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Ethiraju Thandapani; K. Mahalingam

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OSCILLATION AND NONOSCILLATION OF SECOND ORDER NEUTRAL DELAY DIFFERENCE EQUATIONS

E. Thandapani and K. Mahalingam, Tamilnadu

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Abstract. Some new oscillation and nonoscillation criteria for the second order neutral delay difference equation

$$\Delta(c_n\Delta(y_n + p_n y_{n-k})) + q_n y^{\beta}_{n+1-m} = 0, \quad n \geqslant n_0$$

where k, m are positive integers and β is a ratio of odd positive integers are established, under the condition $\sum\limits_{n=n_0}^{\infty} 1/c_n < \infty$.

Keywords: neutral delay, difference equation, oscillation

MSC 2000: 39A10, 39A12

1. Introduction

Consider a neutral delay difference equation of the form

$$\Delta(c_n\Delta(y_n + p_n y_{n-k})) + q_n y^{\beta}_{n+1-m} = 0, \tag{E}$$

where $n \in \mathbb{N}(n_0) = \{n_0, n_0+1, n_0+2, \ldots\}$, n_0 is a nonnegative integer, Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, k and m are nonnegative integers and β is a ratio of odd positive integers and $\{c_n\}$, $\{p_n\}$ and $\{q_n\}$ are real sequences. Throughout this paper we assume without further mention that

(C_1) $\{p_n\}$ is non-decreasing such that $0 \leqslant p_n \leqslant p < 1$ and $q_n \geqslant 0$ and $\{q_n\}$ is not identically zero for infinitely many values of n,

(C_2) $c_n > 0$ and $\sum\limits_{n=n_0}^{\infty} 1/c_n < \infty$.

Let $\theta = \max\{k, m\}$. By a solution of equation (E), we mean a real sequence $\{y_n\}$ defined for $n \geqslant n_0 - \theta$ and satisfying equation (E) for $n \in \mathbb{N}(n_0)$. Such a solution of (E) is said to be oscillatory if for every $N \in \mathbb{N}(n_0)$ there exist integers $n_1, n_2 \geqslant N$ such that $y_{n_1} y_{n_2} \leqslant 0$.

Equations of this type arise in a number of important applications such as problems in population dynamics when maturation and gestation are included, in cobweb models in economics where demand depends on current price but supply depends on the price at an earlier time, and in electric networks containing lossless transmission lines. Hence it is important and useful to study the oscillatory properties of solutions of equation (E).

In most of the papers [1, 2 contains more than one hundred references], one considers the case that $c_n > 0$ and $\sum\limits_{n=n_0}^{\infty} 1/c_n = \infty$ in (E). In [4] the authors consider the cases $-1 < p \leqslant p_n \leqslant 0$, $c_n > 0$ and

$$\sum_{n=n_0}^{\infty} \frac{1}{c_n} = \infty \tag{1}$$

and discuss the asymptotic behavior of nonoscillatory solutions of (E).

Motivated by this observation in this paper our aim is to establish conditions for the oscillation of all solutions and existence of nonoscillatory solutions of (E) when condition (C_2) is satisfied. Examples illustrating the results are included in the text of the paper.

2. Some prelimiry lemmas

In order to prove our main results we need the following lemmas. Throughout we use the following notations without further mention:

$$\begin{aligned} z_n &= y_n + p_n y_{n-k}, \\ \varrho_n &= \sum_{s=n}^{\infty} \frac{1}{c_s}. \end{aligned} \tag{2}$$

Note that from the assumptions it is enough to state and prove the lemmas for the case of $\{y_n\}$ is eventually positive since the case of $\{y_n\}$ is eventually negative is similar.

Lemma 2.1. *Let* $\{y_n\}$ *be an eventually positive solution of* (E). *Then one of the following two cases holds for all sufficiently large* n.

(I) $z_n > 0$, $c_n\Delta z_n > 0$,

(II) $z_n > 0$, $c_n\Delta z_n < 0$.

