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Almost π-lattices

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ALMOST π-LATTICES

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Abstract. In this paper we establish some conditions for an almost π-domain to be a π-domain. Next π-lattices satisfying the union condition on primes are characterized. Using these results, some new characterizations are given for π-rings.

Keywords: π-domain, almost π-domain, π-ring, d-prime element

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1. Introduction

By a C-lattice we mean a (not necessarily modular) complete multiplicative lattice, with least element 0 and compact greatest element 1 (a multiplicative identity), which is generated under joins by a multiplicatively closed subset C of compact elements. Throughout this paper L denotes a principally generated C-lattice. C-lattices can be localized. For any prime element p of L, L_p denotes the localization at $F = \{x \in C \mid x \nleq p\}$. For details on C-lattices and their localization theory, the reader is referred to [11]. We note that in a C-lattice $a = b$ if and only if $a_m = b_m$ for all maximal prime elements m of L.

Recall that an element $e \in L$ is said to be *principal* [6], if it satisfies the dual identities (i) $a \wedge be = ((a : e) \wedge b)e$ and (ii) $(ae \vee b) : e = (b : e) \vee a$. Elements satisfying the weaker identity (i$'$) $a \wedge e = (a : e)e$ obtained from (i) by setting $b = 1$ are called *weak meet principal* and elements satisfying the weaker identity (ii$'$) $ae : e = (0 : e) \vee a$ obtained from (ii) by setting $b = 0$ are called *weak join principal.* Elements satisfying both (i$'$) and (ii$'$) are called *weak principal.* Note that weak principal elements are compact in L [2, Theorem 1.3].

An element $a \in L$ is said to be a *complemented element* if $a \vee b = 1$ and $ab = 0$ for some $b \in L$ and a is called *invertible* if a is principal and $(0 : a) = 0$. An element

$a \in L$ is called a *σ-element* if for every compact element $x \leqslant a$, $a \vee (0 : x) = 1$ and a is called *nilpotent* if $a^n = 0$ for some positive integer n. Note that a compact element is a σ-element if and only if it is a complemented element. For more information on σ-elements, the reader is referred to [13]. A prime element p of L is said to be *unbranched* if p is the only p-primary element, and p is called an *ℓ-prime* if the set of all p-primary elements of L is linearly ordered. A prime element p of L is said to be a *d-prime* [12] if L_p is a discrete valuation lattice (i.e., consists just of the elements 0, 1 and the powers of p all of which are distinct).

L is said to be a *principal element lattice* if every element is principal. Similarly, L is said to be an *almost principal element lattice* if L_m is a principal element lattice for every maximal prime element m of L. For various characterizations of almost principal element lattices and principal element lattices, the reader is referred to [4], [9] and [10]. L is said to be a *special principal element lattice* if it has a unique maximal element which is principal and every element is a power of the maximal element. L is said to be *reduced* if 0 is the only nilpotent element of L. L is said to be an *M-normal lattice* if every prime element contains a unique minimal prime element. For more information on M-normal lattices, the reader is referred to [3] and [13]. It is well known that L is a reduced M-normal lattice if and only if L_m is a domain for every maximal prime element m of L [13, Theorem 1].

L is said to be a *π-lattice* if L is generated by a set S of elements (not necessarily principal) each of which is a finite product of prime elements. L is said to be an *almost π-lattice* if L_m is a π-lattice for every maximal prime element m of L. L is a π-domain if L is a π-lattice and a domain. L is said to be an almost π-domain if L_m is a π-domain for every maximal prime element m of L. π-lattices and almost π-lattices have been studied in [2], [4] and [10]. Note that if L is a π-domain, then L is an almost π-domain. But the converse need not be true. For example, if L is an almost principal element domain which is not a principal element domain, then L is an almost π-domain. But by Theorem 4 of [10], L is not a π-domain.

