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# ON SUPER HAMILTONIAN SEMIGROUPS

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*Abstract.* The concept of super hamiltonian semigroup is introduced. As a result, the structure theorems obtained by A. Cherubini and A. Varisco on quasi commutative semigroups and quasi hamiltonian semigroups respectively are extended to super hamiltonian semigroups.

*Keywords*: quasi hamiltonian semigroups, super hamiltonian semigroups, quasi commutative semigroups, quasi-groups, strong semilattices of semigroups

*MSC 2000*: 20M10

## 1. Introduction

A semigroup $S$ is called quasi commutative if for all $a, b \in S, ab = b^r a$ holds for some positive integer $r \geqslant 1$. The concept of quasi commutativity was first introduced by N. P. Mukherjee [6] in 1971. Later on, M. Chacron and G. Thierrin [1] called a semigroup $S$ a $\sigma$-reflexive semigroup if and only if $S$ satisfies the following condition:

$$\forall\, a, b \in S,\ \exists\, m = m(a, b) \geqslant 1,\ \ ab = (ba)^m.$$

Quasi commutative semigroups, cyclic communicative semigroups and $\sigma$-reflexive semigroups were then studied by a number of authors, for example, see [2], [4], [5] and [7].

In generalizing the concept of quasi commutativity, A. Cherubini and A. Varisco [3] in 1983 called a semigroup $S$ a quasi hamiltonian semigroup if for every $a, b \in S$, there exists two positive integers $r$, $s$ such that $ab = b^r a^s$. Thus, it is clear that the class of quasi hamiltonian semigroups contains the class of quasi commutative

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semigroups as its special subclass. On the other hand, B. Pondělíček [9] in 1975 called a semigroup $S$ a weakly commutative semigroup if for every $a, b \in S$, $(ab)^n \in bSa$ for some positive integer $n \geqslant 1$. Thus, quasi commutative semigroups, quasi hamiltonian semigroups and $\sigma$-reflexive semigroups can be regarded as special weakly commutative semigroups. It was stated by M. Petrich [8, Corollary II.5.6] that a weakly commutative semigroup is a semilattice of archimedean semigroups. Along with this direction, A. Cherubini and A. Varisco proved in [2] and [3] respectively the following structure theorems:

(i) A semigroup $S$ is $\sigma$-reflexive if and only if $S$ is a semilattice of $\sigma$-reflexive archimedean semigroups $S_\alpha$ and for every $a, b \in S$ with $ab \neq ba$, $ab$ belongs to a subgroup of $S$. (See [3, Theorem 2.8].)

(ii) A semigroup $S$ is a quasi hamiltonian semigroup if and only if $S$ is a semilattice of archimedean quasi hamiltonian semigroups, and every subsemigroup of $S$, generated by two elements, is a duo semigroup. (See [3, Theorem 1.9].)

Also, it was shown by N. P. Mukherjee in [6] that:

(iii) Every quasi commutative semigroup $S$ is uniquely expressible as a semilattice of archimedean semigroups $S_\alpha$. (See [6, Theorem 4].)

Inspired by the above definitions and results in the literature, we now call a semigroup $S$ a *generalized quasi hamiltonian* semigroup if for every $a, b \in S$, it holds $(ab)^n = b^r a^s$ for some positive integers $n$, $r$, $s$ satisfying $2n < r + s$. In view of the results mentioned by M. Chacron and G. Thirerrin (see [1, Theorems 1 and 2]), it is natural to call the generalized quasi hamiltonian semigroups with central idempotents the *super hamiltonian* semigroups. What we are going to show is that for a generalized quasi hamiltonian semigroup $S$, $S$ is super hamiltonian if and only if $S$ is a strong semilattice of quasi-groups. This theorem describes the structure of super hamiltonian semigroups and as a consequence, we observe that the quasi-groups are the basic building blocks for super hamiltonian semigroups. Thus, the structure theorem for the quasi hamiltonian semigroups in [3] is extended.

The reader is referred to M. Petrich [8] for notations if it is necessary.

## 2. Super hamiltonian semigroups

We shall first give some definitions that will be used throughout the paper.