P r o o f. Assume that $y_{n-k-m} > 0$ for $n \geqslant N_0 \in \mathbb{N}(n_0)$. Then by the condition (C_1), we have $z_n > 0$ and $\Delta(c_n\Delta z_n) \leqslant 0$ for $n \geqslant N_0$. Hence $\{c_n\Delta z_n\}$ is eventually of one sign. This completes the proof. □

Lemma 2.2. *Let* $\{y_n\}$ *be a positive solution of* (E) *and suppose Case* (I) *of Lemma* 2.1 *holds. Then there exists an integer* $N \in \mathbb{N}(n_0)$ *such that*

$$(1-p)z_n \leqslant y_n \leqslant z_n, \tag{3}$$

for $n \geqslant N$.

P r o o f. Clearly (2) and (C_1) imply $z_n \geqslant y_n$. Moreover from (2) we have

$$y_n = z_n - p_n y_{n-k} \geqslant z_n - p_n z_{n-k} \geqslant z_n(1-p),$$

since $\{z_n\}$ is nondecreasing. This completes the proof. □

Lemma 2.3. *Let* $\{y_n\}$ *be a positive solution of* (E) *and suppose Case* (II) *of Lemma* 2.1 *holds. Then there exists an integer* $N \in \mathbb{N}(n_0)$ *such that* $y_{n-k} \geqslant \frac{z_n}{1+p} \geqslant (1-p)z_n$ *for* $n \geqslant N$

P r o o f. Since $\{z_n\}$ is nonincreasing and from (2) we may assume without loss of generality that $\{y_n\}$ is also nonincreasing (see [3]). Hence

$$z_n \leqslant y_{n-k} + p y_{n-k} = (1+p)y_{n-k}. \tag{4}$$

Since $1 \geqslant 1-p^2$, we have $\frac{1}{1+p} \geqslant 1-p$. Therefore $\frac{z_n}{1+p} \geqslant (1-p)z_n$. This completes the proof. □

Lemma 2.4. *Let* $\{y_n\}$ *be an eventually positive solution of* (E). *Then* $\{z_n\}$ *is bounded above and satisfies*

$$z_n \geqslant -\varrho_n c_n \Delta z_n \tag{5}$$

for all sufficiently large n.

P r o o f. Proceeding as in the proof of Lemma 2.1, we obtain $\Delta(c_n\Delta z_n) \leqslant 0$ for $n \geqslant N \in \mathbb{N}(n_0)$. Then $c_n\Delta z_n \leqslant c_N\Delta z_N$ for $n \geqslant N$. Dividing the last inequality by c_n and summing we obtain $z_n - z_N \leqslant c_N\Delta z_N \sum\limits_{s=N}^{n-1} 1/c_s < \infty$. Hence $\{z_n\}$ is bounded above. Letting $n \to \infty$, we obtain $z_N \geqslant -c_N\Delta z_N\varrho_N$ for sufficiently large N. This completes the proof. □

3. Oscillation results

In this section we obtain sufficient conditions for the oscillation of all solutions of (E). We begin with the following theorem for the case $\beta = 1$.

Theorem 3.1. *Assume $\beta = 1$ and $m > k$ in equation* (E). *If*

$$\sum_{n=n_0}^{\infty} q_n(1 - p_{n+1-m}) = \infty \tag{6}$$

and

$$\sum_{s=n}^{n+m-k-1} q_s \varrho_s > 1 + p \tag{7}$$

are satisfied, then all solutions of equation (E) *are oscillatory.*

Proof. Assume to the contrary that there exists a nonoscillatory solution $\{y_n\}$. Without loss of generality we may assume that $y_{n-\theta} > 0$ for $n \geqslant N \in \mathbb{N}(n_0)$, where N is chosen so that two cases of Lemma 2.1 hold for $n \geqslant N$. We shall show that in each case we are led to a contradiction.