The goal of this paper is to establish some conditions for an almost π-domain to be a π-domain. We prove that if L is an almost π-domain satisfying the condition $(*)$ (see Definition 1), then every principal element is a finite product of primes which are either complemented minimal primes or invertible d-primes. Next we show that if L is an almost π-domain in which every prime minimal over a principal element is compact, then every principal element is a finite product of primes which are either complemented minimal primes or invertible d-primes. Using these results, π-lattices which are also locally domains and π-domains are characterized (see Theorem 3 and Theorem 4). Further, we establish some equivalent conditions in terms of almost π-lattices for a lattice L satisfying the union condition on primes to be a π-lattice (see Theorem 5). As a consequence of these results, we obtain some new characterizations for π-rings (see Theorem 6).

For general background and terminology, the reader may consult [2] and [11].

We shall begin with the following definition.

Definition 1. A multiplicative lattice L_0 is said to satisfy the condition $(*)$ if there exists a multiplicatively closed set S of (not necessarily principal) elements which generate L_0 under joins such that every element of S is a finite meet of primary elements.

Noether lattices [6], Dedekind domains [4] and one dimensional quasi-local domains are examples of multiplicative lattices satisfying the condition $(*)$. Obviously if R is a Laskerian ring [8] or a Krull domain [15, p. 195, Ex. 2], then $L(R)$, the lattice of all ideals of R, satisfies the condition $(*)$.

Lemma 1. *Suppose L satisfies the condition $(*)$. Let x be a principal element of L. Then x has only finitely many minimal primes over x.*

Proof. Let S be the set which generates L under joins such that every element of S is a finite meet of primary elements. Let p be a prime minimal over x. Then x_p is p-primary [11, Property 0.5]. Also by Proposition 2 of [5], x_p is completely join irreducible in L_p, so $x_p = y_p$ for some $y \in S$. Therefore p is minimal over y. As x is the join of a finite number of elements of S and every element of S has only finitely many minimal primes, it follows that x has only finitely many minimal primes and the proof is complete. □

Lemma 2. *Suppose L is a π-lattice. Then every principal element has only finitely many minimal primes.*

Proof. The proof of the lemma is similar to that of Lemma 1. □

Lemma 3. *Suppose L satisfies the condition $(*)$. If $a \in L$ is locally principal, then a is principal.*

Proof. Suppose a is locally principal. Let

$$\theta(a) = \bigvee\{(x : a) \mid x \leqslant a \text{ and } x \text{ is principal}\}.$$

We claim that $\theta(a) = 1$. Let $\theta(a) \leqslant m$ for some maximal prime element m of L. Since a is locally principal, by [5, Proposition 2(d)], it follows that $a_m = y_m$ for some principal element $y \leqslant a$. Again $y_m = x_m$ and $x \leqslant y$ for some $x \in S$, where S is the set which generates L under joins such that every element of S is a finite meet of primary elements. By hypothesis, $x = \bigwedge\limits_{i=1}^{n} q_i$ where q_i are primary elements. Note that $x_m = \bigwedge\{(q_i)_m \mid q_i \leqslant m\}$. If $q_i \leqslant m$ for $i = 1, 2, \ldots, n$, then $x = x_m = a_m$, so

$a = x = y$ and therefore $\theta(a) = 1 \leqslant m$, a contradiction. So assume that $q_i \leqslant m$ for $i = 1, 2, \ldots, k$ and $q_j \nleqslant m$ for $j = k+1, k+2, \ldots, n$. Choose principal elements $x_j \leqslant q_j$ such that $x_j \nleqslant m$ for $j = k+1, k+2, \ldots, n$. Note that $x_m = \bigwedge\limits_{i=1}^{k} (q_i)_m$. Put $z = x_{k+1}x_{k+2}\ldots x_n$. Since $a \leqslant \bigwedge\limits_{i=1}^{k} q_i$ and $z \leqslant \bigwedge\limits_{i=k+1}^{n} q_i$, it follows that $az \leqslant \bigwedge\limits_{i=1}^{n} q_i = x \leqslant y$ and hence $z \leqslant (y : a) \leqslant \theta(a) \leqslant m$, a contradiction. Therefore $\theta(a) = 1$. Since 1 is compact, it follows that $1 = \bigvee\limits_{i=1}^{n} \{(y_i : a) \mid y_i \leqslant a$ and y_i is principal$\}$. Again $a = a.1 = \bigvee\limits_{i=1}^{n} (y_i : a)a \leqslant \bigvee\limits_{i=1}^{n} yi \leqslant a$, so $a = \bigvee\limits_{i=1}^{n} y_i$ and hence a is compact. As a is compact and locally principal, by [5, Theorem 1], it follows that a is principal and the proof is complete. □

