Krzysztof Ciepliński

The structure of disjoint iteration groups on the circle

Czechoslovak Mathematical Journal, Vol. 54 (2004), No. 1, 131–153

Persistent URL: http://dml.cz/dmlcz/127871

Czechoslovak Mathematical Journal, 54 (129) (2004), 131–153

THE STRUCTURE OF DISJOINT ITERATION GROUPS ON THE CIRCLE

Krzysztof Ciepliński, Kraków

(Received May 30, 2001)

Abstract. The aim of the paper is to investigate the structure of disjoint iteration groups on the unit circle $\mathbb{S}^1$, that is, families $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ of homeomorphisms such that

$$F^{v_1} \circ F^{v_2} = F^{v_1+v_2}, \quad v_1, v_2 \in V,$$

and each F^v either is the identity mapping or has no fixed point ($(V,+)$ is an arbitrary 2-divisible nontrivial (i.e., $\operatorname{card} V > 1$) abelian group).

Keywords: (disjoint, non-singular, singular, non-dense, dense, discrete) iteration group, degree, periodic point, orientation-preserving homeomorphism, rotation number, limit set, orbit, system of functional equations

MSC 2000: 37E10, 20F38, 30D05, 37E45, 39B12, 39B32, 39B72

1. Introduction

Let X be a topological space and $(V,+)$ a 2-divisible nontrivial (i.e., $\operatorname{card} V > 1$) abelian group.

Recall that a family $\{F^v\colon X \longrightarrow X;\ v \in V\}$ of homeomorphisms such that

$$F^{v_1} \circ F^{v_2} = F^{v_1+v_2}, \quad v_1, v_2 \in V$$

is called an *iteration group* or a *flow* (on X).

An iteration group $\{F^v\colon X \longrightarrow X;\ v \in V\}$ is said to be *disjoint* if each of its elements either is the identity mapping or has no fixed point (see [2] and also [18]).

Such iteration groups on open real intervals have been examined by M. Bajger and M. C. Zdun in [3] and M. C. Zdun in [16], [17], [18] and [19]. Some results concerning disjoint iteration groups on the unit circle $\mathbb{S}^1$ can be found in [2], [6] and [7].

In this paper we give a complete description of disjoint iteration groups on the unit circle. Our results generalize those obtained by M. Bajger in [2], where a special case of disjoint iteration groups has been studied.

In Section 2 we recall the basic definitions as well as a result from [10, Lemma 1] which serves as the main tool in the proof of the important Proposition 3. Next (Section 3), we introduce the notion of a limit set of iteration groups under study. This enables us to divide these iteration groups into three classes, which will be considered separately in Section 4. The third class will be handled in much the same way as in [2].

2. Preliminaries

We begin by recalling the basic definitions and introducing some notation.

Throughout the paper $\mathbb{N}$ denotes the set of all positive integers. The closure of a set $A \subset \mathbb{S}^1$ will be denoted by $\operatorname{cl} A$ while A^{d} stands for the set of all cluster points of A.

Following [2] and [6], we write

$$\tilde{\Pi}\colon\ \mathbb{R} \ni t \longmapsto \mathrm{e}^{2\pi \mathrm{i} t} \in \mathbb{S}^1 \quad \text{and} \quad \Pi := \tilde{\Pi}_{|[0,1)}.$$

For any $v, w, z \in \mathbb{S}^1$ there exist unique $t_1, t_2 \in [0,1)$ such that $w\Pi(t_1) = z$ and $w\Pi(t_2) = v$, so we can put

$$v \prec w \prec z \quad \text{if and only if} \quad 0 < t_1 < t_2$$

and

$$v \preceq w \preceq z \quad \text{if and only if} \quad t_1 \leqslant t_2 \quad \text{or} \quad t_2 = 0.$$

A set $A \subset \mathbb{S}^1$ is said to be an *open arc* if there are distinct $v, z \in \mathbb{S}^1$ for which

$$A = \overrightarrow{(v,z)} := \{w \in \mathbb{S}^1\colon\ v \prec w \prec z\} = \{\tilde{\Pi}(t);\ t \in (t_v, t_z)\},$$

where $t_v, t_z \in \mathbb{R}$ are such that $\tilde{\Pi}(t_v) = v$, $\tilde{\Pi}(t_z) = z$ and $0 < t_z - t_v < 1$.

Given a subset A of $\mathbb{S}^1$ with $\operatorname{card} A \geqslant 3$ and a function F mapping A into $\mathbb{S}^1$ we say that F is *increasing* (*strictly increasing*) if for any $v, w, z \in A$ such that $v \prec w \prec z$ we have $F(v) \preceq F(w) \preceq F(z)$ (respectively, $F(v) \prec F(w) \prec F(z)$).

For every homeomorphism $F\colon\ A \longrightarrow B$, where $A = \{\tilde{\Pi}(t);\ t \in (a,b)\}$ and $B = \{\tilde{\Pi}(t);\ t \in (c,d)\}$ are open arcs, there exists a unique homeomorphism $f\colon\ (a,b) \longrightarrow (c,d)$ with

$$(F \circ \tilde{\Pi})(x) = (\tilde{\Pi} \circ f)(x), \quad x \in (a,b).$$

We say that f *represents* F, and if f is strictly increasing, then we say that the homeomorphism F *preserves orientation.*

It is well-known (see for instance [1], [4] and [13]) that for every continuous mapping $F\colon\ \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ there is a continuous function $f\colon\ \mathbb{R} \longrightarrow \mathbb{R}$, which is unique up to translation by an integer, and a unique integer k such that

$$F(\tilde{\Pi}(x)) = \tilde{\Pi}(f(x)), \quad x \in \mathbb{R}$$

and

$$f(x+1) = f(x) + k, \quad x \in \mathbb{R}.$$

The function f is called a *lift* of F and the integer k is called the *degree* of F, and is denoted by $\deg F$.

If $F\colon\ \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is a homeomorphism, then so is its lift. Furthermore, $|\deg F| = 1$.

We say that a homeomorphism $F\colon\ \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ *preserves orientation* if $\deg F = 1$, which is clearly equivalent to the fact that the lift of F is increasing.

For every such homeomorphism F the number $\alpha(F) \in [0,1)$ defined by

$$\alpha(F) := \lim_{n\to\infty} \frac{f^n(x)}{n} \pmod 1, \quad x \in \mathbb{R}$$

is called the *rotation number* of F. This number always exists and does not depend on x and the choice of the lift f. Furthermore, $\alpha(F)$ is rational (equal to zero) if and only if F has a periodic (respectively, fixed) point (see for instance [11] and [12]).

Finally, for the convenience of the reader we repeat the relevant, slightly modified, material from [10] without proofs.

Let M be an arbitrary non-empty set and let $F_t\colon\ \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ for $t \in M$ be orientation-preserving homeomorphisms. Denote by N the set of all $t \in M$ such that F_t have no fixed point and for any $z \in \mathbb{S}^1$ set

$$C(z) := \{(F_{t_1}^{n_1} \circ \ldots \circ F_{t_k}^{n_k})(z);\ t_1, \ldots, t_k \in N,\ n_1, \ldots, n_k \in \mathbb{Z},\ k \in \mathbb{N}\}.$$

Lemma 1 (see [10]). *Let $F_t\colon\ \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ for $t \in M$ be orientation-preserving homeomorphisms such that*

$$F_t \circ F_s = F_s \circ F_t, \quad s \in M,\ t \in N. \tag{1}$$

Suppose also that

$$\bigcap_{z\in\mathbb{S}^1} \operatorname{cl} C(z) \neq \emptyset$$

and for every $z \in \mathbb{S}^1$ the set $C(z)$ is infinite. Then for every $l \in \mathbb{Z}$ there exists a unique pair (Φ, c) such that $\Phi\colon\ \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is a continuous function of degree l with $\Phi(1) = 1$ and $c\colon\ M \longrightarrow \mathbb{S}^1$ satisfying the system of Schröder equations

$$\Phi(F_t(z)) = c(t)\Phi(z), \quad z \in \mathbb{S}^1,\ t \in M.$$

Moreover,

$$c(t) = \mathrm{e}^{2\pi \mathrm{i} l \alpha(F_t)}, \quad t \in M.$$

Remark 1 (see [10])**.** Let $F_t\colon\ \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ for $t \in M$ be orientation-preserving homeomorphisms satisfying condition (1). If there exists a $t_0 \in M$ for which $\alpha(F_{t_0}) \notin \mathbb{Q}$, then the assumptions of Lemma 1 are fulfilled.

3. Limit sets

For any orientation-preserving homeomorphism $F\colon\ \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ put

$$C_F(z) := \{F^n(z);\ n \in \mathbb{Z}\}, \quad z \in \mathbb{S}^1.$$

If $\alpha(F) \notin \mathbb{Q}$, then the non-empty set

$$L_F := C_F(z)^{\mathrm{d}}$$

(the *limit set* of F) does not depend on $z \in \mathbb{S}^1$, is invariant with respect to F (that is $F[L_F] = L_F$) and either $L_F = \mathbb{S}^1$ or L_F is a perfect nowhere dense subset of $\mathbb{S}^1$ (see for instance [11]).