Case (I): From Lemma (2.2) and equation (E), we have

$$\Delta(c_n \Delta z_n) + q_n(1 - p_{n+1-m}) z_{n+1-m} \leqslant 0, \quad n \geqslant N. \tag{8}$$

Define $W_n = c_n \Delta z_n / z_{n-m}$ for $n \geqslant N \geqslant n_0 + m$, then we have

$$\Delta W_n \leqslant -q_n(1 - p_{n+1-m}) - \frac{c_n \Delta z_n \Delta z_{n-m}}{z_{n-m} z_{n+1-m}} \leqslant q_n(1 - p_{n+1-m})$$

Summing the last inequality, we obtain $\sum\limits_{s=N}^{n} q_s(1 - p_{s+1-m}) < W_N$.

Letting $n \to \infty$, we have $\sum\limits_{n=N}^{\infty} q_n(1 - p_{n+1-m}) < \infty$, a contradiction to (6).

Case (II): Summing equation (E) from N to $n - 1$, we obtain

$$c_n \Delta z_n - c_N \Delta z_N + \sum_{s=N}^{n-1} q_s y_{s+1-m} = 0$$

or

$$\Delta z_n + \frac{1}{c_s} \sum_{s=N}^{n-1} q_s y_{s+1-m} \leqslant 0.$$

Summing again from n to $j-1$ and rearranging, we obtain

$$\sum_{s=n}^{\infty} q_s \varrho_s y_{s+1-m} < z_n. \tag{9}$$

Using (4) in (9) and using the fact that $\{z_n\}$ is nonincreasing, we obtain

$$\sum_{s=n}^{n+m-k+1} q_s \varrho_s \leqslant (1+p),$$

which contradicts (7). This completes the proof of the theorem. $\square$

Example 3.1. *Consider the following neutral difference equation*

$$\Delta\Big(2^n \Delta\Big(y_n + \frac{1}{2} y_{n-2}\Big)\Big) + 9(2^n) y_{n-4} = 0, \quad n \geqslant 4. \tag{E$_1$}$$

With $\varrho_n = 2^{-n+1}$, *all conditions of Theorem* 3.1 *are satisfied and hence all solutions of* (E_1) *are oscillatory. In fact* $\{y_n\} = \{(-1)^n\}$ *is such a solution of* (E_1).

Theorem 3.2. *Assume* $\beta > 1$ *and* $m \geqslant k$ *in equation* (E_1). *If*

$$\sum_{n=n_0}^{\infty} q_n \varrho_{n+1}^{\beta} = \infty \tag{10}$$

then all solutions of equation (E) *are oscillatory.*

Proof. Proceeding as in the proof of Theorem 3.1 we see that Lemma 2.1 holds for $n \geqslant N \in \mathbb{N}(n_0)$.

Case (I): Summing equation (E) from $N \in \mathbb{N}(n_0)$ to $n-1$, yields

$$c_n \Delta z_n - c_N \Delta z_N + \sum_{s=N}^{n-1} q_s y_{s+1-m}^{\beta} = 0, \quad n \geqslant N. \tag{11}$$

or

$$\sum_{s=N}^{n-1} q_s y_{s+1-m}^{\beta} < c_N \Delta z_N.$$

Letting $n \to \infty$, we obtain

$$\sum_{s=N}^{\infty} q_s y_{s+1-m}^{\beta} < \infty. \tag{12}$$

For this case $\{z_n\}$ is increasing, so there exists a positive constant c such that $z_n > c$ for $n \geqslant N$. This together with (3) yields $y_n \geqslant c(1-p)$ for $n \geqslant N$. Thus there exists an integer $N_1 \geqslant N$ such that

$$y_n \geqslant c(1-p)\varrho_n, \quad n \geqslant N_1,$$

since $\varrho_n \to 0$ as $n \to \infty$. Because $\{\varrho_n\}$ is decreasing, the last inequality implies that

$$y_{n+1-m} \geqslant c(1-p)\varrho_{n+1-m} \geqslant c(1-p)\varrho_{n+1} \tag{13}$$

Combining (12) with (13) we see that

$$\sum_{s=N_1}^{\infty} q_s \varrho_{s+1}^{\beta} < \infty \tag{14}$$

a contradiction to (10).

