Ján Jakubík
On varieties of pseudo MV-algebras

Czechoslovak Mathematical Journal, Vol. 53 (2003), No. 4, 1031–1040

Persistent URL: `http://dml.cz/dmlcz/127858`

Czechoslovak Mathematical Journal, 53 (128) (2003), 1031–1040

ON VARIETIES OF PSEUDO MV-ALGEBRAS

JÁN JAKUBÍK, Košice

(Received March 12, 2001)

Abstract. In this paper we investigate the relation between the lattice of varieties of pseudo MV-algebras and the lattice of varieties of lattice ordered groups.

Keywords: pseudo MV-algebras, lattice ordered group, unital lattice ordered group, variety

MSC 2000: 06D35

1. Introduction and preliminaries

The notion of pseudo MV-algebra has been introduced by Georgescu and Iorgulescu [4], [5] and by Rachůnek [8] (in [8], the term 'generalized MV-algebra' has been used).

We denote by $\mathscr{V}_1$ and $\mathscr{V}_2$ the collection of all varieties of pseudo MV-algebras and the collection of all varieties of lattice ordered groups, respectively. Under the set-theoretical inclusion, $\mathscr{V}_1$ and $\mathscr{V}_2$ are lattices.

In this paper we describe an injective mapping φ of $\mathscr{V}_2$ into $\mathscr{V}_1$ such that for any $Z_1, Z_2 \in \mathscr{V}_2$ we have

$$Z_1 \subseteq Z_2 \Leftrightarrow \varphi(Z_1) \subseteq \varphi(Z_2).$$

If G is a lattice ordered group with a strong unit u, then the pair (G, u) is called a unital lattice ordered group.

We will apply a result of Dvurečenskij [2] on the relations between pseudo MV-algebras and unital lattice ordered groups.

We define the notion of the regular class of unital lattice ordered groups and we denote by $\mathscr{U}$ the collection of all such classes. We consider the partial order on $\mathscr{U}$ defined by the class-theoretical inclusion.

Our method is as follows. First, we prove some auxiliary results concerning neutral ideals of and congruence relations on pseudo MV-algebras.

Then we construct an isomorphism of $\mathscr{U}$ onto $\mathscr{V}_1$. Finally, we describe an injective order-preserving mapping of $\mathscr{V}_2$ into $\mathscr{U}$.

For the results and for the bibliography concerning the varieties of MV-algebras cf. Chapter 8 of the monograph Cignoli, D'Ottaviano and Mundici [1].

2. Preliminaries

For the sake of completeness, we recall the definition of a pseudo MV-algebra.

Let $\mathscr{A} = (A; \oplus, \neg, \sim, 0, 1)$ be an algebra of type $(2,1,1,0,0)$. For $x, y \in A$ we put

$$y \odot x = \sim (\neg x \oplus \neg y).$$

Assume that $\mathscr{A}$ satisfies the following identities:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(A2) $x \oplus 0 = 0 \oplus x = x$;

(A3) $x \oplus 1 = 1 \oplus x = 1$;

(A4) $\sim 1 = 0; \neg 1 = 0$;

(A5) $\sim (\neg x \oplus \neg y) = \neg(\sim x \oplus \sim y)$;

(A6) $x \oplus \sim x \odot y = y \oplus \sim y \odot x = x \odot \neg y \oplus y = y \odot \neg x \oplus x$;

(A7) $x \odot (\neg x \oplus y) = (x \oplus \sim y) \oplus y$;

(A8) $\sim (-x) = x$.

Then $\mathscr{A}$ is called a pseudo MV-algebra.

Let (G, u) be a unital lattice ordered group. Further, let A be the interval $[0, u]$ of G. For $x, y \in A$ we put

$$x \oplus y = (x + y) \wedge u, \quad \neg x = u - x, \quad \sim x = -x + u, \quad 1 = u.$$

Then the algebraic structure

$$\Gamma(G, u) = (A; \oplus, \neg, \sim, 0, u)$$

is a pseudo MV-algebra.