3.1. Non-singular iteration groups.

Let us first observe (see also [15]) that every element of an iteration group $\mathscr{F} = \{F^v\colon\ \mathbb{S}^1 \longrightarrow \mathbb{S}^1, v \in V\}$ preserves orientation (we need the 2-divisibility of the abelian group V only to establish this fact).

Proceeding analogously to the proofs of Lemmas 4 and 5 in [7] we obtain the following two results.

Lemma 2. *Let $\mathscr{F} = \{F^v\colon\ \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ be an iteration group. If $v_0 \in V$ is such that $\alpha(F^{v_0}) \notin \mathbb{Q}$, then*

$$F^v[L_{F^{v_0}}] = L_{F^{v_0}}, \quad v \in V.$$

Proposition 1. *Let* $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ *be an iteration group. If* $v_1, v_2 \in V$ *are such that* $\alpha(F^{v_1})$, $\alpha(F^{v_2}) \notin \mathbb{Q}$, *then*

$$L_{F^{v_1}} = L_{F^{v_2}}.$$

Definition 1 (see also [7]). An iteration group $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ is said to be non-singular if at least one of its elements has no periodic point, otherwise $\mathscr{F}$ is called a singular iteration group.

Clearly, an iteration group $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ is non-singular (singular) if and only if there exists an element of $\mathscr{F}$ with irrational rotation number (respectively, $\alpha(F^v) \in \mathbb{Q}$ for $v \in V$).

Non-singular iteration groups have been investigated in [2], where the structure of such disjoint iteration groups for the case $V = \mathbb{R}$ was described.

If $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ is a non-singular iteration group, then, according to Proposition 1, the set L_{F^v} does not depend on the choice of $F^v \in \mathscr{F}$ such that $\alpha(F^v) \notin \mathbb{Q}$. Thus, we can introduce the following definition.

Definition 2. By a limit set of a non-singular iteration group $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ we mean the set

$$L_{\mathscr{F}} := L_{F^v},$$

where $F^v \in \mathscr{F}$ is an arbitrary homeomorphism with an irrational rotation number.

As an immediate consequence of Definition 2, Lemma 2 and the properties of limit sets of homeomorphisms we obtain

Remark 2. If $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ is a non-singular iteration group, then

(i) $F^v[L_{\mathscr{F}}] = L_{\mathscr{F}}$ for $v \in V$,

(ii) either $L_{\mathscr{F}} = \mathbb{S}^1$ or $L_{\mathscr{F}}$ is a non-empty perfect and nowhere dense subset of $\mathbb{S}^1$.

Denote by $O_{\mathscr{F}}(z)$ the *orbit* of the iteration group $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ at $z \in \mathbb{S}^1$, that is

$$O_{\mathscr{F}}(z) := \{F^v(z);\ v \in V\}, \quad z \in \mathbb{S}^1.$$

With this notation, we have the following lemma.

Lemma 3. *If* $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ *is a non-singular iteration group, then*

$$L_{\mathscr{F}} = O_{\mathscr{F}}(z)^{\mathrm{d}}, \quad z \in L_{\mathscr{F}}.$$

Proof. If $\alpha(F^{v_0}) \notin \mathbb{Q}$ for a $v_0 \in V \setminus \{0\}$, then for each $z \in \mathbb{S}^1$ we have

$$L_{\mathscr{F}} = L_{F^{v_0}} = C_{F^{v_0}}(z)^{\mathrm{d}} \subset O_{\mathscr{F}}(z)^{\mathrm{d}}.$$

Now, fix a $z \in L_{\mathscr{F}}$ and take a $w \in O_{\mathscr{F}}(z)$. Then there is a sequence $(n_k)_{k\in\mathbb{N}}$ of non-zero integers and a $v \in V$ such that $z = \lim\limits_{k\to\infty} F^{n_k v_0}(z)$ and $w = F^v(z)$. The continuity of F^v now gives $w = \lim\limits_{k\to\infty} F^{n_k v_0}(w)$. Therefore $w \in C_{F^{v_0}}(w)^{\mathrm{d}} = L_{F^{v_0}} = L_{\mathscr{F}}$ and, consequently, $O_{\mathscr{F}}(z) \subset L_{\mathscr{F}}$. Finally, the fact that the set $L_{\mathscr{F}}$ is perfect shows that $O_{\mathscr{F}}(z)^{\mathrm{d}} \subset L_{\mathscr{F}}^{\mathrm{d}} = L_{\mathscr{F}}$. □

3.2 Disjoint iteration groups.

The notion of a limit set of a singular iteration group will be introduced in a particular case.

We start with

Lemma 4. *If* $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ *is a disjoint iteration group, then the following conditions are equivalent:*

(i) $\operatorname{card}\mathscr{F} < \aleph_0$,

(ii) *for every* $z \in \mathbb{S}^1$, $\operatorname{card} O_{\mathscr{F}}(z) < \aleph_0$,

(iii) *for every* $z \in \mathbb{S}^1$, $O_{\mathscr{F}}(z)^{\mathrm{d}} = \emptyset$.

Proof. We first show that conditions (i) and (ii) are equivalent. It is obvious that (i) implies (ii). Now, assume that (ii) holds true and suppose, contrary to our claim, that the set $\mathscr{F}$ is infinite. Then there exists a sequence $(F^{v_n})_{n\in\mathbb{N}}$ of elements of $\mathscr{F}$ such that $F^{v_k} \neq F^{v_l}$ for any distinct positive integers k and l. Fix a $z_0 \in \mathbb{S}^1$. Since $\operatorname{card} O_{\mathscr{F}}(z_0) < \aleph_0$, there are distinct $n_1, n_2 \in \mathbb{N}$ with $F^{v_{n_1}}(z_0) = F^{v_{n_2}}(z_0)$. This together with the fact that the iteration group $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ is disjoint gives $F^{v_{n_1}} = F^{v_{n_2}}$, which is impossible. To complete the proof it suffices to observe that conditions (ii) and (iii) are also equivalent. □

The proof of our next proposition is based on ideas similar to those in the proof of Theorem 1 in [17].

Proposition 2. *If $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ is a disjoint iteration group, then*

(i) *the set*

$$L := O_{\mathscr{F}}(z)^{\mathrm{d}}$$

does not depend on $z \in \mathbb{S}^1$,

(ii) *$F^v[L] = L$ for $v \in V$,*

(iii) *either L is a non-empty perfect and nowhere dense subset of $\mathbb{S}^1$ or $L = \mathbb{S}^1$ or $L = \emptyset$.*

Proof. If the set $\mathscr{F}$ is finite, then, by Lemma 4, assertions (i) and (ii) hold true and, moreover, $L = \emptyset$.

We now turn to the case when $\mathscr{F}$ is infinite. Since the iteration group $\mathscr{F}$ is disjoint, there exists a $v \in V \setminus \{0\}$ such that $F^v(u) \neq u$ for $u \in \mathbb{S}^1$. Fix $z, w \in \mathbb{S}^1$ and observe that for any $F^v \in \mathscr{F}$ having no fixed point there is a $v_0 \in V$ with

$$F^{v_0}(w) \in \overrightarrow{\big[z, F^v(z)\big)} := \overrightarrow{\big(z, F^v(z)\big)} \cup \{z\}.$$

Indeed, if it were true that $\overrightarrow{\big[z, F^v(z)\big)} \cap O_{\mathscr{F}}(w) = \emptyset$, then from the fact that F^{nv} is strictly increasing for each non-negative integer n we would have

$$\begin{aligned}\emptyset &= F^{nv}\Big[\overrightarrow{\big[z, F^v(z)\big)} \cap O_{\mathscr{F}}(w)\Big] = F^{nv}\Big[\overrightarrow{\big[z, F^v(z)\big)}\Big] \cap F^{nv}[O_{\mathscr{F}}(w)] \\ &= \overrightarrow{\big[F^{nv}(z), F^{nv}(z), F^{(n+1)v}(z)\big)} \cap O_{\mathscr{F}}(w),\end{aligned}$$

and consequently

$$\bigcup_{n=0}^{\infty} \overrightarrow{\big[F^{nv}(z), F^{(n+1)v}(z)\big)} \cap O_{\mathscr{F}}(w) = \emptyset,$$

which contradicts the equality

$$\bigcup_{n=0}^{\infty} \overrightarrow{\big[F^{nv}(z), F^{(n+1)v}(z)\big)} = \mathbb{S}^1.$$

Now, it is easy to check that

(P) for any $z, w \in \mathbb{S}^1$ and $v_1, v_2 \in V$ with $F^{v_1} \neq F^{v_2}$ there is a $v \in V$ such that $F^v(w) \in \overrightarrow{\big[F^{v_2}(z), F^{v_1}(z)\big)}$.