Lemma 4. *Suppose L is an almost π-domain satisfying the condition $(*)$. If p is a rank one prime, then p is an invertible d-prime.*

Proof. As L is an almost π-domain, by [4, Theorem 2.2 and Corollary 2.3], p is locally principal and hence by Lemma 3, p is principal. Obviously, $0 : p = 0$ and so p is invertible. Again by [4, Lemma 3.2(d)], p is an ℓ-prime. Therefore by [12, Theorem 1 and Theorem 2], p is a d-prime. □

Lemma 5. *Suppose L is a quasi-local π-domain in which p is a prime minimal over a non zero principal element $a \in L$. Then p is a rank one principal prime.*

Proof. By [4, Corollary 2.3], p is principal. Again by [4, Lemma 1.4], there exists a prime $q < p$ such that $pq = q$ and any prime properly contained in p is contained in q. If $q \neq 0$, then by [4, Theorem 2.2 and Corollary 2.3], there exists a non zero principal prime $q_1 \leqslant q$. Since $q_1 < p$ and p is principal, it follows that $q_1 = q_1 p$, so by [2, Theorem 1.4], $q_1 = 0$, a contradiction. Therefore $q = 0$ and hence p is a rank one principal prime. □

Lemma 6. *Let L be a π-lattice which is also locally a domain. If p is a rank one prime, then p is an invertible d-prime.*

Proof. As L is an M-normal lattice, every prime element contains a unique minimal prime element. Suppose $p_1 < p$ is a minimal prime element contained in p. Choose any principal element $a \leqslant p$ such that $a \nleqslant p_1$. Let $m \geqslant p$ be a maximal prime element of L. Then L_m is a π-domain. Since p_m is a prime minimal over a non zero principal element $a_m \in L_m$, by Lemma 5, p_m is principal in L_m. Therefore p is locally principal. It can be easily verified that p is weak join principal. Now we show that p is weak meet principal. Note that $p_1 p = p_1$ locally and hence globally. Let S be the set which generates L under joins such that every element of S is a finite product of prime elements. Let $x \leqslant p$ be any element of S. Again there exist prime elements $q_1, q_2, \ldots, q_n$ such that $x = q_1 q_2 \ldots q_n$. As $x \leqslant p$, it follows that $q_i \leqslant p$ for some i, say $q_1 \leqslant p$. Then either $q_1 = p$ or $q_1 = p_1$. In either case x is a multiple of p. As S generates L under joins, it follows that p is weak meet principal and hence weak principal. Again by [2, Theorem 1.3], p is compact and hence by [5, Theorem 1], p is principal. Obviously, $0 : p = 0$. Again by [4, Lemma 3.2(d)] and [12, Theorem 1 and Theorem 2], p is an invertible d-prime. □

Lemma 7. *Let L be a π-lattice which is also locally a domain. If p is a prime minimal over a principal element a, then p is either a complemented minimal prime or an invertible d-prime.*

Proof. Suppose p is a prime minimal over a principal element a. Note that in a π-lattice, there are only a finite number of minimal primes. As L is a reduced M-normal lattice, it follows that the minimal primes are complemented elements. Therefore if p is a minimal prime, then p is a complemented element. Suppose p is non minimal. Let $m \geqslant p$ be a maximal prime element of L. As L_m is a π-domain and p_m is minimal over a non zero principal element a_m of L_m, by Lemma 5 and Lemma 6, p is an invertible d-prime. □