Dvurečenskij [2] proved that for each pseudo MV-algebra $\mathscr{A}$ there exists a unital lattice ordered group (G, u) such that $\mathscr{A} = \Gamma(G, u)$.

Let $\operatorname{Con} \mathscr{A}$ and $\operatorname{Con} G$ be the lattice of all congruence relations on $\mathscr{A}$ and on G, respectively. For $\varrho \in \operatorname{Con} G$ we denote by $\psi_0(\varrho)$ the equivalence on A defined by

$$a_1 \psi_0(\varrho) a_2 \quad \text{iff} \quad a_1 \varrho a_2, \tag{1}$$

where $a_1, a_2 \in A$.

The relations between $\operatorname{Con}\mathscr{A}$ and $\operatorname{Con}G$ for the particular case when $\mathscr{A}$ is an MV-algebra have been dealt with in [6, Section 1]; cf. also Cignoli, D'Ottaviano and Mundici [1, Chapter 7].

Let us now consider the case when $\mathscr{A}$ is a pseudo MV-algebra. Then G need not be abelian. In this case we have to modify the method from [6] in the following two points:

1) Let $\varrho_1 \in \operatorname{Con}\mathscr{A}$ and $0(\varrho_1) = \{a' \in A\colon 0\varrho_1 a'\}$. Further, let X_0 be the convex ℓ-subgroup of G generated by the set $0(\varrho_1)$. We apply Theorem 6.10 from [3] to obtain the fact that X_0 is an ℓ-ideal of G.

2) The expressions

$$t = \neg(a_2 \oplus \neg a_3), \quad t\varrho_1(a_2 \oplus \neg a_2)$$

in the proof of 1.5 in [6] are to be replaced by

$$t = \neg(a_2 \oplus \sim a_3), \quad t\varrho_1 \neg(a_2 \oplus \sim a_2).$$

The remaining arguments and the results of Section 1 in [6] remain valid for the pseudo MV-algebra $\mathscr{A}$. Thus we have

2.1. Lemma. *The mapping ψ_0 is an isomorphism of the lattice* $\operatorname{Con}G$ *onto the lattice* $\operatorname{Con}\mathscr{A}$.

Let ϱ be as above; put $\varrho_1 = \psi_0(\varrho)$. For $g \in G$ we denote by $\overline{g}$ the congruence class in ϱ containing the element g. Further, we construct in the usual way the factor structure $G/\varrho = \overline{G}$ which has the underlying set $\{\overline{g}\colon\ g \in G\}$. Then $(\overline{G}, \overline{u})$ is a unital lattice ordered group.

Similarly we can construct the factor structure $\overline{\mathscr{A}}^1 = \mathscr{A}/\varrho_1$; its underlying set is $\{\overline{a}^1\colon\ a \in A\}$, where $\overline{a}^1$ is the congruence class in ϱ_1 containing the element a of A. Hence $\overline{\mathscr{A}}^1$ is a factor pseudo MV-algebra of $\mathscr{A}$.

In view of [6, 1.5 and 1.8], for each $a \in A$ we have

$$\overline{a}^1 = A \cap \overline{a}. \tag{2}$$

For each $a \in A$ we put

$$\tau(\overline{a}^1) = \overline{a}.$$

Then in view of (2), τ is a correctly defined mapping of the set $\overline{A}^1$ onto the interval $[\overline{0}, \overline{u}]$ of $\overline{G}$. Clearly $\tau(\overline{0}^1) = \overline{0}$, $\tau(\overline{u}^1) = \overline{u}$.

Consider the pseudo MV-algebras $\overline{\mathscr{A}}^1$ and $\Gamma(\overline{G}, \overline{u})$. Let $x, y \in A$. Then we have

$$\overline{x} \oplus \overline{y} = (\overline{x} + \overline{y}) \wedge \overline{u} = \overline{(x+y) \wedge u},$$
$$\overline{x}^1 \oplus \overline{y}^1 = \overline{x \oplus y}^1 = \overline{(x+y) \wedge u}^1,$$

whence $\tau(\overline{x}^1 \oplus \overline{y}^1) = \overline{x} \oplus \overline{y}$.