Next, fix $z, u \in \mathbb{S}^1$ and observe that the fact that the set $\mathscr{F}$ is infinite yields $O_{\mathscr{F}}(z)^{\mathrm{d}} \neq \emptyset$. Take a $w \in O_{\mathscr{F}}(z)^{\mathrm{d}}$ and let a sequence $(v_n)_{n\in\mathbb{N}}$ of elements of V

be such that $\lim_{n\to\infty} F^{v_n}(z) = w$,

$$F^{v_n}(z) \neq w, \quad n \in \mathbb{N},$$

and either

$$F^{v_{n+1}}(z) \in \overrightarrow{\big(F^{v_n}(z), w\big)}, \quad n \in \mathbb{N}$$

or

$$F^{v_{n+1}}(z) \in \overrightarrow{\big(w, F^{v_n}(z)\big)}, \quad n \in \mathbb{N}.$$

Since $F^{v_n}(z) \neq F^{v_{n+1}}(z)$ for $n \in \mathbb{N}$, from (P) it may be concluded that there exists a sequence $(\bar{v}_n)_{n\in\mathbb{N}}$ of elements of V for which either

$$F^{\bar{v}_n}(u) \in \overrightarrow{\big[F^{v_n}(z), F^{v_{n+1}}(z)\big)}, \quad n \in \mathbb{N}$$

or

$$F^{\bar{v}_n}(u) \in \overrightarrow{\big[F^{v_{n+1}}(z), F^{v_n}(z)\big)}, \quad n \in \mathbb{N}.$$

The fact that $\lim_{n\to\infty} F^{v_n}(z) = w$ now leads to $\lim_{n\to\infty} F^{\bar{v}_n}(u) = w$. As it is easily seen that $F^{\bar{v}_n}(u) \neq w$ for $n \in \mathbb{N}$, we finally obtain $w \in O_{\mathscr{F}}(u)^{\mathrm{d}}$. Therefore $O_{\mathscr{F}}(z)^{\mathrm{d}} \subset O_{\mathscr{F}}(u)^{\mathrm{d}}$, and (i) is proved.

Fix $v \in V$, $z \in \mathbb{S}^1$. Since F^v is a homeomorphism, we have

$$F^v[L] = F^v[O_{\mathscr{F}}(z)^{\mathrm{d}}] = (F^v[O_{\mathscr{F}}(z)])^{\mathrm{d}} = O_{\mathscr{F}}(z)^{\mathrm{d}} = L,$$

which completes the proof of (ii).

In order to show (iii) fix $v \in V$, $w \in \mathbb{S}^1$ and take a $z \in L = O_{\mathscr{F}}(w)^{\mathrm{d}}$. Then there is a sequence $(v_n)_{n\in\mathbb{N}}$ of elements of V such that $\lim_{n\to\infty} F^{v_n}(w) = z$ and

$$F^{v_n}(w) \neq z, \quad n \in \mathbb{N}.$$

Consequently,

$$\lim_{n\to\infty} F^v(F^{v_n}(w)) = \lim_{n\to\infty} F^{v+v_n}(w) = F^v(z)$$

and

$$F^{v+v_n}(w) \neq F^v(z), \quad n \in \mathbb{N}.$$

We conclude from this that $F^v(z) \in O_{\mathscr{F}}(w)^{\mathrm{d}} = L$, hence $O_{\mathscr{F}}(z) \subset L$, and finally $L = O_{\mathscr{F}}(z)^{\mathrm{d}} \subset L^{\mathrm{d}}$. On the other hand we also get

$$L^{\mathrm{d}} = (O_{\mathscr{F}}(z)^{\mathrm{d}})^{\mathrm{d}} \subset O_{\mathscr{F}}(z)^{\mathrm{d}} = L,$$

and therefore the set L is perfect.

Next, assume that L is not nowhere dense. Then L being closed it is not a border set, and therefore it contains an open arc Δ. Fix a $z_0 \in \mathbb{S}^1$. Since $O_{\mathscr{F}}(z_0)^{\mathrm{d}} = L$, there are $v_1, v_2 \in V$ such that $F^{v_1}(z_0) \neq F^{v_2}(z_0)$ and $\overrightarrow{[F^{v_1}(z_0), F^{v_2}(z_0))} \subset \Delta$. With the notation $z := F^{v_1}(z_0)$, $v := v_2 - v_1$ we have $\overrightarrow{[z, F^v(z))} \subset \Delta$, which together with (ii) gives

$$\overrightarrow{[F^{nv}(z), F^{(n+1)v}(z))} = F^{nv}\left[\overrightarrow{[z, F^v(z))}\right] \subset F^{nv}[\Delta] \subset F^{nv}[L] = L, \quad n \in \mathbb{N} \cup \{0\}.$$

We thus get

$$\mathbb{S}^1 = \bigcup_{n=0}^{\infty} \overrightarrow{[F^{nv}(z), F^{(n+1)v}(z))} \subset L$$

and, consequently, $L = \mathbb{S}^1$. □

Proposition 2 enables us to introduce the following definition.

Definition 3. By the limit set of a disjoint iteration group $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ we mean the set

$$L_{\mathscr{F}} := O_{\mathscr{F}}(z)^{\mathrm{d}},$$

where z is an arbitrary element of $\mathbb{S}^1$.

3.3. Classification of iteration groups which are non-singular or disjoint.

Let us first note that although Definitions 2 and 3 are different, Lemma 3 shows that in the case when the iteration group $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ is both disjoint and non-singular they determine the very same set.

Definition 4 (see also [7]). A non-singular or disjoint iteration group $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ is called

- *dense*, if $L_{\mathscr{F}} = \mathbb{S}^1$,
- *non-dense*, if $\emptyset \neq L_{\mathscr{F}} \neq \mathbb{S}^1$,
- *discrete*, if $L_{\mathscr{F}} = \emptyset$.

We will consider the above three classes of iteration groups separately.

Observe also that Remark 2 makes it obvious that every discrete iteration group is both disjoint and singular.

Lemma 5. *If* $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ *is a dense or non-dense iteration group, then*

$$L_{\mathscr{F}} = \operatorname{cl} O_{\mathscr{F}}(z), \quad z \in L_{\mathscr{F}}.$$

P r o o f. Assume that $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ is a non-singular (disjoint) iteration group and fix a $z \in L_{\mathscr{F}}$. By Lemma 3 (respectively, Definition 3) we see at once that

$$L_{\mathscr{F}} = O_{\mathscr{F}}(z)^{\mathrm{d}} \subset \operatorname{cl} O_{\mathscr{F}}(z).$$

On the other hand from Remark 2 (respectively, Proposition 2) we deduce that the set $L_{\mathscr{F}}$ is closed and contains $O_{\mathscr{F}}(z)$, and therefore

$$\operatorname{cl} O_{\mathscr{F}}(z) \subset \operatorname{cl} L_{\mathscr{F}} = L_{\mathscr{F}}.$$

□

4. Main results

We start with three auxiliary lemmas. The first is valid without any assumption on the iteration group $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$.

Lemma 6. *If* $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \to \mathbb{S}^1;\ v \in V\}$ *is an iteration group, then the set*

$$H_{\mathscr{F}} := \{\alpha(F^v);\ v \in V\}$$

is either dense in $[0,1)$ *or equal to* $\{k/n;\ k = 0,\dots,n-1\}$ *for an* $n \in \mathbb{N}$.

P r o o f. We first show that the set

$$G := \{\alpha(F^v) + k;\ v \in V;\ k \in \mathbb{Z}\}$$

is a subgroup of the group $(\mathbb{R},+)$. From Theorem 1 in [8] it follows that for any $v_1, v_2 \in V$, $k_1, k_2 \in \mathbb{Z}$ there exists an integer k_3 such that

$$\begin{aligned}\alpha(F^{v_1}) + k_1 + \alpha(F^{v_2}) + k_2 &= \alpha(F^{v_1} \circ F^{v_2}) + k_1 + k_2 + k_3\\ &= \alpha(F^{v_1+v_2}) + k_1 + k_2 + k_3,\end{aligned}$$

which gives $\alpha(F^{v_1}) + k_1 + \alpha(F^{v_2}) + k_2 \in G$. Now, fix $v \in V$, $k \in \mathbb{Z}$. Clearly, if $\alpha(F^v) = 0$, then $-(\alpha(F^v)+k) = -k \in G$. If $\alpha(F^v) \neq 0$, then Corollary 1 in [8] leads to

$$-(\alpha(F^v) + k) = -(1 - \alpha(F^{-v}) + k) = \alpha(F^{-v}) - k - 1 \in G.$$

Next, note that G being a subgroup of $\mathbb{R}$ it is either a dense subset of $\mathbb{R}$ or equal to $p\mathbb{Z} := \{pk;\ k \in \mathbb{Z}\}$ for a non-negative real number p (see for instance [5]). Since $1 \in G$, in the latter case there is a positive integer n for which $p = 1/n$. The equality $G \cap [0,1) = H_{\mathscr{F}}$ now completes the proof. □

Lemma 7. *If $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \to \mathbb{S}^1;\ v \in V\}$ is a disjoint iteration group, then $F^{v_1} = F^{v_2}$ for any $v_1, v_2 \in V$ such that $\alpha(F^{v_1}) = \alpha(F^{v_2})$.*

P r o o f. Let $v_1, v_2 \in V$ be such that $\alpha(F^{v_1}) = \alpha(F^{v_2})$. If $\alpha(F^{v_1}) = \alpha(F^{v_2}) = 0$, then both F^{v_1} and F^{v_2} have fixed points, which together with the fact that the iteration group $\mathscr{F}$ is disjoint gives $F^{v_1} = \mathrm{id} = F^{v_2}$.

