure, there exists a finite chain $a = z_0 \leqslant z_1 \leqslant \ldots \leqslant z_n = b$ in L such that for each $1 \leqslant i \leqslant n$, $(z_{i-1}, z_i) \in T^*$ holds. Now $a \leqslant z_{i-1} \leqslant z_i \leqslant b$ gives $(z_{i-1}, z_i) \in T \wedge T^* = \Delta$, i.e. $z_{i-1} = z_i$ for all $1 \leqslant i \leqslant n$. Hence we get $a = b$, contrary to our assumption. □

To make our proofs self-consistent we need some additional notions. Let $L = \prod_{i \in I} L_i$ be the direct product of lattices L_i, $i \in I$ and let x_i denote the i-th component (coordinate) of an $x \in L$. The identity and the all relation on L_i are denoted by Δ_i and ∇_i, respectively. A tolerance $\varphi \in \operatorname{Tol} L$ is called the *product of the tolerances* $\varphi_i \in \operatorname{Tol} L_i$ if $(a, b) \in \varphi \Leftrightarrow (a_i, b_i) \in \varphi_i$ for all $i \in I$ (where $a, b \in L$). We write $\varphi = \prod_{i \in I} \varphi_i$ or $\varphi = \varphi_1 \times \ldots \times \varphi_n$ (when $I = \{1, \ldots, n\}$).

Remark 2.2. The class of lattices has *directly decomposable tolerances* (see e.g. [3]), that is $L \cong \prod_{i=1}^{n} L_i$ implies $\operatorname{Tol} L \cong \prod_{i=1}^{n} \operatorname{Tol} L_i$.

P r o o f of Main Theorem. (i) Let L be an arbitrary lattice and assume that $\operatorname{Tol} L$ is complemented. First, we show that $\operatorname{Tol} L = \operatorname{Con} L$.

Take any $T \in \operatorname{Tol} L$. Denoting the complement of T by $\overline{T}$, we prove $(\overline{T})^* = T$. As $T \wedge \overline{T} = \Delta$ implies $T \leqslant (\overline{T})^*$, and since $\overline{T} \wedge (\overline{T})^* = \Delta$ and $\operatorname{Tol} L$ is 0-modular, we obtain $(\overline{T})^* = (T \vee \overline{T}) \wedge (\overline{T})^* = T$. Now Proposition 2.1 gives that $T \in \operatorname{Con} L$ and this proves $\operatorname{Tol} L = \operatorname{Con} L$.

As now $\operatorname{Con} L$ is also complemented and so it is a Boolean lattice, in view of [21], L is a discrete subdirect product of some simple lattices L_i, $i \in I$. Since $\operatorname{Tol} L = \operatorname{Con} L$ and since any L_i is a homomorphic image of L, we deduce that any L_i must be tolerance simple: Indeed, if a tolerance $T \in \operatorname{Tol} L_i \setminus \operatorname{Con} L_i$ existed, then, in view of [4] Theorem 7, there would exist also a tolerance $T' \in \operatorname{Tol} L \setminus \operatorname{Con} L$, contrary to our assumption. Therefore we get $\operatorname{Tol} L_i = \operatorname{Con} L_i = \{\Delta_i, \nabla_i\}$, $i \in I$ and hence L is a discrete subdirect product of tolerance simple lattices.

Conversely, assume that L is tolerance-trivial and it is a discrete subdirect product of tolerance simple lattices L_i, $i \in I$. Since each L_i is (congruence) simple as well, $\operatorname{Con} L$ is a Boolean lattice according to [21]. As $\operatorname{Tol} L = \operatorname{Con} L$, $\operatorname{Tol} L$ is complemented.

(ii) Let L be a bounded lattice and assume that $\operatorname{Tol} L$ is complemented. Then, in view of the above (i), L is tolerance-trivial and $\operatorname{Con} L$ is a Boolean lattice. Moreover, since any congruence-boolean bounded lattice has a finite congruence lattice (see e.g. [5]), $\operatorname{Con} L$ is finite. As by [14] any tolerance-trivial algebra is congruence permutable, L is also congruence permutable. On the other hand, [6] Theorem 3.1 implies that any bounded lattice with a finite Boolean congruence lattice and permutable congruences is a finite direct product of simple lattices. (See also [11], Theorem 6 (iii).) Hence we obtain $L = \prod_{i=1}^{n} L_i$ with $n \in \mathbb{N}$ and all L_i congruence simple. As L is tolerance-trivial, we can repeat the argument in the "if" part of the proof of assertion (i) providing that all the lattices L_i, $1 \leqslant i \leqslant n$ are tolerance simple.