Similarly we can verify the relations

$$\tau(\neg \overline{x}^1) = \neg \overline{x}, \quad \tau(\sim \overline{x}^1) = \sim \overline{x}.$$

Summarizing, we obtain

2.2. Lemma. *The mapping τ is an isomorphism of the pseudo MV-algebra $\overline{\mathscr{A}}^1$ onto the pseudo MV-algebra $\Gamma(\overline{G}, \overline{u})$.*

For the related result concerning MV-algebras cf. Theorem 7.4.2 in [1].

2.3. Lemma. *Let G_0 be a lattice ordered group and let $\emptyset \neq X \subseteq G_0^+$. Assume that the following conditions are valid:*
(i) *X is closed with respect to the operation $+$;*
(ii) *X is a sublattice of the lattice G_0^+;*
(iii) *$x + X = X + x$ for each $x \in X$;*
(iv) *if $x_1, x_2 \in X$ and $x_1 \leqslant x_2$, then $-x_1 + x_2 \in X$ and $x_2 - x_1 \in X$.*
Put $Y = \{x_1 - x_2 \colon x_1, x_2 \in X\}$. Then Y is an ℓ-subgroup of G_0 and $Y^+ = X$.

Proof. a) Let $y, y' \in Y$. Hence there are $x_1, x_2, x_1', x_2' \in X$ such that $y = x_1 - x_2$, $y' = x_1' - x_2'$. Then

$$y + y' = x_1 - x_2 + x_1' - x_2'.$$

In view of (iii) there is $x_1'' \in X$ such that $-x_2 + x_1' = x_1'' - x_2$, whence according to (i) we have

$$y + y' = (x_1 + x_1'') - (x_2' + x_2) \in Y.$$

Further, $-y = x_2 - x_1 \in Y$. Hence Y is a subgroup of the group G_0.

b) Let $y \in Y$, $y \geqslant 0$. Under the notation as above we have $x_1 \geqslant x_2$. Then in view of (iv), $y \in X$.

c) Let y and y' be as in a). Denote $z = -x_2 - x_2'$. Hence $y \geqslant z$, $y' \geqslant z$. Then in view a) and b) we obtain $y - z \in X$, $y' - z \in X$. Thus according to (ii) we have

$$(y - z) \vee (y' - z) = v \in X.$$

By applying a) we get $v + z \in Y$, whence $y \vee y' \in Y$. Analogously we obtain the relation $y \wedge y' \in Y$. Hence Y is an ℓ-subgroup of G_0. Further, from $X \subseteq G_0^+$ and from b) we conclude that $Y^+ = X$. $\square$

Now let us suppose that G_0 is a lattice ordered group with a strong unit u and that $\mathscr{A}_1$ is a subalgebra of the pseudo MV-algebra $\Gamma(G_0, u)$. Let A_1 be the underlying set of $\mathscr{A}_1$. Hence $A_1 \subseteq G_0^+$.

We will apply some results of Section 2 of [2]. We denote by X the set of all elements $g \in G_0$ which can be expressed in the form

$$g = a_1 + a_2 + \ldots + a_n \quad (a_1, a_2, \ldots, a_n \in A_1,\ \ n \geqslant 1).$$

Then X satisfies the condition (i) from 2.3. Further, from Proposition 3.7 and Proposition 3.8 in [2] we conclude that the conditions (ii), (iii) and (iv) from 2.3 are satisfied as well. Let Y be as in 2.3; thus Y is an ℓ-subgroup of G_0.

We denote by $[0, u]_2$ the interval with the endpoints 0 and u in Y.