Now, assume that $\alpha(F^{v_1}) = \alpha(F^{v_2}) \neq 0$. Then Corollary 1 in [8] implies that

$$\alpha(F^{v_1}) + \alpha(F^{-v_2}) - 1 = 0,$$

and Theorem 1 in [8] leads to

$$0 = \alpha(F^{v_1} \circ F^{-v_2}) = \alpha(F^{v_1 - v_2}).$$

Therefore $F^{v_1 - v_2} = \mathrm{id}$, and consequently $F^{v_1} = F^{v_2}$. □

Lemma 8. *If $c\colon V \longrightarrow \mathbb{S}^1$ is a function such that $\operatorname{card} \operatorname{Im} c = \aleph_0$ and*

$$c(v_1 + v_2) = c(v_1)c(v_2), \quad v_1, v_2 \in V,$$

then $\operatorname{cl} \operatorname{Im} c = \mathbb{S}^1$.

P r o o f. An easy computation shows that the set $\operatorname{cl} \operatorname{Im} c$ is a closed subgroup of the group $(\mathbb{S}^1, \cdot)$. Since (see for instance [14]) every such subgroup is either finite or equal to $\mathbb{S}^1$, and $\operatorname{cl} \operatorname{Im} c$ is infinite, we get $\operatorname{cl} \operatorname{Im} c = \mathbb{S}^1$. □

4.1. Discrete iteration groups.

First, we shall consider the easiest case, namely we will deal with discrete iteration groups.

Lemma 9. *Let $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ be a disjoint iteration group. Then the iteration group $\mathscr{F}$ is discrete if and only if $H_{\mathscr{F}} = \big\{k/n;\ k = 0, \dots, n-1\big\}$ for a positive integer n.*

P r o o f. From Lemma 4 it follows that $L_{\mathscr{F}} = \emptyset$ if and only if the set $\mathscr{F}$ is finite. Since, by Lemma 7, $\alpha\colon \mathscr{F} \longrightarrow H_{\mathscr{F}}$ is a bijection, this is equivalent to the fact that $\operatorname{card} H_{\mathscr{F}} < \aleph_0$, which, on account of Lemma 6, holds if and only if $H_{\mathscr{F}} = \big\{k/n;\ k = 0, \dots n-1\big\}$ for a positive integer n. □

Theorem 1. *Assume that $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ is a discrete iteration group and let n be a positive integer for which $H_{\mathscr{F}} = \big\{k/n;\ k = 0,\dots,n-1\big\}$. Then there exists a mapping $m\colon V \longrightarrow \{0,\dots,n-1\}$ such that $F^v = G^{m(v)}$ for all $v \in V$ and a homeomorphism $G \in \mathscr{F}$ with $\alpha(G) = 1/n \pmod 1$.*

Proof. Fix a $G \in \mathscr{F}$ for which $\alpha(G) = 1/n \pmod 1$ and take a $k \in \{0,\dots,n-1\}$. Obviously, $G^k \in \mathscr{F}$ and, by Theorem 1 in [8], $\alpha(G^k) = k/n$. This together with Lemma 7 shows that $\mathscr{F} = \{G^k;\ k = 0,\dots,n-1\}$, and therefore there is a function $m\colon V \longrightarrow \{0,\dots,n-1\}$ such that $F^v = G^{m(v)}$ for $v \in V$. □

4.2. Dense iteration groups.

Our next proposition serves as an important tool in the investigation of iteration groups which are dense or non-dense.

Proposition 3. *If $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ is a dense or non-dense iteration group, then there exists a unique pair $(\varphi_{\mathscr{F}}, c_{\mathscr{F}})$ such that $\varphi_{\mathscr{F}}\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is a continuous function of degree 1 with $\varphi_{\mathscr{F}}(1) = 1$ and $c_{\mathscr{F}}\colon V \longrightarrow \mathbb{S}^1$ satisfying the system of functional equations*

$$\varphi_{\mathscr{F}}(F^v(z)) = c_{\mathscr{F}}(v)\varphi_{\mathscr{F}}(z), \quad z \in \mathbb{S}^1;\ v \in V. \tag{2}$$

The mapping $c_{\mathscr{F}}$ is given by

$$c_{\mathscr{F}}(v) = \Pi(\alpha(F^v)), \quad v \in V$$

and fulfils the equation

$$c_{\mathscr{F}}(v_1 + v_2) = c_{\mathscr{F}}(v_1)c_{\mathscr{F}}(v_2), \quad v_1, v_2 \in V. \tag{3}$$

The function $\varphi_{\mathscr{F}}$ is increasing and

$$\varphi_{\mathscr{F}}[L_{\mathscr{F}}] = \mathbb{S}^1. \tag{4}$$

Moreover, $\varphi_{\mathscr{F}}$ is a homeomorphism if and only if the iteration group $\mathscr{F}$ is dense.

Proof. We first show that the assumptions of Lemma 1 are fulfilled. Since $L_{\mathscr{F}} \neq \emptyset$, we see at once that $N \neq \emptyset$. If $\mathscr{F}$ is non-singular, then our assertion follows immediately from Remark 1. If the iteration group $\mathscr{F}$ is disjoint, then it is easy to check that for every $z \in \mathbb{S}^1$ we have $O_{\mathscr{F}}(z) = C(z)$, and consequently each set $C(z)$ is infinite and

$$\emptyset \neq L_{\mathscr{F}} \subset \bigcap_{z \in \mathbb{S}^1} \operatorname{cl} C(z).$$

Thus the assumptions of Lemma 1 are satisfied, and therefore there is a unique pair $(\varphi_{\mathscr{F}}, c_{\mathscr{F}})$ such that $\varphi_{\mathscr{F}} \colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is a continuous function of degree 1 with $\varphi_{\mathscr{F}}(1) = 1$ and $c_{\mathscr{F}} \colon V \longrightarrow \mathbb{S}^1$ satisfying system (2). Moreover, the mapping $c_{\mathscr{F}}$ is of the desired form, and Theorem 1 in [8] now shows that (3) holds true.

In the proof of Lemma 1 (see [10]) it is also shown that the function $\varphi_{\mathscr{F}}$ has an increasing lift $\beta \colon \mathbb{R} \longrightarrow \mathbb{R}$ with $\beta(0) = 0$. To see that the mapping $\varphi_{\mathscr{F}}$ is increasing, fix $u, w, z \in \mathbb{S}^1$ for which $u \prec w \prec z$ and let $t_u, t_w, t_z \in [0, 1)$ be such that $\Pi(t_u) = u$, $\Pi(t_w) = w$ and $\Pi(t_z) = z$. From Lemma 2 and Remark 3 in [6] it follows that we may assume that $0 \leqslant t_u < t_w < t_z < 1$. Then, using the facts that the function $\beta \colon \mathbb{R} \longrightarrow \mathbb{R}$ is increasing, $\beta(0) = 0$ and $\deg \varphi_{\mathscr{F}} = 1$, we get

$$0 = \beta(0) \leqslant \beta(t_u) \leqslant \beta(t_w) \leqslant \beta(t_z) \leqslant \beta(1) = 1,$$

and consequently $\varphi_{\mathscr{F}}(u) \preceq \varphi_{\mathscr{F}}(w) \preceq \varphi_{\mathscr{F}}(z)$ as claimed.

Now, fix a $z \in L_{\mathscr{F}}$. Since the pair $(\varphi_{\mathscr{F}}, c_{\mathscr{F}})$ satisfies system (2), we have

$$\varphi_{\mathscr{F}}[O_{\mathscr{F}}(z)] = \varphi_{\mathscr{F}}(z) c_{\mathscr{F}}[V] = \varphi_{\mathscr{F}}(z)\Pi[H_{\mathscr{F}}],$$

which together with the compactness of $\mathbb{S}^1$ and the continuity of the functions Π and $\varphi_{\mathscr{F}}$ gives

$$\begin{aligned} \varphi_{\mathscr{F}}(z)\Pi[\operatorname{cl} H_{\mathscr{F}}] &\subset \varphi_{\mathscr{F}}(z) \operatorname{cl} \Pi[H_{\mathscr{F}}] = \operatorname{cl}(\varphi_{\mathscr{F}}(z)\Pi[H_{\mathscr{F}}]) \\ &= \operatorname{cl} \varphi_{\mathscr{F}}[O_{\mathscr{F}}(z)] = \varphi_{\mathscr{F}}[\operatorname{cl} O_{\mathscr{F}}(z)]. \end{aligned}$$

As $L_{\mathscr{F}} \neq \emptyset$, it follows from Lemmas 9 and 6 that the set $H_{\mathscr{F}}$ is dense in $[0, 1)$. Therefore Lemma 5 now leads to

$$\begin{aligned} \mathbb{S}^1 = \varphi_{\mathscr{F}}(z)\mathbb{S}^1 &= \varphi_{\mathscr{F}}(z)\Pi[[0, 1)] = \varphi_{\mathscr{F}}(z)\Pi[\operatorname{cl} H_{\mathscr{F}}] \\ &\subset \varphi_{\mathscr{F}}[\operatorname{cl} O_{\mathscr{F}}(z)] = \varphi_{\mathscr{F}}[L_{\mathscr{F}}], \end{aligned}$$

and (4) is proved.