Conversely, assume that $L = \prod_{i=1}^{n} L_i$ with all L_i tolerance simple. Then, in view of Remark 2.3, we have $\operatorname{Tol} L = \prod_{i=1}^{n} \operatorname{Tol} L_i$. Therefore $\operatorname{Tol} L$, as a direct product of two-element chains, is complemented. □

A bounded lattice L is called *semicomplemented* if for any $x \in L$, $x \neq 1$ there exists a $y \in L$ such that $x \wedge y = 0$ and $y \neq 0$. In view of [20], any semicomplemented lattice which is pseudocomplemented is complemented, too. It is also known that any tolerance simple lattice L is locally order-polynomially complete (see e.g. [15]), i.e. every order-preserving function $f\colon L^n \to L$ is a local polynomial of L. Hence we obtain

Corollary 2.3. *If L is a bounded lattice, then* $\operatorname{Tol} L$ *is semicomplemented if and only if L is a finite direct product of locally order-polynomially complete lattices.*

Remark 2.4. (i) Notice that, implicit in the proof of Main Theorem (ii) is the following assertion: If L is a bounded lattice and $\operatorname{Tol} L$ is complemented, then $\operatorname{Tol} L$ is finite.

(ii) We also note that the statement (i) of our Main Theorem is a generalization of the results of [12] and [13].

Since any simple distributive lattice is a two-element Boolean lattice we obtain the following

Corollary 2.5. *The tolerance lattice of a bounded distributive lattice L is semicomplemented if and only if L is a finite Boolean lattice.*

3. Application to weakly atomic modular lattices

Let $u < v$ be elements of a lattice L. If u is covered by v (i.e. when there is no $z \in L$ with $u < z < v$), then we write $u \prec v$. L is called *weakly atomic* if for any $a, b \in L$, $a < b$ there exist $c, d \in L$ such that $a \leqslant c \prec d \leqslant b$. A congruence $\theta \in \operatorname{Con} L$ is called *separable* [7] if for any $a, b \in L$, $a < b$, there exists a chain $a = z_0 \leqslant z_1 \leqslant \ldots \leqslant z_n = b$ such that for each $i = 1, \ldots, n$ either $(z_{i-1}, z_i) \in \theta$ holds or there are no elements $r, s \in L$ satisfying $z_{i-1} \leqslant r < s \leqslant z_i$ and $(r, s) \in \theta$.

Now we are able to formulate the second main result of this paper:

Theorem 3.1. *Let L be a bounded weakly atomic modular lattice. Then the following statements are equivalent:*

(i) *L is tolerance-trivial and every congruence of L is definable.*
(ii) $\operatorname{Tol} L$ *is a Boolean lattice.*
(iii) *L is a finite direct product of tolerance simple lattices.*
(iv) *L is complemented and has a finite height.*

Proof. (i) ⇒ (ii). By the Grätzer-Schmidt well-known theorem [8] the congruence lattice of a lattice L is Boolean if and only if L is weakly modular and every congruence of it is definable. Since any modular lattice is also weakly modular, (i) implies that the lattice $\operatorname{Tol} L = \operatorname{Con} L$ is Boolean.

(ii) ⇒ (iii). If $\operatorname{Tol} L$ is a Boolean lattice, then it is complemented as well, and hence by applying Main Theorem (ii) we obtain (iii).

(iii) ⇒ (iv). Assume that $L = \prod\limits_{i=1}^{n} L_i$ with all L_i tolerance simple. Then each L_i, as a direct factor of the bounded lattice L, is isomorphic to a principal ideal of L.

Therefore each L_i is also a bounded and weakly atomic modular lattice. Since, in view of [5] Theorem 4.2, any weakly atomic modular lattice with a Boolean congruence lattice is locally finite and since each $\operatorname{Con} L_i$ is a two-element Boolean lattice, we conclude that every L_i is locally finite. As each L_i is bounded and modular, by the Jordan-Dedekind chain condition it has a finite height. On the other hand, any tolerance simple modular lattice with a finite height is complemented, according to [17]. (See also [16].) Therefore all the lattices L_i, $1 \leqslant i \leqslant n$, are complemented and have finite height, and hence their finite direct product L is also complemented and has a finite height.

(iv) $\Rightarrow$ (i). Since any complemented modular lattice is relatively complemented, we can now apply [4] Theorem 5 which asserts that any relatively complemented lattice is tolerance-trivial. As all congruences of a lattice with a finite height are definable (since every chain of it is finite), we get (i), and our proof is completed. □

Since every direct factor of a lattice with a finite height has also a finite height and since, according to [17], a modular lattice with a finite height is tolerance simple if and only if it is a (finite dimensional) irreducible projective geometry, we deduce

Corollary 3.2. *A bounded weakly atomic modular lattice is tolerance-boolean if and only if it is a finite direct product of finite dimensional irreducible projective geometries.*

As any lattice with a finite height is bounded, weakly atomic, and each congruence of it is definable, by applying Theorem 3.1 we obtain

Corollary 3.3. *For any modular lattice of finite height the following statements are equivalent*[1]*:*

(i) *L is tolerance-trivial.*

(ii) *L is a finite direct product of tolerance simple lattices.*

(iii) *L is complemented.*

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