2.4. Lemma. $[0, u]_2 = A_1$.

Proof. Let $a \in A_1$. Then $0 \leqslant a \leqslant u$. Further, $a \in X \subseteq Y$, whence $a \in [0, u]_2$. Conversely, let $t \in [0, u]_2$. Then $0 \leqslant t \leqslant u$ and $t \in Y$. Thus in view of 2.3, $t \in X$. Hence there are $a_1, a_2, \ldots, a_n \in A_1$ with $t = a_1 + \ldots + a_n$. Because $t \leqslant u$, by considering the pseudo MV-algebra $\Gamma(G_0, u)$ we conclude that we have

$$(*) \qquad t = a_1 \oplus \ldots \oplus a_n$$

in $\Gamma(G_0, u)$. Since $\mathscr{A}_1$ is a subalgebra of $\Gamma(G_0, u)$, the equality $(*)$ holds in $\mathscr{A}_1$ as well. Therefore $t \in A_1$. □

In view of 2.3, 2.4 and of the fact that $\mathscr{A}_1$ is a subalgebra of $\Gamma(G_0, u)$ we obtain

2.5. Lemma. *Under the notation as above,* $\mathscr{A}_1 = \Gamma(Y, u)$.

3. Regular classes of unital lattice ordered groups

We denote by $\mathscr{G}_0$ the class of all unital lattice ordered groups. Let $(G_i, u_i)_{i \in I}$ be an indexed system of elements of $\mathscr{G}_0$. Consider the direct product

$$G^0 = \prod_{i \in I} G_i.$$

For $g \in G^0$ and $i \in I$ we denote by $g(G_i)$ the component of the element g in G_i. There exists $u^0 \in G^0$ such that $u^0(G_i) = u_i$ for each $i \in I$. Let G^1 be the convex

ℓ-subgroup of G^0 which is generated by the element u^0. Then u^0 is a strong unit of G^1, whence $(G^1, u^0) \in \mathscr{G}_0$. We denote

$$G^1 = \prod_{i \in I}^{1} G_i.$$

Assume that (G_1, u_1) belongs to $\mathscr{G}_0$ and let φ be a homomorphism of G_1 into a lattice ordered group G_2. Then $\varphi(u_1)$ is a strong unit of $\varphi(G_1)$, hence $(\varphi(G_1), \varphi(u_1)) \in \mathscr{G}_0$. We say that $((\varphi(G_1), \varphi(u_1))$ is a homomorphic image of (G_1, u_1) (under the homomorphism φ).

Let X_0 be the kernel of φ and let ϱ be the congruence relation on G_1 determined by the ℓ-ideal X_0. For $x \in G_1$ we denote by $\overline{x}$ the class of the partition of G_1 corresponding to ϱ such that $x \in \overline{x}$. Hence $\overline{u}_1$ is a strong unit of $G_1/\varrho = \overline{G}_1$ and $(\overline{G}_1, \overline{u}_1)$ is isomorphic to $(\varphi(G_1), \varphi(u_1))$.

3.1. Definition. A nonempty subclass Y of $\mathscr{G}_0$ is called regular if it satisfies the following conditions:

(i) Let $(H_1, u_1) \in Y$ and let H_2 be an ℓ-subgroup of H_1 such that $u_1 \in H_2$. Then $(H_2, u_1) \in Y$.

(ii) The class Y is closed with respect to homomorphisms.

(iii) Assume that $(G_i, u_i)_{i \in I}$ is an indexed system of elements of Y. Let u^0 and G^1 be as above. Then $(G^1, u^0) \in Y$.

Let $X \in \mathscr{V}_1$. Each element $\mathscr{A} \in X$ can be written as $\mathscr{A} = \Gamma(G, u)$ with $(G, u) \in \mathscr{G}_0$. We denote by Y the class of all such (G, u).

3.2. Lemma. *The class Y satisfies the condition* (i) *from* 3.1.

P r o o f. Assume that H_1, H_2 and u_1 are as in the condition (i) of 3.1. There exists $\mathscr{A}_1 \in X$ with $\mathscr{A}_1 = \Gamma(H_1, u_1)$.