If $\varphi_{\mathscr{F}}$ is a homeomorphism, then (4) makes it obvious that $L_{\mathscr{F}} = \mathbb{S}^1$. Now, let $L_{\mathscr{F}} = \mathbb{S}^1$ and suppose, contrary to our claim, that the mapping $\varphi_{\mathscr{F}}$ is not invertible. Then there exist distinct $z_1, z_2 \in \mathbb{S}^1$ such that $\varphi_{\mathscr{F}}(z_1) = \varphi_{\mathscr{F}}(z_2)$. We shall show that the function $\varphi_{\mathscr{F}}$ is constant on $\overrightarrow{[z_1, z_2)}$ or $\overrightarrow{[z_2, z_1)}$. If this assertion were false then, by (4), there would be $w \in \overrightarrow{(z_1, z_2)}$, $u \in \overrightarrow{(z_2, z_1)}$ for which $\varphi_{\mathscr{F}}(z_1) \neq \varphi_{\mathscr{F}}(w) \neq \varphi_{\mathscr{F}}(u) \neq \varphi_{\mathscr{F}}(z_1)$. Therefore we would have $w \prec z_2 \prec u$, $u \prec z_1 \prec w$, and, by virtue of the fact that $\varphi_{\mathscr{F}}$ is increasing,

$$\varphi_{\mathscr{F}}(w) \preceq \varphi_{\mathscr{F}}(z_2) \preceq \varphi_{\mathscr{F}}(u) \quad \text{and} \quad \varphi_{\mathscr{F}}(u) \preceq \varphi_{\mathscr{F}}(z_1) \preceq \varphi_{\mathscr{F}}(w).$$

Since $\varphi_{\mathscr{F}}(z_1) = \varphi_{\mathscr{F}}(z_2)$ and $\varphi_{\mathscr{F}}(z_1) \neq \varphi_{\mathscr{F}}(w) \neq \varphi_{\mathscr{F}}(u) \neq \varphi_{\mathscr{F}}(z_1)$, from Lemma 3 in [6] we would thus get

$$\varphi_{\mathscr{F}}(w) \prec \varphi_{\mathscr{F}}(z_1) \prec \varphi_{\mathscr{F}}(u) \quad \text{and} \quad \varphi_{\mathscr{F}}(u) \prec \varphi_{\mathscr{F}}(z_1) \prec \varphi_{\mathscr{F}}(w),$$

which is impossible. Denote by Δ an open arc such that $\varphi_{\mathscr{F}}$ is constant on Δ and fix a $z_0 \in \mathbb{S}^1$. Assuming that the iteration group $\mathscr{F}$ is disjoint (non-singular) we see that since $\Delta \subset \mathbb{S}^1 = L_{\mathscr{F}}$ and $L_{\mathscr{F}} = O_{\mathscr{F}}(z_0)^{\mathrm{d}}$ ($L_{\mathscr{F}} = C_{F^{v_0}}(z_0)^{\mathrm{d}}$ for an $F^{v_0} \in \mathscr{F}$ with $\alpha(F^{v_0}) \notin \mathbb{Q}$), there exist $u, w \in \Delta \cap O_{\mathscr{F}}(z_0)$ (respectively, $u, w \in \Delta \cap C_{F^{v_0}}(z_0)$) for which $u \neq w$. u and w, being elements of $O_{\mathscr{F}}(z_0)$ ($C_{F^{v_0}}(z_0)$), may be written as $u = F^{v_u}(z_0)$, $w = F^{v_w}(z_0)$ for some $v_u, v_w \in V$ (respectively, $v_u = nv_0$ and $v_w = mv_0$ for some integers n, m). But $\varphi_{\mathscr{F}}(u) = \varphi_{\mathscr{F}}(w)$ since we also have $u, w \in \Delta$. Therefore it follows from (2) that

$$\begin{aligned}\varphi_{\mathscr{F}}(z_0)c_{\mathscr{F}}(v_w) &= \varphi_{\mathscr{F}}(F^{v_w}(z_0)) = \varphi_{\mathscr{F}}(w) = \varphi_{\mathscr{F}}(u) = \varphi_{\mathscr{F}}(F^{v_u}(z_0)) \\ &= \varphi_{\mathscr{F}}(z_0)c_{\mathscr{F}}(v_u),\end{aligned}$$

and, consequently,

$$\Pi(\alpha(F^{v_u})) = c_{\mathscr{F}}(v_u) = c_{\mathscr{F}}(v_w) = \Pi(\alpha(F^{v_w})).$$

This clearly forces $\alpha(F^{v_u}) = \alpha(F^{v_w})$, and Lemma 7 (or Theorem 1 in [8]) now yields $F^{v_u} = F^{v_w}$, which contradicts the fact that $u \neq w$. □

The mappings $\varphi_{\mathscr{F}}$ and $c_{\mathscr{F}}$ guaranteed by the above lemma enable us, among other things, to give a necessary and sufficient condition for conjugacy of disjoint and dense (or non-dense) iteration groups (see [7] for the case when $V = \mathbb{R}$ and the iteration groups are non-singular, and [9] for the general case).

Regarding the structure of dense iteration groups we have the following theorem, which follows immediately from Proposition 3.

Theorem 2. *If $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ is a dense iteration group, then there exists a unique orientation-preserving homeomorphism $\varphi_{\mathscr{F}}\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ having a fixed point* 1 *such that*

$$F^v(z) = \varphi_{\mathscr{F}}^{-1}(\Pi(\alpha(F^v))\varphi_{\mathscr{F}}(z)), \quad z \in \mathbb{S}^1,\ v \in V.$$

Corollary 1. *Every dense iteration group* $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ *is disjoint.*

Proof. Fix $z_0 \in \mathbb{S}^1$, $v_0 \in V$ such that $F^{v_0}(z_0) = z_0$ and note that Theorem 2 implies that $z_0 = \varphi_{\mathscr{F}}^{-1}(\Pi(\alpha(F^{v_0}))\varphi_{\mathscr{F}}(z_0))$. Therefore $\Pi(\alpha(F^{v_0})) = 1$, and applying Theorem 2 again we obtain $F^{v_0} = \mathrm{id}$. □

4.3 Non-dense iteration groups.

Finally, we turn to non-dense iteration groups.

In this case, according to Definition 4, Remark 2 and Proposition 2, the limit set of the iteration group $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ is a non-empty perfect and nowhere dense subset of $\mathbb{S}^1$, and therefore we have the decomposition

$$\mathbb{S}^1 \setminus L_{\mathscr{F}} = \bigcup_{q\in\mathbb{Q}} I_q, \tag{5}$$

where I_q for $q \in \mathbb{Q}$ are open pairwise disjoint arcs.

Lemma 10. *Assume that* $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ *is a non-dense iteration group and let the pair* $(\varphi_{\mathscr{F}}, c_{\mathscr{F}})$ *be such that* $\varphi_{\mathscr{F}}\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ *is a continuous function of degree* 1 *with* $\varphi_{\mathscr{F}}(1) = 1$ *and* $c_{\mathscr{F}}\colon V \longrightarrow \mathbb{S}^1$ *satisfy system* (2)*. Then*

(i) *for every* $q \in \mathbb{Q}$ *the mapping* $\varphi_{\mathscr{F}}$ *is constant on* I_q,

(ii) *if* $A \subset \mathbb{S}^1$ *is an open arc such that* $\varphi_{\mathscr{F}}$ *is constant on* A*, then* $A \subset I_q$ *for a* $q \in \mathbb{Q}$,

(iii) *for any distinct* $p, q \in \mathbb{Q}$, $\varphi_{\mathscr{F}}[I_p] \cap \varphi_{\mathscr{F}}[I_q] = \emptyset$,

(iv) *for any* $q \in \mathbb{Q}$, $v \in V$ *there exists a* $p \in \mathbb{Q}$ *with* $F^v[I_q] = I_p$,

(v) *the sets* $\operatorname{Im} c_{\mathscr{F}}$ *and*

$$K_{\mathscr{F}} := \varphi_{\mathscr{F}}[\mathbb{S}^1 \setminus L_{\mathscr{F}}]$$

are countable,

(vi) $K_{\mathscr{F}} \cdot \operatorname{Im} c_{\mathscr{F}} = K_{\mathscr{F}}$,

(vii) *the sets* $\operatorname{Im} c_{\mathscr{F}}$ *and* $K_{\mathscr{F}}$ *are dense in* $\mathbb{S}^1$.

Proof. Analysis similar to that in the proof of Proposition 2(a) in [2] shows that (i) holds true.

