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TOTALLY REAL SUBMANIFOLDS IN A QUATERNION SPACE FORM

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Abstract. In this paper, we prove a theorem for n-dimensional totally real minimal submanifold immersed in quaternion space form.

Keywords: totally real submanifold, quaternion space form

MSC 2000: 53C40, 53C56

1. Introduction

Let $M(c)$ denote a $4n$-dimensional quaternion space form of quaternion sectional curvature c and let $P(H)$ denote the $4n$-dimensional quaternion projective space of constant quaternion sectional curvature 4. Let N be an n-dimensional Riemannian manifold isometrically immersed in $M(c)$. We call N a totally real submanifold of $M(c)$ if each tangent 2-plane of N is mapped into a totally real plane in $M(c)$.Chen and Houh [3] gave an inequality for totally real submanifolds of $M(c)$. In particular they considered the case when N was totally geodesic. Later, Sun [6] considered the case when N was pseudo-umbilical and extended the theorem of Chen and Houh; he obtained two integral inequalities for compact totally real submanifolds of $M(c)$. There is other literature studying totally real submanifolds [4], [8], [9].

In this paper, we make use of Yau's maximum principle to study the complete totally real minimal submanifolds with Ricci curvature bounded from below and use the method of proof which is given in [7]. Thus we obtain the following result.

Main theorem. *Let $M(c)$ denote a $4n$-dimensional quaternion space form of quaternion sectional curvature c. Let N be an n-dimensional totally real minimal manifold immersed in $M(c)$ with Ricci curvature bounded from below. Then* *either N is totally geodesic or*

$$\inf \tau \leqslant \frac{1}{12}(3n-2)(n+1)c \tag{1.1}$$

where τ is the scalar curvature of N.

2. Local formulas

We use the same notations and terminology as in [3], [6] unless otherwise stated. Let $M(c)$ denote a $4n$-dimensional quaternion space form of quaternion sectional curvature c and let N be an n-dimensional totally real submanifold of $M(c)$. We choose a local field of orthonormal frames,

$$\begin{gathered} e_1, \dots, e_n, \quad e_{I(1)} = Ie_1, \ \dots, \ e_{I(n)} = Ie_n, \quad e_{J(1)} = Je_1, \ \dots, \ e_{J(n)} = Je_n, \\ e_{K(1)} = Ke_1, \ \dots, \ e_{K(n)} = Ke_n, \end{gathered}$$

is such a way that when restricted to N, $e_1, \dots, e_n$ are tangent to N. Here I, J, K are the almost Hermitian structure and satisfy

$$IJ = -JI = K, \quad JK = -KJ = I, \quad KI = -IK = J, \quad I^2 = J^2 = K^2 = -1.$$

We shall use the following convention on the range of indices:

$$\begin{gathered} A, B, \dots = 1, \dots, n, \ I(1), \dots, I(n), \ J(1), \dots, J(n), \ K(1), \dots, K(n), \\ \alpha, \beta, \dots = I(1), \ \dots, \ I(n), \ J(1), \dots, J(n), \ K(1), \ \dots, \ K(n), \\ i, j, \dots = 1, \dots, n, \quad \Phi = I, J, K. \end{gathered}$$

Let $\{w_A\}$ be the dual frame field. Then the structure equations of $M(c)$ are given by

$$\begin{aligned} dw_A &= -\sum_B w_{AB} \wedge w_B, \quad w_{AB} + w_{BA} = 0, \\ dw_{AB} &= -\sum_C w_{AC} \wedge w_{CB} + \frac{1}{2} \sum_{CD} K_{ABCD} w_C \wedge w_D, \qquad (2.1) \\ K_{ABCD} &= \frac{c}{4}\big(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + I_{AC}I_{BD} - I_{AD}I_{BC} + 2I_{AB}I_{CD} + J_{AC}J_{BD} \\ &\quad - J_{AD}J_{BC} + 2J_{AB}J_{CD} + K_{AC}K_{BD} - K_{AD}K_{BC} + 2K_{AB}K_{CD}\big). \end{aligned}$$