**Definition 2.1.** An element $a$ in a semigroup $S$ is called a quasi regular element if there exists an integer $n \geqslant 1$ and $x \in S$ such that $a^n = a^n x a^n$. We shall call a semigroup a quasi regular semigroup if all elements of $S$ are quasi regular.

Quasi regular semigroups have been extensively investigated (see [12]). It is trivial to see that a quasi regular semigroup contains some idempotents.

**Definition 2.2** [11]. A quasi regular semigroup $S$ is called a quasi-group if there exists only one idempotent in $S$, that is, $|E| = 1$.

**Definition 2.3** [10]. A semigroup $S$ is called a $t$-archimedean semigroup if for all $a, b \in S$, there exists a positive integer $n \geqslant 1$ such that $b^n \in aS \cap Sa$.

The following result is stated in [3] for quasi-hamiltonian semigroups. In fact, it can be easily verified that the result holds for generalized quasi hamiltonian semigroups as well.

**Proposition 2.4.** *A generalized quasi hamiltonian semigroup $S$ is a semilattice of $t$-archimedean semigroups $S_\alpha$.*

Hereafter, we shall write the generalized quasi hamiltonian semigroup $S$ as $S = \bigcup_{\alpha \in Y} S_\alpha$, where $S_\alpha$ is a $t$-archimedean subsemigroup of $S$ for every $\alpha \in Y$. We also call $S_\alpha$ the $t$-archimedean component of the semigroup $S$.

We now prove the following lemmas which are the crucial lemmas in the establishment of our main theorem.

**Lemma 2.5.** *Let $S$ be a generalized quasi hamiltonian semigroup. Then every $t$-archimedean component of $S$ contains a unique idempotent of $S$.*

Proof. Since $S$ is a generalized quasi hamiltonian semigroup, by Proposition 2.4, $S = \bigcup_{\alpha \in Y} S_\alpha$, where each $S_\alpha$ is a $t$-archimedean subsemigroup of $S$. Now, let $a \in S$. Then $a \in S_\alpha$ for some $\alpha \in Y$. Since $S_\alpha$ is a subsemigroup of $S$, we have $\langle a \rangle \subseteq S_\alpha$. Since $S$ is a generalized quasi hamiltonian semigroup, we have $(a^2)^n = a^r a^s$ for some positive integers $n$, $r$ and $s$, that is, $a^{2n} = a^{r+s}$ with $2n \neq r + s$. It is now clear that $a$ is periodic and hence there exists a positive integer $m \geqslant 1$ such that $a^m = e \in S_\alpha$. We claim that the idempotent $e$ in $S_\alpha$ is unique. In fact, if we let $e$, $f$ be idempotents of $S_\alpha$, then, since $S_\alpha$ is a $t$-archimedean semigroup, there exists $x$, $y$ in $S_\alpha$ such that $e = fx$ and $f = ye$. This leads to $e = fe = ye = f$. Thus, $S_\alpha$ contains exactly one idempotent of $S$. The proof is completed. □

**Lemma 2.6.** *The $t$-archimedean semigroup $S_\alpha$ of a generalized quasi hamiltonian semigroup $S$ is a quasi-group.*

Proof. In view of Lemma 2.5, we only need to prove that the $t$-archimedean component $S_\alpha$ of $S$ is quasi regular. To this end, we let $a \in S_\alpha$. Then we consider the elements $a$, $e$ in $S_\alpha$, where $e$ is the unique idempotent in $S_\alpha$, by Lemma 2.5. Since $S_\alpha$ is a $t$-archimedean semigroup, there exist elements $u$, $v$, $x$ in $S_\alpha$ and a positive integer $n \geqslant 1$ such that $a^n = eu = ve$ and $e = a^n x$. These equalities now lead to $a^n = ea^n = a^n x a^n$. Thereby, $S_\alpha$ is a quasi-group. □

Summing up Lemma 2.5 and Lemma 2.6, we obtain the following result.

**Lemma 2.7.** *If $S$ is a generalized quasi hamiltonian semigroup then $S$ is a semilattice of quasi-groups $S_\alpha$.*

The above lemma improves the result of A. Cherubini and A. Varisco [3] for quasi hamiltonian semigroups. Also, we observe that for every element $a$ in a quasi-group $S_\alpha$, $a^n$ is an idempotent for some positive integer $n$. Thus, it is trivial to see that in the quasi-group $S_\alpha$, the idempotent is in the center of $S_\alpha$.