Lemma 8. *Let $p_1, p_2, \ldots, p_n$ be distinct prime elements of L and let q_i be p_i-primary elements. If each q_i is weak meet principal, then $q_1 \wedge q_2 \wedge \ldots \wedge q_n = q_1 q_2 \ldots q_n$.*

Proof. Rearrange $p_1, p_2, \ldots, p_n$, if necessary, so that $p_i \nleqslant p_j$ for $i < j$. We prove the result by induction on n. Since $p_1 \nleqslant p_2$ and q_1 is weak meet principal, it follows that $q_1 \wedge q_2 = q_1 q_2$. Therefore the result is true for $n = 2$. Now assume that $q_1 \wedge q_2 \wedge \ldots \wedge q_{n-1} = q_1 q_2 \ldots q_{n-1}$. Since each q_i is weak meet principal, by [5, Proposition 1(a) and Theorem 6] $q_1 q_2 \ldots q_{n-1}$ is weak meet principal. Again since $q_1 q_2 \ldots q_{n-1}$ is weak meet principal, it follows that $(q_1 q_2 \ldots q_{n-1}) \wedge q_n = q_1 q_2 \ldots q_{n-1} x$ for some $x \in L$. As $q_1 q_2 \ldots q_{n-1} x \leqslant q_n$ and $q_i \nleqslant p_n$ for $1 \leqslant i \leqslant n-1$, it follows that $x \leqslant q_n$. Therefore $q_1 \wedge q_2 \wedge \ldots \wedge q_n = (q_1 \wedge q_2 \wedge \ldots \wedge q_{n-1}) \wedge q_n = (q_1 q_2 \ldots q_{n-1}) \wedge q_n \leqslant q_1 q_2 \ldots q_{n-1} q_n$ and hence $q_1 \wedge q_2 \wedge \ldots \wedge q_n = q_1 q_2 \ldots q_{n-1} q_n$. This completes the proof of the lemma. □

Lemma 9. *Let L be a π-lattice which is also locally a domain. Then every principal element is a finite product of primes which are either complemented minimal primes or invertible d-primes.*

Proof. By Lemma 2, every principal element has only finitely many minimal primes. Again by Lemma 7, every prime minimal over a principal element is either a complemented minimal prime or an invertible d-prime. Let a be a principal element of L. Let $p_1, p_2, \ldots, p_n$ be the minimal primes over a. Without loss of generality, assume that $p_1, p_2, \ldots, p_s$ are the invertible d-primes and $p_{s+1}, p_{s+2}, \ldots, p_n$ are the complemented minimal primes. Since $p_1, p_2, \ldots, p_s$ are d-primes, by [4, Lemma 3.2(c)], there exist positive integers n_i for $i = 1, 2, \ldots, s$ such that $a \leqslant p_i^{n_i}$ and $a \nleqslant p_i^{n_i+1}$. Observe that by [4, Lemma 3.2(d)], the powers of p_i $(1 \leqslant i \leqslant s)$ are p_i-primary elements. Let $b = p_1^{n_1} p_2^{n_2} \ldots p_s^{n_s} p_{s+1} p_{s+2} \ldots p_n$. We claim that $a = b$. Let m be a maximal prime element of L. If $p_j \leqslant m$ for some $j \in \{s+1, s+2, \ldots n\}$, then $a_m = b_m = 0_m$. Without loss of generality, assume that $p_1, p_2, \ldots, p_t \leqslant m$ for $(1 \leqslant t < s)$ and $p_j \nleqslant m$ for $(t+1 \leqslant j \leqslant s)$. Note that L_m is a π-domain and a_m is a non zero principal element of L_m. Therefore by [4, Lemma 2.3] and Lemma 5, a_m is a finite product of the rank one principal prime elements minimal over it. Again using Lemma 8, it can be easily shown that $a_m = (p_{1_m})^{n_1} (p_{2_m})^{n_2} \ldots (p_{t_m})^{n_t}$. Therefore $a_m = (p_{1_m})^{n_1} (p_{2_m})^{n_2} \ldots (p_{t_m})^{n_t} \ldots (p_{s_m})^{n_s} p_{s+1_m} p_{s+2_m} \ldots p_{n_m} = b_m$ since $(p_{j_m})^{n_j} = 1_m$ for $(t+1 \leqslant j \leqslant s)$ and $p_{k_m} = 1_m$ for $(s+1 \leqslant k \leqslant n)$. This shows that $a_m = b_m$ for all maximal prime elements m containing a. Further, if $a \nleqslant m$, then $a_m = b_m = 1_m$. Consequently, $a = b$ and the proof is complete. □