Restricting these forms to N, we get the following structure equations of the immersion:

$$w_\alpha = 0, \quad w_{\alpha i} = \sum_j h^\alpha_{ij} w_j, \quad h^\alpha_{ij} = h^\alpha_{ji}, \quad h^{\Phi(i)}_{jk} = h^{\Phi(j)}_{ik} = h^{\Phi(k)}_{ij},$$

$$dw_{ij} = -\sum_k w_{ik} \wedge w_{kj} + \frac{1}{2}\sum_k R_{ijkl} w_k \wedge w_l,$$

$$R_{ijkl} = K_{ijkl} + \sum_\alpha (h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk}), \tag{2.2}$$

$$dw_{\alpha\beta} = -\sum_\gamma w_{\alpha\gamma} \wedge w_{\gamma\beta} + \frac{1}{2}\sum_{ij} R_{\alpha\beta ij} w_i \wedge w_j,$$

$$R_{\alpha\beta ij} = K_{\alpha\beta ij} + \sum_k (h^\alpha_{ik} h^\beta_{kj} - h^\beta_{ik} h^\alpha_{kj}). \tag{2.3}$$

We call $h = \sum_{ij\alpha} h^\alpha_{ij} w_i w_j e_\alpha$ the second fundamental form of the immersed manifold N. We denote by $S = \sum_{ij\alpha} (h^\alpha_{ij})^2$ the square of the length of h.

If N is minimal in $M(c)$, i.e., *trace* $h = 0$, then by the equations (2.1) and (2.2), we have

$$\tau = \frac{c}{4} n(n+1) - S \tag{2.4}$$

where S is the scalar curvature of N. We define h^α_{ijk} and h^α_{ijkl} by

$$\sum_k h^\alpha_{ijk} w_k = dh^\alpha_{ij} - \sum_l h^\alpha_{il} w_{lj} - \sum_l h^\alpha_{lj} w_{li} + \sum_\beta h^\beta_{ij} w_{\alpha\beta} \tag{2.5}$$

and

$$\sum_l h^\alpha_{ijkl} w_l = dh^\alpha_{ijk} - \sum_l h^\alpha_{ljk} w_{li} - \sum_l h^\alpha_{ilk} w_{lj} - \sum_l h^\alpha_{ijl} w_{lk} + \sum_\beta h^\beta_{ijk} w_{\alpha\beta},$$

respectively, where

$$h^\alpha_{ijk} = h^\alpha_{ikj}$$

and

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{im} R_{mjkl} + \sum_m h^\alpha_{mj} R_{mikl} - \sum_\beta h^\beta_{ij} R_{\alpha\beta kl}.$$

Let H_α and Δ denote the $n \times n$ matrix (h^α_{ij}) and the Laplacian on N, respectively. By a simple calculation, we have (cf. [1], [2], [5])

$$\frac{1}{2}\Delta S = \sum_{ijk\alpha} (h^\alpha_{ijk})^2 + \frac{c}{4}(n+1)S + \sum_{\alpha\beta} \mathrm{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha\beta} (\mathrm{tr}\, H_\alpha H_\beta)^2. \tag{2.6}$$

In order to prove the main theorem, we need the following Lemmas.

Lemma 1 [6]. *Let H_i, $i \geqslant 2$ be symmetric $n \times n$ matrices, $S_i = \operatorname{tr} H_i^2$, $S = \sum\limits_i S_i$. Then*

$$\sum_{ij} \operatorname{tr}(H_iH_j - H_jH_i)^2 - \sum_{ij} (\operatorname{tr} H_iH_j)^2 \geqslant -\frac{3}{2}S^2 \tag{2.7}$$

and the equality holds if and only if either all $H_i = 0$ or there exist two H_i different from zero. Morever, if $H_1 \neq 0$, $H_2 \neq 0$, $H_i = 0$, $i \neq 1, 2$, then $S_1 = S_2$ and there exists an orthogonal $n \times n$ matrix T such that

$$TH_1^tT = \begin{pmatrix} \sqrt{\frac{S_1}{2}} & 0 & \dots & 0 \\ 0 & -\sqrt{\frac{S_1}{2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad TH_2^tT = \begin{pmatrix} 0 & \sqrt{\frac{S_1}{2}} & \dots & 0 \\ \sqrt{\frac{S_1}{2}} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Lemma 2 [7]. *Let N be a complete Riemannian manifold with Ricci curvature bounded from below and let f be a C^2-function bounded from above on N, then for all $\varepsilon > 0$, there exists a point $x \in N$ at which*

(i) $\sup f - \varepsilon < f(x)$,
(ii) $\|\nabla f(x)\| < \varepsilon$,
(iii) $\Delta f(x) < \varepsilon$.