The element u_1 is a strong unit of H_2, hence we can construct the pseudo MV-algebra $\mathscr{A}_2 = \Gamma(H_2, u_1)$.

Let us denote by $\oplus_i, \neg_i$ and $\sim_i$ the corresponding operations in $\mathscr{A}_i$ $(i = 1, 2)$. If $+$, $-$ and $\wedge$ are the operations in H_1, then from the fact that H_2 is an ℓ-subgroup of H_1 we conclude that for $h, h' \in H_2$ we have

$$h \oplus_1 h' = (h + h') \wedge u_1 = h \oplus_2 h',$$
$$\neg_1 h = u_1 - h = \neg_2 h, \quad \sim_1 h = -h + u_1 = \sim_2 h.$$

Hence $\mathscr{A}_2$ is an subalgebra of $\mathscr{A}_2$. Since $\mathscr{A}_1 \in X$, we get $\mathscr{A}_2 \in X$. Thus $(H_2, u_1) \in Y$. $\square$

3.3. Lemma. *The class Y satisfies the condition* (ii) *from* 3.1.

P r o o f. Let $(G,u)\in Y$ and let $(\varphi(G),\varphi(u))$ be a homomorphic image of (G,u). Then without loss of generality we can assume that $(\varphi(G),\varphi(u))=(\overline{G},\overline{u})$, where $\overline{G}=G/\varrho$ for some congruence relation ϱ on G. Thus in view of 2.2, $\Gamma(\overline{G},\overline{u})$ is isomorphic to a pseudo MV-algebra $\overline{\mathscr{A}}^1=\Gamma(G,u)\in X$. Then $\overline{\mathscr{A}}^1\in X$, whence $(\overline{G},\overline{u})\in Y$. □

3.4. Lemma. *The class Y satisfies the condition* (iii) *from* 3.1.

P r o o f. Suppose that the assumptions of the condition (iii) of 3.1 are satisfied. For each $i\in I$ there exists $\mathscr{A}_i\in X$ with $\mathscr{A}_i=\Gamma(G_i,u_i)$. Put

$$\mathscr{A}=\Gamma(G^1,u^0).$$

From the relation

$$G^1=\prod_{i\in I}^{1}G_i$$

we conclude that the interval $[0,u^0]$ of G^1 can be written as a direct product

$$[0,u^0]=\prod_{i\in I}[0,u_i].$$

Thus in view of the results of [6], the pseudo MV-algebra $\mathscr{A}$ is isomorphic to the direct product of the pseudo MV-algebras $\mathscr{A}_i$ $(i\in I)$. Therefore $\mathscr{A}$ belongs to the variety X. This yields that (G^1,u^0) is an element of Y. □

Under the notation as above we put $Y=\psi_1(X)$. Thus according to 3.2, 3.3 and 3.4 we have

3.5. Lemma. *ψ_1 is a mapping of the collection $\mathscr{V}_1$ into $\mathscr{U}$.*

Now let $Y_1\in\mathscr{U}$. We denote by X_1 the class of all pseudo MV-algebras $\mathscr{A}$ such that $\mathscr{A}=\Gamma(G,u)$ for some $(G,u)\in Y_1$.

3.6. Lemma. *The class X_1 is closed with respect to subalgebras.*

P r o o f. Let $\mathscr{A}\in X_1$. Thus there is $(G,u)\in Y_1$ with $\mathscr{A}=\Gamma(G,u)$. Let $\mathscr{A}_1$ be a subalgebra of $\mathscr{A}$. In view of 2.5 there exists an ℓ-subgroup G_1 of G such that u is a strong unit of G_1 and $\mathscr{A}_1=\Gamma(G_1,u)$. Then we have $(G_1,u)\in Y_1$, whence $\mathscr{A}_1\in X_1$. □

3.7. Lemma. *The class X_1 is closed with respect to homomorphic images.*

P r o o f. Let $\mathscr{A} \in X_1$. It suffices to verify that, whenever ϱ_1 is a congruence relation on $\mathscr{A}$, then $\mathscr{A}/\varrho_1$ belongs to X_1.