Lemma 10. *Let L be a π-lattice which is also locally a domain. Then L satisfies the condition $(*)$.*

P r o o f. Note that by [1, Lemma 2.2], complemented elements are idempotent principal elements and by [4, Lemma 3.2(d)], powers of invertible prime elements are primary. Now the result follows from Lemma 8 and Lemma 9. □

Lemma 11. *Let $a \in L$. Suppose every prime minimal over a is compact. Then a has only finitely many minimal primes.*

P r o o f. Note that by [9, Lemma 1], a finite product of compact elements is compact. Therefore by hypothesis and [18, Theorem 3.4], a has only finitely many minimal primes. □

Lemma 12. *Suppose L is an almost π-domain satisfying the condition $(*)$. Let p be a prime minimal over a principal element $a \in L$. Then p is either a complemented minimal prime or an invertible d-prime.*

P r o o f. Suppose p is minimal. Then p is locally principal and hence by Lemma 3, p is principal. As L is a reduced M-normal lattice, by [13, Theorem 1], p is a principal σ-element and hence complemented. Suppose p is non minimal. Let $m \geqslant p$ be a maximal prime element of L. As L_m is a π-domain and p_m is minimal over a non zero principal element a_m in L_m, by Lemma 5, $\operatorname{rank} p_m = 1$ and hence $\operatorname{rank} p = 1$. Again by Lemma 4, p is an invertible d-prime. □

If $\{p_\alpha\}_{\alpha\in I}$ is the collection of prime elements minimal over a, then by the isolated primary component of a belonging to p_β (or the isolated p_β-primary component of a) we mean the meet q_β of all p_β-primary elements which contain a. The kernel a^* of a is the meet of all q_β's. If p is a prime minimal over a, then a_p is p-primary and contained in any p-primary element which contains a. Hence a_p is the isolated p-primary component of a and $a^* = \bigwedge\limits_{\alpha\in I} a_{p_\alpha}$. The kernel of an element was studied in [10].

Lemma 13. *Suppose L is an almost π-domain satisfying the condition $(*)$. Then the kernel of a principal element is a finite product of primes which are either complemented minimal primes or invertible d-primes.*

P r o o f. Let a be a principal element of L. Then

$$a^* = \bigwedge\{a_p \mid p \text{ is a prime minimal over } a\}.$$

By Lemma 1, a has only finitely many minimal primes. Let $p_1, p_2, \ldots, p_n$ be the minimal primes of a. By Lemma 12, each p_i is either a complemented minimal prime or an invertible d-prime. Without loss of generality, assume that $p_1, p_2, \ldots, p_s$ are the invertible d-primes and $p_{s+1}, p_{s+2}, \ldots, p_n$ are the complemented minimal primes. Note that each a_{p_i} $(1 \leqslant i \leqslant n)$ is p_i-primary. Since the minimal primes are complemented, it follows that the minimal primes are unbranched, so $a_{p_i} = p_i$ for $i = s+1, s+2, \ldots, n$. As each p_i $(1 \leqslant i \leqslant s)$ is invertible, by [4, Lemma 3.2(d)], each p_i-primary element is a power of p_i. Therefore $a_{p_i} = p_i^{n_i}$ (for $i = 1, 2, \ldots, s$) for some positive integer n_i. Again by Lemma 8, $a^* = p_1^{n_1} \wedge \ldots \wedge p_s^{n_s} \wedge p_{s+1} \wedge \ldots \wedge p_n = p_1^{n_1} \ldots p_s^{n_s} p_{s+1} \ldots p_n$. This completes the proof of the lemma. □