Case (II): From (4) and (5) and the facts $m \geqslant k$, and z_n is decreasing we have

$$\begin{aligned} y_{n+1-m} &= y_{n+1-m+k-k} \\ &\geqslant (1-p)z_{n+1-m+k} \\ &\geqslant (1-p)z_{n+1} \\ &\geqslant -(1-p)c_{n+1}\Delta z_{n+1}\varrho_{n+1} \end{aligned} \tag{15}$$

for $n \geqslant N$.Consider the difference

$$\begin{aligned} \Delta((c_n\Delta z_n)^{-\beta+1}) &= (-\beta+1)t^{-\beta}\Delta(c_n\Delta z_n) \\ &= (\beta-1)(-q_n y_{n+1-m}^{\beta})t^{-\beta} \end{aligned} \tag{16}$$

where $c_{n+1}\Delta z_{n+1} < t < c_n\Delta c_n$. Now equation (16), in view of (15) implies that

$$\begin{aligned} &\Delta((c_n\Delta z_n)^{1-\beta}) \\ &\qquad \leqslant (-\beta+1)(-q_n y_{n+1-m}^{\beta})(c_{n+1}\Delta z_{n+1})^{-\beta} \\ &\qquad \leqslant (-\beta+1)q_n(1-p)^{\beta}\varrho_{n+1}^{\beta}(c_{n+1}\Delta z_{n+1})^{\beta}(c_{n+1}\Delta z_{n+1})^{-\beta}. \end{aligned}$$

Hence

$$\Delta((c_n\Delta z_n)^{1-\beta}) \leqslant -(\beta-1)(1-p)^{\beta}q_n\varrho_{n+1}^{\beta}$$

for $n \geqslant N$. Summing (17) from N to n, we have

$$(c_{n+1}\Delta z_{n+1})^{1-\beta} - (c_N\Delta z_N)^{1-\beta} \leqslant -(1-p)^{\beta}(\beta-1)\sum_{s=N}^{n} q_s\varrho_{s+1}^{\beta} \tag{17}$$

and so letting $n \to \infty$, we obtain $\sum\limits_{s=N}^{n} q_s\varrho_{s+1}^{\beta} < \infty$, which contradicts (10). This completes the proof of the theorem. □

Example 3.2. Consider the following neutral difference equation

$$(\mathrm{E}_2)\qquad \Delta\Big(2_n\Delta\Big(y_n+\frac{1}{2^4}y_{n-4}\Big)\Big)+32^{3n-14}y_{n-5}^3=0,\quad n\geqslant 5.$$

With $\varrho_n = 1/2^{n-1}$, all conditions of Theorem 3.2 are satisfied and hence all solutions of (E_2) are oscillatory. One such solution of (E_2) is $\{y_n\}=\{(-1)^n/2^n\}$.

Theorem 3.3. *Assume* $0<\beta<1$ *and* $m\geqslant k$ *in equation* (E). *If*

$$(18)\qquad \sum_{n=n_0}^{\infty} q_n\varrho_{n+1}=\infty$$

then all solutions of equation (E) *are oscillatory.*

P r o o f. Proceeding as in the proof of Theorem 3.1, we see that Lemma 2.1 holds and in Case (I) we have (13) and (14). For large n, we have $\varrho_n\leqslant 1$ and $\varrho_n^\beta\geqslant\varrho_N$. Therefore from (14) we obtain $\sum\limits_{n=N}^{\infty} q_n\varrho_{n+1}\leqslant\sum\limits_{n=N}^{\infty} q_n\varrho_{n+1-m}^\beta<\infty$, a contradiction to (18). For the Case (II), from (11) $-\Delta z_{n+1}\geqslant(1/c_{n+1})\sum\limits_{s=n}^{\infty} q_s y_{s+1-m}^\beta$, for $n\geqslant N$. We consider the difference $\Delta(z_n^{2\varepsilon})$, where $\varepsilon>0$ is such that $2\varepsilon<1-\beta$, $-\Delta(z_{n+1}^{2\varepsilon})=-2\varepsilon t^{2\varepsilon-1}\Delta z_{n+1}\geqslant 2\varepsilon t^{2\varepsilon-1}(1/c_{n+1})\sum\limits_{s=N}^{n} q_s y_{s+1-m}^\beta\geqslant 2\varepsilon z_{n+1}^{2\varepsilon-1}(1/c_{n+1})\sum\limits_{s=N}^{n} q_s y_{s+1-m}^\beta$, where $z_{n+2}<t<z_{n+1}$, and $\{z_n\}$ is decreasing. Hence