Let (G, u) be as in the proof of 3.6 and let ϱ_1 be a congruence relation on $\mathscr{A}$. Put $\mathscr{A}/\varrho_1 = \overline{\mathscr{A}}^1$. Let $(\overline{G}, \overline{u})$ be as in 2.2. Since Y_1 is closed with respect to homomorphisms, we get $(\overline{G}, \overline{u}) \in Y_1$ and hence $\Gamma(\overline{G}, \overline{u}) \in X_1$. Then according to 2.2 we obtain that $\mathscr{A}/\varrho_1$ belongs to X_1. □

3.8. Lemma. *The class X_1 is closed with respect to direct products.*

P r o o f. Let $(\mathscr{A})_{i\in I}$ be an indexed system of elements of X_1. For each $i \in I$ there exists $(G_i, u_i) \in Y_1$ with $\Gamma(G_i, u_i) = \mathscr{A}_i$. Put

$$(*) \qquad \mathscr{A} = \prod_{i\in I} \mathscr{A}_i.$$

Further, let (G^1, u^0) be as above. Since $Y_1 \in \mathscr{U}$ and $(G_i, u_i) \in Y_1$ we get $(G^1, u^0) \in Y_1$. The relation $(*)$ yields that $\mathscr{A} = \Gamma(G^1, u^0)$. Thus $\mathscr{A} \in X_1$. □

In view of 3.6, 3.7 and 3.8 we have

3.9. Lemma. *The class X_1 is a variety of pseudo MV-algebras.*

Let us put $X_1 = \chi_1(Y_1)$ for each $Y_1 \in \mathscr{U}$. From the definitions of ψ_1 and χ_1 we immediately obtain

3.10. Lemma.

(i) $\chi_1 = \psi_1^{-1}$.

(ii) *If $X_1, X_2 \in \mathscr{V}_1$ and $Y_1, Y_2 \in \mathscr{U}$, then*

$$X_1 \subseteq X_2 \Leftrightarrow \psi_1(X_1) \subseteq \psi_1(X_2),$$
$$Y_1 \subseteq Y_2 \Leftrightarrow \chi_1(Y_1) \subseteq \chi_1(Y_2).$$

Hence we get as a corollary

3.11. Theorem. *ψ_1 is an isomorphism of the partially ordered set $\mathscr{V}_1$ onto the partially ordered collection $\mathscr{U}$.*

4. The relation between $\mathscr{U}$ and $\mathscr{V}_2$

Assume that Z is a variety of lattice ordered groups. We denote by Y the class of all unital lattice ordered groups (G, u) such that G belongs to Z.

4.1. Lemma. *The class Y is regular.*

Proof. It is obvious that Y is nonempty. We have to verify that the conditions (i), (ii) and (iii) from 3.1 are satisfied.

The validity of (i) and of (ii) is obvious. Let $(G_i, u_i)_{i\in I}$, u^0 and G^1 be as in the condition (iii) of 3.1. Further, let G^0 be as above. Then $G_i \in Z$ for each $i \in I$, hence $G^0 \in Z$ and thus G^1 belongs to Z as well. Also, u^0 is a strong unit of G^1. Therefore $(G^1, u^0) \in Y$. Thus the condition (iii) from 3.1 is satisfied. □

If Z and Y are as above, then we write $Y = \psi_2(Z)$. Hence ψ_2 is a mapping of $\mathscr{V}_2$ into $\mathscr{U}$. It is clear that if Z_1, Z_2 are elements of $\mathscr{V}_2$, then

$$Z_1 \subseteq Z_2 \Rightarrow \psi_2(Z_1) \subseteq \psi_2(Z_2).$$

4.2. Lemma. *Let $Z_1, Z_2 \in \mathscr{V}_2$. Assume that Z_1 is not a subclass of Z_2. Then $\psi_2(Z_1)$ is not a subclass of $\psi_2(Z_2)$.*