Lemma 14. *Suppose L is an almost π-domain satisfying the condition $(*)$. Then every principal element is equal to its kernel.*

P r o o f. Let a be a principal element of L. By the proof of Lemma 13, there exist prime elements $p_1, p_2, \ldots, p_m$ minimal over a such that $a_{p_i} = p_i^{n_i}$ for some positive integer n_i and $a^* = p_1^{n_1} p_2^{n_2} \ldots p_m^{n_m}$, where each p_i is either a complemented minimal prime or an invertible d-prime. Again by imitating the proof of Lemma 9, it can be easily shown that $a = a^*$ and the proof is complete. □

Lemma 15. *Suppose L is an almost π-domain. If p is a compact prime of rank less than or equal to one, then p is either a complemented minimal prime or an invertible d-prime.*

P r o o f. Suppose p is a compact minimal prime. As L is a reduced M-normal lattice, it follows that p is complemented. If $\operatorname{rank} p = 1$, then by the proof of

Lemma 4, p is locally principal and hence by [5, Theorem 1], p is principal. The remaining proof is similar to that of Lemma 4. □

Lemma 16. *Suppose L is an almost π-domain. If p is a compact prime minimal over a principal element $a \in L$, then p is either a complemented minimal prime or an invertible d-prime.*

Proof. If p is minimal, then by Lemma 15, p is complemented. Suppose p is non minimal. Then by Lemma 5, $\operatorname{rank} p = 1$ and hence by Lemma 15, p is an invertible d-prime. □

Lemma 17. *Suppose L is an almost π-domain. Let $a \in L$ be a principal element such that every prime minimal over a is compact. Then a^* is a finite product of primes which are either complemented minimal primes or invertible d-primes.*

Proof. By Lemma 11, a has only finitely many minimal primes. Again by Lemma 16, each prime minimal over a is either a complemented minimal prime or an invertible d-prime. Now by imitating the proof of Lemma 13, we can get the result. □

Lemma 18. *Suppose L is an almost π-domain. Let $a \in L$ be a principal element such that every prime minimal over a is compact. Then a is equal to its kernel.*

Proof. Using Lemma 17 and by imitating the proof of Lemma 14, we can get the result. □

Theorem 1. *Suppose L is an almost π-domain satisfying the condition $(*)$. Then every principal element is a finite product of primes which are either complemented minimal primes or invertible d-primes.*

Proof. The proof of the theorem follows from Lemma 13 and Lemma 14. □

Theorem 2. *Suppose L is an almost π-domain. Let every prime minimal over a principal element is compact. Then every principal element is a finite product of primes which are either complemented minimal primes or invertible d-primes.*

Proof. The proof of the theorem follows from Lemma 17 and Lemma 18. □

Theorem 3. *The following statements on L are equivalent:*

(i) *L is a π-lattice which is also locally a domain.*

(ii) *L is an almost π-domain satisfying the condition $(*)$.*

(iii) *L is a reduced lattice in which every principal element is a finite product of primes which are either complemented minimal primes or invertible d-primes.*

(iv) *L is an almost π-domain in which every prime of rank less than or equal to one is compact.*

(v) *L is a reduced lattice in which every prime minimal over a principal element is either a complemented minimal prime or an invertible d-prime.*

Proof. (i)⇒(ii) follows from Lemma 10 and (ii)⇒(iii) follows from Theorem 1.

(iii)⇒(iv). Suppose (iii) holds. By (iii), L is a reduced π-lattice and an M-normal lattice. Therefore L is an almost π-domain. The remaining proof is obvious.