By using Lemma 2.7, we now establish the following theorem for super hamiltonian semigroups.

**Theorem 2.8** (Main Theorem). *Let $S$ be a generalized quasi hamiltonian semigroup. Then $S$ is a super hamiltonian semigroup if and only if $S$ is a strong semilattice of quasi-groups $S_\alpha$.*

Proof. ($\Rightarrow$) Let $S$ be a super hamiltonian semigroup. Then by Lemma 2.7, $S = \bigcup_{\alpha \in Y} S_\alpha$, where $Y$ is a semilattice and each $S_\alpha$ is a quasi-group for every $\alpha \in Y$. Now, let $e_\alpha$ be the identity element of the quasi-group $S_\alpha$. Define a mapping $\varphi_{\alpha,\beta}\colon S_\alpha \longrightarrow S_\beta$ by $a\varphi_{\alpha,\beta} = ae_\beta$ for any $a \in S_\alpha$ and $\alpha, \beta \in Y$ with $\alpha \geqslant \beta$. It is trivial to see that $\varphi_{\alpha,\alpha}$ is the identity mapping on $S_\alpha$. Also, by the multiplication on $S$, $ae_\beta \in S_\beta$. Suppose that $a$, $b$ are two arbitrary elements of $S_\alpha$. Then we have

$$\begin{aligned}(ab)\varphi_{\alpha,\beta} &= (ab)e_\beta \\ &= ae_\beta be_\beta \quad (\text{since } e_\beta \text{ is in the center of } S_\beta) \\ &= a\varphi_{\alpha,\beta} b\varphi_{\alpha,\beta}.\end{aligned}$$

This shows that $\varphi_{\alpha,\beta}$ is a homomorphism. Moreover, since $S$ is a super hamiltonian semigroup, we can easily verify that $e_\alpha e_\beta = e_\gamma$ for $\alpha, \beta, \gamma \in Y$ with $\alpha\beta = \gamma$, where $e_\gamma$ is the idempotent in $S_\gamma$. Thus, for $\alpha \geqslant \beta \geqslant \gamma$ with $a \in S_\alpha$, we have

$$a\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = (ae_\beta)e_\gamma = ae_\gamma = a\varphi_{\alpha,\gamma}.$$

Because the element $a$ is arbitrarily chosen in $S$, we have $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$. Hence, the maps $\varphi_{\alpha,\beta}$ are structure homomorphisms of the strong semilattice of quasi-groups $S_\alpha$, that is, $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$ is a strong semilattice of the quasi-groups $S_\alpha$.

($\Leftarrow$) Let $S$ be a strong semilattice of the quasi-groups $S_\alpha$. Then for any element $a \in S$ and any idempotent $e \in E \subseteq S$, there exist some $\alpha, \beta \in Y$ such that $a \in S_\alpha$ and

$e \in S_\beta \cap E$. Since $S_{\alpha\beta} \in S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$ and $\varphi_{\alpha,\beta}$ are the structure homomorphisms of the strong semilattice $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$, we have

$$\begin{aligned} ae &= a\varphi_{\alpha,\alpha\beta} e\varphi_{\beta,\alpha\beta} \\ &= e\varphi_{\beta,\alpha\beta} a\varphi_{\alpha,\alpha\beta} \quad \text{(since } S_{\alpha\beta} \text{ is a quasi-group)} \\ &= ea. \end{aligned}$$

This shows that $e$ lies in the center of $S$ as well. Since $S$ itself has been assumed to be a generalized quasi hamiltonian semigroup, we deduce that $S$ is super hamiltonian. This finishes the proof. □

**Remark.** A. Cherubini and A. Varisco have remarked in [2] that the idempotents of a quasi commutative semigroup $S$ are in the center of $S$. Our Theorem 2.8 extends their remark from quasi commutative semigroups to super hamiltonian semigroups.

**Example 2.9.** Let $S_\alpha = \{a, b, e\}$, $S_\beta = \{c, f\}$ and $S_{\alpha\beta} = \{u, v, w, x, y, z\}$ be respectively quasi-groups on a semilattice $Y = \{\alpha, \beta, \alpha\beta\}$.