(ii) Let A be an open arc such that $\varphi_{\mathscr{F}}$ is constant on A. Since $L_{\mathscr{F}}$ is a border set, there exists a $q \in \mathbb{Q}$ for which $I_q \cap A \neq \emptyset$. Write $I_q = \overrightarrow{(a,b)}$, take a $z_0 \in A \setminus \{a\}$ and assume that the iteration group $\mathscr{F}$ is disjoint (non-singular). As $a \in L_{\mathscr{F}}$ and $L_{\mathscr{F}} = O_{\mathscr{F}}(z_0)^{\mathrm{d}}$ ($L_{\mathscr{F}} = C_{F^{v_0}}(z_0)^{\mathrm{d}}$ for an $F^{v_0} \in \mathscr{F}$ with $\alpha(F^{v_0}) \notin \mathbb{Q}$), there is a sequence $(v_k)_{k\in\mathbb{N}}$ of elements of V (respectively, $v_k = n_k v_0$ for some integers n_k)

such that $F^{v_k}(z_0) \neq a$ for $k \in \mathbb{N}$ and $\lim_{k\to\infty} F^{v_k}(z_0) = a$. We shall show that $a \notin A$. To obtain a contradiction, suppose that $a \in A$. Then there exists a positive integer k_0 with $F^{v_k}(z_0) \in A$ for $k > k_0$, and the fact that the function $\varphi_{\mathscr{F}}$ is constant on the arc A leads to

$$\varphi_{\mathscr{F}}(F^{v_k}(z_0)) = \varphi_{\mathscr{F}}(z_0), \quad k > k_0.$$

On the other hand, (2) shows that

$$\varphi_{\mathscr{F}}(F^{v_k}(z_0)) = \Pi(\alpha(F^{v_k}))\varphi_{\mathscr{F}}(z_0), \quad k \in \mathbb{N}.$$

Combining these equalities we deduce that $\Pi(\alpha(F^{v_k})) = 1$ for any $k > k_0$ and, consequently, $\alpha(F^{v_k}) = 0$. The disjointness of the iteration group $\mathscr{F}$ (or the irrationality of $\alpha(F^{v_0})$ together with the fact that $\alpha(F^{v_k}) = n_k\alpha(F^{v_0}) (\mathrm{mod}\ 1)$, which follows from Theorem 1 in [8]) now gives $F^{v_k} = \mathrm{id}$, and therefore $a = \lim_{k\to\infty} F^{v_k}(z_0) = z_0$, which is impossible. Similarly, $b \notin A$, and so $A \subset I_q$.

(iii) Fix two distinct $p, q \in \mathbb{Q}$. Obviously, $I_p \cap I_q = \emptyset$ and moreover, $\operatorname{card} \varphi_{\mathscr{F}}[I_p] = \operatorname{card} \varphi_{\mathscr{F}}[I_q] = 1$, which is clear from (i). Suppose, contrary to our claim, that $\varphi_{\mathscr{F}}[I_p] = \varphi_{\mathscr{F}}[I_q] = \{z_0\}$ for a $z_0 \in \mathbb{S}^1$ and choose $z_1 \in I_p$, $z_2 \in I_q$. Since $\overrightarrow{(z_1, z_2)} \cap L_{\mathscr{F}} \neq \emptyset \neq L_{\mathscr{F}} \cap \overrightarrow{(z_2, z_1)}$, (ii) makes it obvious that the function $\varphi_{\mathscr{F}}$ is constant neither on $\overrightarrow{(z_1, z_2)}$ nor on $\overrightarrow{(z_2, z_1)}$. This together with the equality $\varphi_{\mathscr{F}}\big[\overrightarrow{(z_1, z_2)}\big] \cup \varphi_{\mathscr{F}}\big[\overrightarrow{(z_2, z_1)}\big] = \mathbb{S}^1$ shows that there are $z_3 \in \overrightarrow{(z_1, z_2)}$, $z_4 \in \overrightarrow{(z_2, z_1)}$ for which

$$\varphi_{\mathscr{F}}(z_3) \neq z_0, \quad \varphi_{\mathscr{F}}(z_4) \neq z_0, \quad \varphi_{\mathscr{F}}(z_3) \neq \varphi_{\mathscr{F}}(z_4).$$

Therefore we have

$$z_4 \prec z_1 \prec z_3, \quad z_3 \prec z_2 \prec z_4,$$

and using the facts that the mapping $\varphi_{\mathscr{F}}$ is increasing and $\varphi_{\mathscr{F}}(z_1) = \varphi_{\mathscr{F}}(z_2) = z_0$ and Lemma 3 in [6],

$$\varphi_{\mathscr{F}}(z_4) \prec \varphi_{\mathscr{F}}(z_1) \prec \varphi_{\mathscr{F}}(z_3), \quad \varphi_{\mathscr{F}}(z_3) \prec \varphi_{\mathscr{F}}(z_2) \prec \varphi_{\mathscr{F}}(z_4).$$

Consequently,

$$\varphi_{\mathscr{F}}(z_1) \in \overrightarrow{(\varphi_{\mathscr{F}}(z_4), \varphi_{\mathscr{F}}(z_3))}, \quad \varphi_{\mathscr{F}}(z_2) \in \overrightarrow{(\varphi_{\mathscr{F}}(z_3), \varphi_{\mathscr{F}}(z_4))},$$

which contradicts the equality $\varphi_{\mathscr{F}}(z_1) = \varphi_{\mathscr{F}}(z_2)$.

Proceeding analogously to the proof of Proposition 2 in [2] we obtain (iv), the equality $\operatorname{card} K_{\mathscr{F}} = \aleph_0$ and the inequality $\operatorname{card} \operatorname{Im} c_{\mathscr{F}} \leqslant \aleph_0$. To complete the proof of (v) it suffices to check that $\operatorname{card}\operatorname{Im} c_{\mathscr{F}} \geqslant \aleph_0$. For this purpose, let us first observe that since $L_{\mathscr{F}} \neq \emptyset$, Lemmas 9 and 6 show that the set $H_{\mathscr{F}}$ is dense in $[0,1)$. This together with Proposition 3 and the fact that the mapping Π is bijective gives

$$\operatorname{card}\operatorname{Im} c_{\mathscr{F}} = \operatorname{card}\Pi[H_{\mathscr{F}}] = \operatorname{card} H_{\mathscr{F}} \geqslant \aleph_0,$$

and (v) is proved.

As in the proof of Proposition 2(f) in [2] we can see that (vi) holds true.

(vii) Since the function $c_{\mathscr{F}}\colon V \longrightarrow \mathbb{S}^1$ satisfies equation (3) and $\operatorname{card}\operatorname{Im} c_{\mathscr{F}} = \aleph_0$, it follows from Lemma 8 that the set $\operatorname{Im} c_{\mathscr{F}}$ is dense in $\mathbb{S}^1$, and consequently, by (vi), so is the set $K_{\mathscr{F}}$. □

Assume that $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ is a non-dense iteration group and let I_q for $q \in \mathbb{Q}$ be open pairwise disjoint arcs for which (5) holds true. According to Lemma 10 we can correctly define the bijection $\Phi_{\mathscr{F}}\colon \mathbb{Q} \longrightarrow K_{\mathscr{F}}$ and the mapping $T_{\mathscr{F}}\colon \mathbb{Q}\times V \longrightarrow \mathbb{Q}$ by putting

$$\{\Phi_{\mathscr{F}}(q)\} := \varphi_{\mathscr{F}}[I_q], \quad q \in \mathbb{Q} \tag{6}$$

and

$$T_{\mathscr{F}}(q,v) := \Phi_{\mathscr{F}}^{-1}(\Phi_{\mathscr{F}}(q)c_{\mathscr{F}}(v)), \quad q \in \mathbb{Q};\ v \in V. \tag{7}$$

The following lemma is an immediate consequence of (7) and the properties of the function $c_{\mathscr{F}}$.

Lemma 11. *If $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ is a non-dense iteration group, then*

(i) $T_{\mathscr{F}}(T_{\mathscr{F}}(q,v_1),v_2) = T_{\mathscr{F}}(q, v_1+v_2)$, $q \in \mathbb{Q}$, $v_1, v_2 \in V$,

(ii) $T_{\mathscr{F}}(q,0) = q$, $q \in \mathbb{Q}$.

Proceeding analogously to the proof of Lemma 6 in [2] we also obtain

Lemma 12. *If $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ is a non-dense iteration group, then*

$$F^v[I_q] = I_{T_{\mathscr{F}}(q,v)}, \quad q \in \mathbb{Q},\ v \in V.$$

Before we state our next result we recall one more definition from [2].