(iv)⇒(v). Suppose (iv) holds. Let p be a prime minimal over a principal element $a \in L$. By Lemma 5, $\operatorname{rank} p \leqslant 1$ and hence by Lemma 15, p is either a complemented minimal prime or an invertible d-prime. Thus (v) holds.

(v)⇒(i). Suppose (v) holds. By (v), every prime minimal over a principal element is compact. Therefore by Theorem 2, it is enough if we show that L is an almost π-domain. Note that by [13, Theorem 1(v)], L is a reduced M-normal lattice and so every prime element contains a unique minimal prime element. Therefore by (v), every non minimal prime contains an invertible d-prime. Now we show that L is an almost π-domain. Let m be a maximal prime element of L. If m is minimal, then L_m is a two element chain. Suppose m is non minimal. Let p_m be a non zero prime element of L_m. Since p is a non minimal prime, there exists an invertible d-prime p_1 such that $p_1 \leqslant p$. Clearly, ${p_1}_m$ is a non zero principal prime element contained in p_m and hence by [4, Theorem 2.3 and Corollary 2.3], L_m is a π-domain. Consequently, L is an almost π-domain and the proof is complete. □

Theorem 4. *Suppose L is a domain. Then the following statements on L are equivalent:*

(i) *L is a π-domain.*

(ii) *L is an almost π-domain satisfying the condition $(*)$.*

(iii) *Every non zero principal element is a finite product of invertible d-primes.*

(iv) *L is an almost π-domain in which every rank one prime element is compact.*

(v) *Every prime minimal over a non zero principal element is an invertible d-prime.*

Proof. The proof of the theorem follows from Theorem 3. □

L is said to satisfy the union condition on primes if for any set $p_1, \ldots, p_n$ of primes in L and any $a \in L$ with $a \nleqslant p_1, \ldots, p_n$ there exists a principal element $e \leqslant a$ with $e \nleqslant p_1, \ldots, p_n$.

Theorem 5. *Suppose L satisfies the union condition on primes. Then the following statements on L are equivalent:*

(i) *L is a π-lattice.*

(ii) *L is an almost π-lattice in which every principal element is a finite meet of primary elements.*

(iii) *L is an almost π-lattice satisfying the condition $(*)$.*

(iv) *L is an almost π-lattice in which every prime of rank less than or equal to one is compact.*

(v) *Every minimal prime is principal and every non minimal prime contains a non minimal principal prime.*

Proof. (i)$\Rightarrow$(ii). Suppose (i) holds. Clearly, L is an almost π-lattice. As L satisfies the union condition on primes, by [4, Corollary 2.1], for every maximal prime element m of L, L_m is either a domain or a special principal element lattice. Therefore every prime contains a unique minimal prime and non maximal minimal primes are unbranched and idempotent. Using these facts and by imitating the proofs of Lemma 5 and Lemma 6, it can be easily shown that every prime minimal over a principal element is either a minimal prime or an invertible d-prime. Let a be a principal element of L. By Lemma 2, a has only finitely many minimal primes. Let $p_1, p_2, \ldots, p_m$ be the primes minimal over a. Without loss of generality, assume that $p_1, p_2, \ldots, p_s$ are the invertible d-primes, $p_{s+1}, p_{s+2}, \ldots, p_{s+t}$ are the non maximal minimal primes and $p_{s+t+1}, p_{s+t+2}, \ldots, p_m$ are the minimal primes which are also maximal. Since $p_1, p_2, \ldots, p_s$ are the invertible d-primes minimal over a, there exist positive integers n_i for $i = 1, 2, \ldots, s$, such that $a \leqslant p_i^{n_i}$ and $a \nleqslant p_i^{n_i+1}$. Since each L_{p_i} $(s+t+1 \leqslant i \leqslant m)$ is a special principal element lattice, there exist positive integers n_j (for $s+t+1 \leqslant j \leqslant m$) such that $a_{p_j} = (p_j^{n_j})_{p_j}$. Observe that the powers of p_i $(1 \leqslant i \leqslant m)$ are p_i-primary elements. Let $b = p_1^{n_1} \wedge p_2^{n_2} \wedge \ldots \wedge p_s^{n_s} \wedge p_{s+1} \wedge \ldots \wedge p_{s+t} \wedge p_{s+t+1}^{n_{s+t+1}} \wedge \ldots \wedge p_m^{n_m}$. Now by imitating the proof of Lemma 9, it can be easily shown that $a = b$. Therefore (ii) holds.