$$(19)\qquad -\Delta(z_{n+1}^{2\varepsilon})\geqslant\frac{2\varepsilon}{c_{n+1}}\sum_{s=N}^{n} q_s y_{s+1-m}^\beta z_{s+1}^{2\varepsilon-1}.$$

Since $c\geqslant z_n>0$ and $c\geqslant y_n>0$ where c is a constant, from (4) and in view of $m\geqslant k$, we obtain

$$(20)\qquad y_{n+1-m}\geqslant(1-p)z_{n+1}$$

for $n\geqslant N$. From (19) and (20) we obtain

$$-\Delta(z_{n+1}^{2\varepsilon})\geqslant\frac{2\varepsilon(1-p)^\beta}{c_{n+1}}\sum_{s=N}^{n} q_s z_{s+1}^{\beta+2\varepsilon-1}\geqslant\frac{K}{c_{n+1}}\sum_{s=N}^{n} q_s,$$

where $K=2\varepsilon(1-p)^\beta c^\beta+2\varepsilon-1$. Summing the last inequality and rearranging, we obtain $z_{N+1}^{2\varepsilon}-z_{n+2}^{2\varepsilon}\geqslant K\sum\limits_{i=N}^{n} q_i\sum\limits_{s=i}^{n}1/c_{s+1}$ and so letting $n\to\infty$, we obtain $\sum\limits_{i=N}^{n} q_i\varrho_{i+1}<\infty$, which again contradicts (18). This completes the proof of the theorem. □

Example 3.3. Consider the following neutral difference equation

$$(\mathrm{E}_3)\qquad \Delta\Big(n(n+1)\Delta\Big(y_n+\frac{1}{2}y_{n-2}\Big)\Big)+\frac{3(n+1)(4n^2+4n+3)}{2(n-3)^{\frac{1}{3}}}y_{n-3}^{\frac{1}{3}}=0, n\geqslant 4.$$

With $\varrho_n=\frac{1}{n}$, all conditions of Theorem (3.3) are satisfied and hence all solutions of (E_3) are oscillatory. In fact $\{y_n\}=\{(-1)^n n\}$ is such a solution of (E_3).

4. Existence of nonoscillatory solutions

In this section we provide sufficient conditions for the existence of nonoscillatory solutions of equation (E) when $\beta>1$ or $0<\beta<1$.

Theorem 4.1. *Assume $\beta>1$ in equation* (E). *If*

$$\sum_{n=n_0}^{\infty} q_n\varrho_{n+1-m}^{\beta}<\infty \tag{21}$$

and

$$\lim_{n\to\infty}\frac{\varrho_{n-k}}{\varrho_n}=1 \tag{22}$$

then equation (E) *has a nonoscillatory solution.*

Proof. Choose $N\in\mathbb{N}(n_0)$ sufficiently large so that from (21) and (22) we have

$$\sum_{n=N-1}^{\infty} q_n\varrho_{n+1-m}^{\beta}\leqslant\frac{1-p}{8\beta} \tag{23}$$

$$\frac{p_n\varrho_{n-k}}{\varrho_n}\leqslant\frac{3p+1}{4} \tag{24}$$

for $n\geqslant N$. Consider the Banach space B_N of all real sequences $y=\{y_n\}$, $n\geqslant N$ with the norm defined as $\|y_n\|=\sup\{|y_n/\varrho_n|\}$, $n\geqslant N$. We define a closed bounded subset S of B_N as follows:

$$S=\Big\{y\in B_N\colon\ \frac{3(1-p)}{8}\leqslant\frac{y_n}{\varrho_n}\leqslant 1,\ n\geqslant N\Big\}.$$