Let $A := \overrightarrow{(v,z)}$ be an open arc and let $t_v, t_z \in \mathbb{R}$ be such that $v = \tilde{\Pi}(t_v)$, $z = \tilde{\Pi}(t_z)$ and $0 < t_z - t_v < 1$. A mapping $F\colon A \longrightarrow \mathbb{S}^1$ is said to be *linear* if there are $a, b \in \mathbb{R}$, $a > 0$ with

$$F(\tilde{\Pi}(x)) = \tilde{\Pi}(ax+b), \quad x \in (t_v, t_z).$$

Proposition 4. *Assume that* $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ *is a non-dense iteration group and let the pair* $(\varphi_{\mathscr{F}}, c_{\mathscr{F}})$ *be such that* $\varphi_{\mathscr{F}}\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ *is a continuous function of degree* 1 *with* $\varphi_{\mathscr{F}}(1) = 1$ *and* $c_{\mathscr{F}}\colon V \longrightarrow \mathbb{S}^1$ *fulfil system* (2)*. Then there exists a unique family* $\mathscr{P} = \{P^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ *of continuous mappings such that for any* $q \in \mathbb{Q}$, $v \in V$,

$$P^v \quad \text{is linear on } I_q \tag{8}$$

and

$$P^v[I_q] = I_{T_{\mathscr{F}}(q,v)}, \tag{9}$$

where I_q *for* $q \in \mathbb{Q}$ *are open pairwise disjoint arcs satisfying* (5)*, and* $T_{\mathscr{F}}$ *is given by* (7)*.*

Moreover,

$$\varphi_{\mathscr{F}}(P^v(z)) = c_{\mathscr{F}}(v)\varphi_{\mathscr{F}}(z), \quad z \in \mathbb{S}^1,\ v \in V \tag{10}$$

and the family $\mathscr{P}$ *is a disjoint, non-dense iteration group. If the iteration group* $\mathscr{F}$ *is non-singular, then so is* $\mathscr{P}$*. Otherwise the iteration group* $\mathscr{P}$ *is singular.*

P r o o f. Proceeding analogously to the proof of Lemma 13 in [6] (see also Proposition 3 in [2]) we show the existence and uniqueness of a family $\mathscr{P} = \{P^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ of continuous functions satisfying conditions (8) and (9) as well as the fact that this family is a disjoint iteration group for which (10) and

$$P^v[\mathbb{S}^1 \setminus L_{\mathscr{F}}] = \mathbb{S}^1 \setminus L_{\mathscr{F}}, \quad v \in V \tag{11}$$

hold true (to prove that the functions $B_v\colon \mathbb{S}^1 \setminus L_{\mathscr{F}} \longrightarrow \mathbb{S}^1 \setminus L_{\mathscr{F}}$ for $v \in V$ can be extended to strictly increasing mappings $P^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ it suffices to apply Lemma 12 in [7]). We will now prove that the iteration group $\mathscr{P}$ is non-dense. To do this, let us first observe that $L_{\mathscr{P}} \neq \mathbb{S}^1$. In fact, since $P^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ for $v \in V$ are homeomorphisms, (11) yields

$$P^v[L_{\mathscr{F}}] = L_{\mathscr{F}}, \quad v \in V.$$

Consequently, fixing a $z_0 \in L_{\mathscr{F}}$ we see that $O_{\mathscr{P}}(z_0) = \{P^v(z_0);\ v \in V\} \subset L_{\mathscr{F}}$, which together with Definition 3 and the fact that $L_{\mathscr{F}}$ is a perfect nowhere dense subset of $\mathbb{S}^1$ gives

$$L_{\mathscr{P}} = O_{\mathscr{P}}(z_0)^{\mathrm{d}} \subset L_{\mathscr{F}}^{\mathrm{d}} = L_{\mathscr{F}} \neq \mathbb{S}^1.$$

It remains to show that $L_{\mathscr{P}} \neq \emptyset$. But if this assertion were false, from (10) and Lemma 4 we would have

$$\begin{aligned}\operatorname{card}\operatorname{Im} c_{\mathscr{F}} &= \operatorname{card}(\varphi_{\mathscr{F}}(z)\operatorname{Im} c_{\mathscr{F}}) = \operatorname{card}\varphi_{\mathscr{F}}[O_{\mathscr{P}}(z)] \\ &\leqslant \operatorname{card} O_{\mathscr{P}}(z) < \aleph_0, \quad z \in \mathbb{S}^1,\end{aligned}$$

contrary to Lemma 10(v).

Next, assume that the iteration group $\mathscr{F}$ is non-singular and let $v_0 \in V$ be such that $\alpha(F^{v_0}) \notin \mathbb{Q}$. Suppose, contrary to our claim, that $\alpha(P^v) \in \mathbb{Q}$ for $v \in V$. Then $(P^{v_0})^{n_0}(z_0) = z_0$ for some $n_0 \in \mathbb{N}$, $z_0 \in \mathbb{S}^1$, and (10) gives

$$\varphi_{\mathscr{F}}(z_0) = \varphi_{\mathscr{F}}(P^{n_0 v_0}(z_0)) = \varphi_{\mathscr{F}}(z_0)c_{\mathscr{F}}(n_0 v_0).$$

This clearly forces $c_{\mathscr{F}}(n_0 v_0) = 1$ and (3) now shows that

$$1 = c_{\mathscr{F}}(n_0 v_0) = c_{\mathscr{F}}(v_0)^{n_0} = \Pi(\alpha(F^{v_0}))^{n_0} = \tilde{\Pi}(n_0\alpha(F^{v_0})).$$

Thus $n_0\alpha(F^{v_0}) \in \mathbb{Z}$ which, in view of $n_0 \neq 0$ and $\alpha(F^{v_0}) \notin \mathbb{Q}$, is impossible.

Finally, assume that $\mathscr{F}$ is a singular and disjoint iteration group and suppose, contrary to our claim, that $\alpha(P^{v_0}) \notin \mathbb{Q}$ for a $v_0 \in V$. Since $\alpha(F^{v_0}) \in \mathbb{Q}$, there are $n_0 \in \mathbb{Z}\setminus\{0\}$, $z_0 \in \mathbb{S}^1$ such that $(F^{v_0})^{n_0}(z_0) = z_0$, and therefore $\alpha(F^{n_0 v_0}) = 0$. Proposition 3 now gives $c_{\mathscr{F}}(n_0 v_0) = 1$, which together with (7) leads to $T_{\mathscr{F}}(q, n_0 v_0) = q$ for $q \in \mathbb{Q}$. From this and (9) we deduce that $P^{n_0 v_0}[I_q] = I_q$ for $q \in \mathbb{Q}$. By virtue of the fact that the function $P^{n_0 v_0}$ is linear on each arc I_q we thus get $P^{n_0 v_0} = \mathrm{id}$ and consequently $\alpha(P^{v_0}) \in \mathbb{Q}$, a contradiction. □

The piecewise linear iteration group $\mathscr{P} = \{P^v \colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ described by Proposition 4 is called the *generating iteration group* of a non-dense iteration group $\mathscr{F} = \{F^v \colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ (see [2] and also [18]).

It is worth pointing out that the generating iteration group of $\mathscr{F} = \{F^v \colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ is disjoint even when $\mathscr{F}$ is not.

Proposition 5. *Let $\mathscr{F} = \{F^v \colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ and $\mathscr{G} = \{G^v \colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ be non-singular (or singular but not discrete) disjoint iteration groups and assume that there exists a pair (φ, c) such that $\varphi \colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is a continuous mapping of degree 1 and $c \colon V \longrightarrow \mathbb{S}^1$ satisfy the system*

$$\left.\begin{aligned}\varphi(F^v(z)) &= \varphi(z)c(v) \\ \varphi(G^v(z)) &= \varphi(z)c(v)\end{aligned}\right\} z \in \mathbb{S}^1,\ v \in V.$$

Then $L_{\mathscr{F}} = L_{\mathscr{G}}$ *and there is an orientation-preserving homeomorphism* $\Gamma\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ *with*

$$\Gamma(z) = z, \quad z \in L_{\mathscr{F}} \tag{12}$$

and

$$\Gamma \circ F^v = G^v \circ \Gamma, \quad v \in V.$$

If moreover the iteration groups $\mathscr{F}$ *and* $\mathscr{G}$ *are non-dense, then*

$$\Gamma[I_q] = I_q, \quad q \in \mathbb{Q}, \tag{13}$$

where I_q *for* $q \in \mathbb{Q}$ *are open pairwise disjoint arcs for which* (5) *holds true.*

P r o o f. Let us first observe that if $\varphi(1) = 1$, which we may assume, then Proposition 3 implies that $\varphi = \varphi_{\mathscr{F}} = \varphi_{\mathscr{G}}$ and $c = c_{\mathscr{F}} = c_{\mathscr{G}}$.

If the iteration group $\mathscr{F}$ is dense, then analysis similar to that in the proof of Proposition 4 in [2] shows that φ is a homeomorphism and the iteration group $\mathscr{G}$ is dense. Thus $F^v = G^v$ for $v \in V$ and $\Gamma := \mathrm{id}$ has all the desired properties.

Now, assume that the iteration group $\mathscr{F}$ is non-dense. As in the proof of Proposition 4 in [2] we can see that $L_{\mathscr{F}} = L_{\mathscr{G}}$, which, on account of (7) and (6), allows us to assume that $T_{\mathscr{F}} = T_{\mathscr{G}}$.

Introduce the following equivalence relation on $\mathbb{Q}$:

$$p \mathscr{R} q \quad \text{if and only if there is a } v \in V \text{ such that} \quad p = T_{\mathscr{F}}(q, v),$$

and let E be a subset of $\mathbb{Q}$ having exactly one point in common with each equivalence class with respect to the relation $\mathscr{R}$. Denote by $[q]_{\mathscr{R}}$ the equivalence class of $q \in \mathbb{Q}$ and let $\mathbb{Q} \ni q \longmapsto v_q \in V$ be a mapping such that

$$[q]_{\mathscr{R}} \cap E = \{T_{\mathscr{F}}(q, v_q)\}, \quad q \in \mathbb{Q}.$$

Putting

$$\Gamma(z) := \begin{cases} (G^{-v_q} \circ F^{v_q})(z), & z \in I_q,\ q \in \mathbb{Q}, \\ z, & z \in L_{\mathscr{F}} \end{cases} \tag{14}$$

we conclude from Lemmas 12 and 11 that conditions (12) and (13) hold true. We shall next show that $\Gamma\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is an orientation-preserving homeomorphism. To do this, let us first observe that since for every $q \in \mathbb{Q}$, G^{-v_q} and F^{v_q} are orientation-preserving homeomorphisms, it follows from Lemma 5 in [6] that so is $\Gamma_{|I_q}$. Now, we prove that the mapping $\Gamma\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is strictly increasing. For this purpose, fix $x, w, z \in \mathbb{S}^1$ with $w \in \overrightarrow{(x,z)}$ and consider the following five cases.