Proof. By way of contradiction, assume that

$$\psi_2(Z_1) \subseteq \psi_2(Z_2). \tag{1}$$

Since the varieties can be defined by identities and since the relation $Z_1 \subseteq Z_2$ fails to be valid we conclude that there exists an identity

$$p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n) \tag{2}$$

where p and q are terms constructed by the operations $+$, $-$, $\wedge$, $\vee$ such that

(i) the identity (2) is valid for Z_2,

(ii) the identity (2) fails to be valid for Z_1.

In view of (ii), there exists $G_1 \in Z_1$ such that G_1 does not satisfy the identity (2). Hence there are elements $g_1, g_2, \ldots, g_n \in G_1$ such that

$$p(g_1, \ldots, g_n) \neq q(g_1, \ldots, g_n). \tag{3}$$

Put

$$u = |g_1| \vee |g_2| \vee \ldots \vee |g_n|$$

and let G_1' be the convex ℓ-subgroup of G_1 which is generated by the element u. Then u is a strong unit of G_1', whence

$$(G_1', u) \in \psi_2(Z_1).$$

Thus according to (1) we have $(G_1', u) \in \psi_2(Z_2)$. This yields that $G_1' \in Z_2$ and then, in view of (i), G_1' satisfies the identity (2). Since $g_1, g_2, \ldots, g_n \in G_1'$, according to (3) we have arrived at a contradiction. □

Summarizing, from 4.1 and 4.2 we conclude

4.3. Proposition. *ψ_2 is an injective mapping of $\mathscr{V}_2$ into $\mathscr{U}$ such that for $Z_1, Z_2 \in \mathscr{V}_2$ we have*

$$Z_1 \subseteq Z_2 \Leftrightarrow \psi_2(Z_1) \subseteq \psi_2(Z_2).$$

Hence according to 3.10 we obtain

4.4. Theorem. *There exists an injective mapping φ of $\mathscr{V}_2$ into $\mathscr{V}_1$ such that for $Z_1, Z_2 \in \mathscr{V}_2$ we have*

$$Z_1 \subseteq Z_2 \Leftrightarrow \varphi(Z_1) \subseteq \varphi(Z_2).$$

References

[1] *R. Cignoli, M. I. D'Ottaviano and D. Mundici*: Algebraic Foundations of many-valued Reasoning. Trends in Logic, Studia Logica Library, Vol. 7, Kluwer Academic Publishers, Dordrecht, 2000.

[2] *A. Dvurečenskij*: Pseudo MV-algebras are intervals in ℓ-groups. J. Austral. Math. Soc. (Ser. A) *72* (2002), 427–445.

[3] *A. Dvurečenskij*: States on pseudo MV-algebras. Studia Logica. To appear.

[4] *G. Georgescu and A. Iorgulescu*: Pseudo MV-algebras: a noncommutative extension of MV-algebras. In: The Proceedings of the Fourth International Symposium on Economic Informatics, Buchurest, Romania. 1999, pp. 961–968.

[5] *G. Georgescu and A. Iorgulescu*: Pseudo MV-algebras. Multiple-Valued Logic (a special issue dedicated to Gr. C. Moisil) *6* (2001), 95–135.

[6] *J. Jakubík*: Subdirect product decompositions of MV-algebras. Czechoslovak Math. J. *49* (1999), 163–173.

[7] *J. Jakubík*: Direct product decompositions of pseudo MV-algebras. Arch. Math. *37* (2001), 131–142.

[8] *J. Rachůnek*: A non-commutative generalization of MV-algebras. Czechoslovak Math. J. *52* (2002), 255–273.

[9] *J. Rachůnek*: Prime spectra of non-commutative generalizations of MV-algebras. (Submitted).

Author's address: Matematický ústav SAV, Grešákova 6, 040 01 Košice, Slovakia, e-mail: `musavke@saske.sk`.