Define an operator $T\colon\ S\to B_N$ such that

$$\begin{aligned}Ty_n&=\frac{3p+5}{8}\varrho_n-p_ny_{n-k}\\&\quad+\varrho_n\sum_{s=N-1}^{n-2}q_sy_{s+1-m}^{\beta}+\sum_{s=n}^{\infty}q_{s-1}\varrho_sy_{s-m}^{\beta},\quad n\geqslant N+1\\Ty_N&=\varrho_N.\end{aligned}$$

Clearly, Ty is continuous. For every $x \in S$ and $n \geqslant N$, using (24) we obtain

$$
\begin{aligned}
Ty_n &\geqslant \frac{3p+5}{8}\varrho_n - p_n y_{n-k} \\
&\geqslant \varrho_n\Big(\frac{3p+5}{8} - p_n\frac{\varrho_{n-k}}{\varrho_n}\Big) \\
&\geqslant \frac{3(1-p)}{8}\varrho_n.
\end{aligned}
$$

Also, in view of (23) we have

$$
\begin{aligned}
Ty_n &\leqslant \varrho_n\frac{3p+5}{8} + \varrho_n\sum_{s=N-1}^{n-2} q_s y_{s+1-m}^{\beta} + \sum_{s=n}^{\infty}\varrho_s q_{s-1} y_{s-m}^{\beta} \\
&\leqslant \varrho_n\left(\frac{3p+5}{8} + \sum_{s=N-1}^{\infty} q_s \varrho_{s+1-m}^{\beta}\right) \\
&\leqslant \varrho_n\Big(\frac{3p+5}{8} + \frac{1-p}{8}\Big) < \varrho_n.
\end{aligned}
$$

Thus we proved that $TS \subset S$. Now we shall show that the operator T is a contraction on S. In fact for $x, y \in S$ we have

$$
\begin{aligned}
\frac{1}{\varrho_n}|Ty_n - Tx_n| &\leqslant \frac{p_n\varrho_{n-k}}{\varrho_n}\Big|\frac{y_{n-k}}{\varrho_{n-k}} - \frac{x_{n-k}}{\varrho_{n-k}}\Big| + \sum_{s=N-1}^{n-2}|q_s y_{s+1-m}^{\beta} - x_{s+1-m}^{\beta}| \\
&\quad + \frac{1}{\varrho_n}\sum_{s=n}^{\infty}\varrho_s q_{s-1}|y_{s-m}^{\beta} - x_{s-m}^{\beta}| \\
&\leqslant p_n\frac{\varrho_{n-k}}{\varrho_n}\Big|\frac{y_{n-k}}{\varrho_{n-k}} - \frac{x_{n-k}}{\varrho_{n-k}}\Big| \\
&\quad + \sum_{s=N-1}^{\infty} q_s {\varrho_{s+1-m}}^{\beta}\Big|\Big(\frac{y_{s+1-m}}{\varrho_{s+1-m}}\Big)^{\beta} - \Big(\frac{x_{s+1-m}}{\varrho_{s+1-m}}\Big)^{\beta}\Big|.
\end{aligned}
$$

By the Mean value Theorem applies to the function $\gamma(t) = t^{\beta}$, $\beta > 1$, we see that for any $x, y \in S$, we have $\left|x^{\beta} - y^{\beta}\right| \leqslant 2\beta|x-y|$ for all $n \geqslant N$.

$$
\begin{aligned}
\|Ty - Tx\| &\leqslant \frac{3p+1}{4}\|y-x\| + 2\beta\sum_{s=N}^{\infty} q_s\varrho_{s+1-m}^{\beta}\|y-x\| \\
&\leqslant \Big(\frac{3p+1}{4} + \frac{1-p}{4}\Big)\|y-x\| < \|y-x\|
\end{aligned}
$$

Thus T is a contraction mapping on S. and so T has a unique fixed point y in S. It is not difficult to check that $y = \{y_n\}$ is a solution of (E) and we see that this solution is nonoscillatory. This completes the proof of the theorem. □

An alternate and, as we will see, independent result for the superlinear case $\beta > 1$ is the following.