The Cayley tables of $S_\alpha$, $S_\beta$ and $S_{\alpha\beta}$ are respectively the following

$S_\alpha$:

| * | $a$ | $b$ | $e$ |
|---|---|---|---|
| $a$ | $b$ | $e$ | $e$ |
| $b$ | $e$ | $e$ | $e$ |
| $e$ | $e$ | $e$ | $e$ |

$S_\beta$:

| * | $c$ | $f$ |
|---|---|---|
| $c$ | $f$ | $f$ |
| $f$ | $f$ | $f$ |

$S_{\alpha\beta}$:

| * | $u$ | $v$ | $w$ | $x$ | $y$ | $z$ |
|---|---|---|---|---|---|---|
| $u$ | $u$ | $v$ | $w$ | $x$ | $y$ | $z$ |
| $v$ | $v$ | $u$ | $y$ | $z$ | $w$ | $x$ |
| $w$ | $w$ | $z$ | $u$ | $y$ | $x$ | $v$ |
| $x$ | $x$ | $y$ | $z$ | $u$ | $v$ | $w$ |
| $y$ | $y$ | $x$ | $v$ | $w$ | $z$ | $u$ |
| $z$ | $z$ | $w$ | $x$ | $v$ | $u$ | $y$ |

Define the mapping $\theta_{\alpha,\alpha\beta}\colon S_\alpha \to S_{\alpha\beta}$ by $x \longmapsto u$ for any $x \in S_\alpha$; $\theta_{\beta,\alpha\beta}\colon S_\beta \to S_{\alpha\beta}$ by $y \longmapsto u$ for any $y \in S_\beta$ and let $\theta_{\alpha,\alpha}$ be the identity mapping for any $\alpha \in Y$. Then we can easily check that the mappings $\theta_{\alpha,\beta}$ form a family of structure homomorphisms of the strong semilattice $S = [Y; S_\alpha; \theta_{\alpha,\beta}]$, where $S = \bigcup_{\alpha \in Y} S_\alpha = S_\alpha \cup S_\beta \cup S_{\alpha\beta}$. By using the above structure homomorphisms, we obtain the following Cayley table to the semigroup $S$:

| $*$ | $a$ | $b$ | $e$ | $c$ | $f$ | $u$ | $v$ | $w$ | $x$ | $y$ | $z$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $a$ | $b$ | $e$ | $e$ | $u$ | $u$ | $u$ | $v$ | $w$ | $x$ | $y$ | $z$ |
| $b$ | $e$ | $e$ | $e$ | $u$ | $u$ | $u$ | $v$ | $w$ | $x$ | $y$ | $z$ |
| $e$ | $e$ | $e$ | $e$ | $u$ | $u$ | $u$ | $v$ | $w$ | $x$ | $y$ | $z$ |
| $c$ | $u$ | $u$ | $u$ | $f$ | $f$ | $u$ | $v$ | $w$ | $x$ | $y$ | $z$ |
| $f$ | $u$ | $u$ | $u$ | $f$ | $f$ | $u$ | $v$ | $w$ | $x$ | $y$ | $z$ |
| $u$ | $u$ | $u$ | $u$ | $u$ | $u$ | $u$ | $v$ | $w$ | $x$ | $y$ | $z$ |
| $v$ | $v$ | $v$ | $v$ | $v$ | $v$ | $v$ | $u$ | $y$ | $z$ | $w$ | $x$ |
| $w$ | $w$ | $w$ | $w$ | $w$ | $w$ | $w$ | $z$ | $u$ | $y$ | $x$ | $v$ |
| $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $y$ | $z$ | $u$ | $v$ | $w$ |
| $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $y$ | $x$ | $v$ | $w$ | $z$ | $u$ |
| $z$ | $z$ | $z$ | $z$ | $z$ | $z$ | $z$ | $w$ | $v$ | $v$ | $u$ | $y$ |

Then, we can check that $S$ is a super hamiltonian semigroup, but $S$ is not commutative and in fact not quasi hamiltonian as well (indeed, it suffices to consider e.g. $v, w$).

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