3. Proof of the main theorem

In this section, the method of the proof used by Ximin in [7] is applied to a totally real minimal submanifold immersed in quaternion space form.

From (2.6) and (2.7), we obtain

$$\frac{1}{2}\Delta S \geqslant S\Big(\frac{c}{4}(n+1) - \frac{3}{2}S\Big). \tag{3.1}$$

We know that $S = \frac{c}{4}n(n+1) - \tau$. By the condition of the theorem, we conclude that S is bounded. We define $f = S$ and $F = (f+a)^{\frac{1}{2}}$ (where $a > 0$ is any positive constant number). F is bounded. We have

$$\mathrm{d}F = \frac{1}{2}(f+a)^{-\frac{1}{2}}\,\mathrm{d}f, \tag{3.2}$$

$$\begin{aligned} \Delta F &= \frac{1}{2}\Big(-\frac{1}{2}(f+a)^{-\frac{3}{2}}\,\|\mathrm{d}f\|^2 + (f+a)^{-\frac{1}{2}}\Delta f\Big) \\ &= \frac{1}{2}(-2\,\|\mathrm{d}F\|^2 + \Delta f)(f+a)^{-\frac{1}{2}}, \end{aligned} \tag{3.3}$$

i.e.,

$$\Delta F=\frac{1}{2F}(-2\left\|\mathrm{d}F\right\|^{2}+\Delta f). \tag{3.4}$$

Hence, $F\Delta F=-\left\|\mathrm{d}F\right\|^{2}+\frac{1}{2}\Delta f$ or $\frac{1}{2}\Delta f=F\Delta F+\left\|\mathrm{d}F\right\|^{2}$. Applying Lemma 2 to F, we have for all $\varepsilon>0$, there exists a point $x\in N$ such that at x

$$\|\mathrm{d}F(x)\|\leqslant\varepsilon, \tag{3.5}$$

$$\Delta F(x)<\varepsilon, \tag{3.6}$$

$$F(x)>\sup F-\varepsilon. \tag{3.7}$$

From (3.5), (3.6) and (3.7), we have

$$\frac{1}{2}\Delta f<\varepsilon^{2}+F\varepsilon=\varepsilon\left(\varepsilon+F\right). \tag{3.8}$$

We take a sequence $\{e_m\}$ such that $\varepsilon_m\to 0$ $(m\to\infty)$ and for all m, there exists a point $x_m\in N$ such that (3.5), (3.6) and (3.7) hold. Therefore, $\varepsilon_m\big(\varepsilon_m+F(x_m)\big)\to 0$ $(m\to\infty)$ (because F is bounded).

From (3.7), we have $F\left(x_m\right)>\sup F-\varepsilon_m$, because $\{F\left(x_m\right)\}$ is a bounded sequence. So we get $F\left(x_m\right)\to F_0$ (if necessary, we can choose a subsequence). Hence, $F_0\geqslant\sup F$. So we have

$$F_0=\sup F. \tag{3.9}$$

From the definition of F, we get

$$f\left(x_m\right)\to f=\sup f. \tag{3.10}$$

(3.1) and (3.8) imply that

$$f\Big(\frac{c}{4}(n+1)-\frac{3}{2}S\Big)\leqslant\frac{1}{2}\Delta f\leqslant\varepsilon(\varepsilon+F), \tag{3.11}$$

and

$$f(x_m)\Big(\frac{c}{4}(n+1)-\frac{3}{2}f(x_m)\Big)<\varepsilon_m^{2}+\varepsilon_m F(x_m)\leqslant\varepsilon_m^{2}+\varepsilon_m F_0. \tag{3.12}$$

Let $m\to\infty$, then $\varepsilon_m\to 0$ and $f(x_m)\to f_0$. Hence,

$$f_0\Big(\frac{c}{4}(n+1)-\frac{3}{2}f_0\Big)\leqslant 0. \tag{3.13}$$

(i) If $f_0 = 0$, we have $f = S \equiv 0$. Hence, M is totally geodesic.

(ii) If $f_0 > 0$, we have $\frac{c}{4}(n+1) - \frac{3}{2} f_0 \leqslant 0$ and

$$f_0 \geqslant \frac{c}{6}(n+1),$$

that is, $\sup S \geqslant \frac{c}{6}(n+1)$. Therefore, $\inf \tau \leqslant \frac{1}{12}(3n-2)(n+1)c$. This completes the proof.

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