Theorem 4.2. *Assume* $\beta > 1$. *If*

$$\sum_{n=n_0}^{\infty} q_n \varrho_{n+1} < \infty \tag{25}$$

then equation (E) *has a nonoscillatory solution.*

Proof. Choose $N \in \mathbb{N}\,(n_0)$ sufficiently large so that we have

$$\sum_{n=N-1}^{\infty} q_n \varrho_{n+1} < \frac{1-p}{8\beta}. \tag{26}$$

for $n \geqslant N$. Let B_N be set of all bounded real sequences $y = \{y_n\}$, $n \geqslant N$ with the norm $\|y_n\| = \sup_{n \geqslant N} \{|x_n|\}$ and let

$$S = \Big\{ y \in B_N \colon \ \frac{3(1-p)}{8} \leqslant y_n \leqslant 1, \ n \geqslant N \Big\}.$$

Define the mapping $T\colon\ S \to B_N$ by

$$Ty_n = \begin{cases} \frac{5p+3}{8} - p_n y_{n-k} + \varrho_n \sum\limits_{s=N-1}^{n-2} q_s y^{\beta}_{s+1-m} + \sum\limits_{s=n}^{\infty} q_{s-1} \varrho_s y^{\beta}_{s-m}, & n \geqslant N+1, \\ Ty_N. \end{cases}$$

Clearly, T is continuous. Now for every $y \in S$ and $n \geqslant N$, we have

$$Ty_n \geqslant \frac{5p+3}{8} - py_{n-k} \geqslant \frac{5p+3}{8} - p \geqslant \frac{3(1-p)}{8}.$$

Now

$$Ty_n \leqslant \frac{5p+3}{8} + \sum_{s=N-1}^{\infty} q_s \varrho_{s+1} \leqslant \frac{5p+3}{8} + \frac{1-p}{8\beta} < 1$$

by (26). Thus $TS \subset S$. Moreover since S is bounded, closed and convex subset of B_N, we only need to show that T is a contraction mapping on S in order to apply contraction mapping principle. For $x, y \in S$ and $n \geqslant N$, we have

$$\begin{aligned} |Ty_n - Tx_n| &\leqslant p_n |y_{n-k} - x_{n-k}| + \varrho_n \sum_{s=N-1}^{n-2} q_s |y^{\beta}_{s+1-m} - x^{\beta}_{s+1-m}| \\ &\quad + \sum_{s=n}^{\infty} q_{s-1} \varrho_s |y^{\beta}_{s-m} - x^{\beta}_{s-m}| \\ &\leqslant p_n |y_{n-k} - x_{n-k}| + \sum_{s=N-1}^{\infty} q_s \varrho_s |y^{\beta}_{s+1-m} - x^{\beta}_{s+1-m}|. \end{aligned}$$

Again, the Mean value Theorem applied to the function $\gamma(t) = u^\beta$, $\beta > 1$, shows that for every $x, y \in S$, we have $|y^\beta - x^\beta| \leqslant 2\beta|y - x|$ for all $n \geqslant N$. Hence

$$\begin{aligned}\|Ty - Tx\| &\leqslant p\|y - x\| + 2\beta \sum_{s=N-1}^{\infty} q_s \varrho_s \|y - x\| \\ &\leqslant \Big(p + 2\beta \frac{1-p}{8\beta}\Big)\|y - x\| < \|y - x\|.\end{aligned}$$

Thus, T is a contraction mapping and so T has a unique fixed point y, which is clearly a positive solution of (E). This completes the proof of the theorem. □

The following examples show that Theorem 4.1 and 4.2 are independent of each other.