(ii)$\Rightarrow$(iii) is obvious.

(iii)$\Rightarrow$(iv). Suppose (iii) holds. By [4, Corollary 2.1, Theorem 2.2 and Corollary 2.3], every prime of rank less than or equal to one is locally principal and hence by Lemma 3, principal.

(iv)$\Rightarrow$(v). Suppose (iv) holds. Observe that the rank of every prime minimal over a principal element is less than or equal to one and every prime of rank less than or equal to one is locally principal. Therefore by (iv), every prime minimal over a principal element is a principal prime of rank less than or equal to one. Therefore (v) holds.

(v)$\Rightarrow$(i). Suppose (v) holds. We show that L is an almost π-lattice. Let m be a maximal prime element of L. If m is minimal, then L_m is a special principal element lattice. Suppose m is non minimal. By (v), there exists a non minimal principal prime $p \leqslant m$. Let $q < p$ be a principal minimal prime. As p is principal, it follows that $pq = q$. Therefore by [2, Theorem 1.4], $q_m = 0_m$ in L_m and hence L_m is a domain. Again since by (v), every non zero prime element of L_m contains a non zero principal prime element, by [4, Theorem 2.2 and Corollary 2.3], L_m is a π-domain. This shows that L is an almost π-lattice. Note that by (v) and by Lemma 5, every prime minimal over a principal element is a principal prime of rank less than or equal to one. Therefore every principal element has only finitely many minimal primes. Now by using [4, Lemma 1.4] and Lemma 8 and by imitating the proof of (i)⇒(ii) (or Lemma 9), it can be easily shown that every principal element is a finite product of principal primes of rank less than or equal to one. Therefore L is a π-lattice and the proof is complete. □

Let R be a commutative ring with identity and let $L(R)$ be the lattice of all ideals of R. An ideal M of R is called a *quasi-principal ideal* [15, p. 147] (or a *principal element* of $L(R)$ [17]) if it satisfies the following identities (i) $(A \cap (B : M))M = AM \cap B$ and (ii) $(A + BM) : M = (A : M) + B$, for all $A, B \in L(R)$. It should be mentioned that every quasi-principal ideal is finitely generated and also a finite product of quasi-principal ideals of R is again a quasi-principal ideal [15, Exercise 10, p. 147]. In fact, an ideal I of R is quasi-principal if and only if it is finitely generated and locally principal [17, Theorem 2].

R is said to be a *π-ring* [7, p. 572] if every principal ideal is a finite product of prime ideals. For various characterizations of π-rings which are also domains, the reader is referred to [14] and [16]. We call a ring R an *almost π-ring* if R_M is a π-ring, for every maximal ideal M of R. The following Theorem 6 gives some new characterizations for π-rings in terms of almost π-rings.

Theorem 6. *The following statements on R are equivalent:*

(i) *R is a π-ring.*

(ii) *R is an almost π-ring in which every quasi-principal ideal is a finite intersection of primary ideals.*

(iii) *R is an almost π-ring in which every principal ideal is a finite intersection of primary ideals.*

(iv) *R is an almost π-ring in which every prime ideal of rank less than or equal to one is finitely generated.*

(v) *Every minimal prime ideal is quasi-principal and every non minimal prime ideal contains a non minimal quasi-principal prime ideal.*

Proof. The proof of the theorem follows from Theorem 5 and the fact that the lattice of all ideals of R is a principally generated C-lattice and satisfies the union condition on prime ideals. □

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