(i) $\{x, w, z\} \subset I_q$ for a $q \in \mathbb{Q}$.

Putting

$$t_x := \Pi^{-1}(x), \quad t_w := \Pi^{-1}(w), \quad t_z := \Pi^{-1}(z),$$

we have either $t_x < t_w < t_z$, or $t_w < t_z < t_x$, or $t_z < t_x < t_w$. Assume, for instance, that $t_x < t_w < t_z$ and let f_q represent the homeomorphism $\Gamma_{|I_q}$. Since $\Gamma_{|I_q}$ preserves orientation, we see that $f_q(t_x) < f_q(t_w) < f_q(t_z)$, and (13) leads to $f_q(t_x), f_q(t_w), f_q(t_z) \in (t_1, t_2)$ for some $t_1, t_2 \in \mathbb{R}$ with $0 < t_2 - t_1 < 1$. Therefore $\tilde{\Pi}(f_q(t_w)) \in \overrightarrow{(\tilde{\Pi}(f_q(t_x)), \tilde{\Pi}(f_q(t_z)))}$, and consequently $\Gamma(w) \in \overrightarrow{(\Gamma(x), \Gamma(z))}$.

(ii) $\operatorname{card}(\{x, w, z\} \cap I_q) = 2$ for a $q \in \mathbb{Q}$.

According to Lemmas 1 and 2 in [6] we may assume that $x, w \in I_q$. Fixing a $u \in I_q$ for which $w \in \overrightarrow{(x,u)}$ we conclude from (i), (13) and (14) that

$$\Gamma(w) \in \overrightarrow{(\Gamma(x), \Gamma(u))} \subset I_q \quad \text{and} \quad \Gamma(z) \notin I_q,$$

and, consequently, $\Gamma(w) \in \overrightarrow{(\Gamma(x), \Gamma(z))}$.

(iii) $\operatorname{card}(\{x, w, z\} \cap I_q) = \operatorname{card}(\{x, w, z\} \cap I_p) = 1$ for some distinct $p, q \in \mathbb{Q}$.

We can assume, in view of Lemmas 1 and 2 in [6], that $x \in I_q$, $w \in I_p$. Since $x \prec w \prec z$, we have $I_q \prec I_p \prec I(z)$, where

$$I(z) := \begin{cases} I_r, & z \in I_r,\ r \in \mathbb{Q}, \\ \{z\}, & z \in L_{\mathscr{F}}. \end{cases}$$

Therefore, by (13) and (14), we get $\Gamma[I_q] \prec \Gamma[I_p] \prec \Gamma[I(z)]$, and consequently $\Gamma(x) \prec \Gamma(w) \prec \Gamma(z)$.

(iv) $\operatorname{card}(\{x, w, z\} \cap L_{\mathscr{F}}) = 2$.

On account of Lemmas 1 and 2 in [6] we may assume that $x, z \in L_{\mathscr{F}}$ and $w \in I_q$ for a $q \in \mathbb{Q}$. As I_q is an open arc, we see that $I_q \subset \overrightarrow{(x,z)}$. Hence, by (13) and (14), we obtain

$$\Gamma[I_q] = I_q \subset \overrightarrow{(x,z)} = \overrightarrow{(\Gamma(x), \Gamma(z))},$$

and therefore $\Gamma(w) \in \overrightarrow{(\Gamma(x), \Gamma(z))}$.

(v) $\{x, w, z\} \subset L_{\mathscr{F}}$.

From (14) we have $\Gamma(x) = x$, $\Gamma(w) = w$ and $\Gamma(z) = z$, and our assertion follows.

We have thus proved that the function $\Gamma\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is strictly increasing. Since, by (13) and (14), we also have $\operatorname{Im}\Gamma = \mathbb{S}^1$, Remark 3 in [7] and Lemma 4 in [6] show that Γ is a homeomorphism. Finally, it follows from Lemma 11 in [7] that this homeomorphism preserves orientation.

The rest of the proof runs as in the proof of Proposition 4 in [2]. □

The following facts follow immediately from Propositions 3, 4 and 5.

Corollary 2. *Assume that* $\mathscr{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ *is a non-dense disjoint iteration group and let* $\mathscr{P} = \{P^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ *be its generating iteration group. Then*

$$L_{\mathscr{F}} = L_{\mathscr{P}}$$

and the iteration group $\mathscr{P}$ *is non-singular if and only if so is* $\mathscr{F}$*.*

Theorem 3. *Assume that* $\mathscr{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ *is a non-dense disjoint iteration group and let* $\mathscr{P} = \{P^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1;\ v \in V\}$ *be its generating iteration group. Then there exists an orientation-preserving homeomorphism* $\Gamma : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ *such that*

$$\Gamma \circ F^v = P^v \circ \Gamma, \quad v \in V$$

and conditions (12) *and* (13) *are satisfied, where* I_q *for* $q \in \mathbb{Q}$ *are open pairwise disjoint arcs for which* (5) *holds true.*

References

[1] *L. Alsedà, J. Llibre and M. Misiurewicz*: Combinatorial Dynamics and Entropy in Dimension One. Advanced Series in Nonlinear Dynamics, 5. World Scientific Publishing Co. Inc., River Edge, 1993.

[2] *M. Bajger*: On the structure of some flows on the unit circle. Aequationes Math. *55* (1998), 106–121.

[3] *M. Bajger and M. C. Zdun*: On rational flows of continuous functions. Iteration Theory (Batschuns, 1992). World Sci. Publishing, River Edge, 1996, pp. 265–276.

[4] *L. S. Block and W. A. Coppel*: Dynamics in One Dimension. Lecture Notes in Mathematics, 1513. Springer-Verlag, Berlin, 1992.

[5] *N. Bourbaki*: Éléments de mathématique. Topologie générale. Hermann, Paris, 1971.

[6] *K. Ciepliński*: On the embeddability of a homeomorphism of the unit circle in disjoint iteration groups. Publ. Math. Debrecen *55* (1999), 363–383.

[7] *K. Ciepliński*: On conjugacy of disjoint iteration groups on the unit circle. Ann. Math. Sil. *13* (1999), 103–118; European Conference on Iteration Theory (Muszyna-Złockie, 1998) .

[8] *K. Ciepliński*: The rotation number of the composition of homeomorphisms. Rocznik Nauk.—Dydakt. AP w Krakowie. Prace Mat. *17* (2000), 83–87.

[9] *K. Ciepliński*: Topological conjugacy of disjoint flows on the circle. Bull. Korean Math. Soc. *39* (2002), 333–346.

[10] *K. Ciepliński and M. C. Zdun*: On a system of Schröder equations on the circle. Internat. J. Bifur. Chaos Appl. Sci. Engrg. *13* (2003), 1883–1888.

[11] *I. P. Cornfeld, S. V. Fomin and Ya. G. Sinai*: Ergodic Theory. Grundlehren der Mathematischen Wissenschaften, 245. Springer-Verlag, New York, 1982.

[12] *Z. Nitecki*: Differentiable Dynamics. An Introduction to the Orbit Structure of Diffeomorphisms. The M.I.T. Press Cambridge, Massachusetts-London, 1971.

[13] *C. T. C. Wall*: A Geometric Introduction to Topology. Addison-Wesley Publishing Co., Reading, Massachusetts-London-Don Mills, 1972.

[14] *P. Walters*: An Introduction to Ergodic Theory. Graduate Texts in Mathematics, 79. Springer-Verlag, New York-Berlin, 1982.

[15] *M. C. Zdun*: On embedding of homeomorphisms of the circle in a continuous flow. Iteration theory and its functional equations (Lochau, 1984). Lecture Notes in Math., 1163. Springer, Berlin, 1985, pp. 218–231.

[16] *M. C. Zdun*: On the embeddability of commuting functions in a flow. Selected topics in functional equations and iteration theory (Graz, 1991). Grazer Math. Ber., 316. Karl-Franzens-Univ. Graz, Graz, 1992, pp. 201–212.

[17] *M. C. Zdun*: On the orbits of disjoint groups of continuous functions. Rad. Mat. *8* (1992/96), 95–104.

[18] *M. C. Zdun*: The structure of iteration groups of continuous functions. Aequationes Math. *46* (1993), 19–37.

[19] *M. C. Zdun*: On some invariants of conjugacy of disjoint iteration groups. Results Math. *26* (1994), 403–410.

Author's address: Institute of Mathematics, Pedagogical University, Podchorążych 2, PL-30-084 Kraków, Poland, e-mail: `kc@wsp.krakow.pl` or `smciepli@cyf-kr.edu.pl`.