Example 4.1. Consider the equation

$$(\mathrm{E}_4) \qquad \Delta\Big(n(n-1)\Delta\Big(y_n + \frac{1}{2}y_{n-1}\Big)\Big) + \frac{2(n-2)^3}{(n+1)(n+2)} y_{n-2}^3 = 0, \quad n \geqslant 3$$

Here $\varrho_n = \frac{1}{n-1}$ and we see that all the conditions of Theorem 4.1 are satisfied. Therefore, equation (E_4) has a nonoscillatory solution. Infact, one such solution is $\{y_n\} = \{1/n\}$. Notice that condition (25) does not hold, so Theorem 4.2 does not apply.

Example 4.2. For the equation

$$(\mathrm{E}_5) \qquad \Delta\Big(4^n\Delta\Big(y_n + \frac{1}{2}y_{n-1}\Big)\Big) + 2^n y_{n-2}^3 = 0,$$

we have $\varrho_n = \frac{1}{3}4^{-n+1}$ and we see that the conditions (21) and (25) hold, but (22) does not hold. Thus, by Theorem 4.2, equation (E_5) has a nonoscillatory solution, but Theorem 4.1 does not apply.

Our next result is for the sublinear case $0 < \beta < 1$.

Theorem 4.3. *Assume that* $0 < \beta < 1$. *If*

$$\sum_{n=n_0}^{\infty} q_n \varrho_{n+1} < \infty$$

then the equation (E) *has a nonoscillatory solution.*

Proof. Choose $N \in \mathbb{N}(n_0)$ sufficiently large so that

$$\sum_{n=N}^{\infty} q_n \varrho_{n+1} < \frac{(1-p)^2}{8}.$$

Let B_N be the set of all bounded real sequences $y = \{y_n\}$ with norm $\|y\| = \sup\limits_{n\geqslant N}\{|y_n|\}$ and let

$$S = \Big\{y \in B_N \colon \ \frac{3(1-p)}{8} \leqslant y_n \leqslant 1, \ n \geqslant N\Big\}.$$

Define the operator $T\colon \ S \to B_N$ by

$$Ty_n = \begin{cases} \frac{5p+3}{8} - p_n y_{n-k} + \varrho_n \sum\limits_{s=N-1}^{n-2} q_s y^{\beta}_{s+1-m} + \sum\limits_{s=n}^{\infty} q_{s-1}\varrho_s y^{\beta}_{s-m}, & n \geqslant N+1, \\ Ty_N. \end{cases}$$

It is easy to see that T is continuous, $TS \subset S$, and for $y, x \in S$ and $n \geqslant N$, we have

$$|Ty_n - Tx_n| \leqslant p_n|y_{n-k} - x_{n-k}| + \sum_{s=N-1}^{\infty} q_s\varrho_{s+1}|y^{\beta}_{s+1-m} - x^{\beta}_{s+1-m}|.$$

By the Mean value Theorem applies to the function $\gamma(t) = t^{\beta}$, $0 < \beta < 1$, we see that for every $x, y \in S$, we have $|y^{\beta} - x^{\beta}| \leqslant \frac{8\beta}{3(1-p)}|y - x|$. Thus

$$\|Ty - Tx\| \leqslant \|y - x\|\Big(p + \frac{8\beta}{3(1-p)} \cdot \frac{(1-p)^2}{8}\Big) < \|y - x\|.$$

and we see that T is a contraction on S. Then T has a unique fixed point $\{y_n\}$, which is clearly a positive solution of (E). This completes the proof of the theorem. □

Example 4.3. Consider the equation

$$(\mathrm{E}_6) \qquad \Delta\Big(4^n\Delta\Big(y_n + \frac{1}{2}y_{n-1}\Big)\Big) + 2^{\frac{4n-2}{3}} y^{\frac{1}{3}}_{n-2} = 0, \quad n \geqslant 3.$$

We have $\varrho_n = \frac{1}{3}4^{-n+1}$ and we see that the conditions of Theorem 4.3 are satisfied. Here, $\{y_n\} = \{1/2^n\}$ is a nonoscillatory solution of E_6.

We conclude this paper with the following remarks.

Remark 1. Let $p_n \equiv 0$ and $m = 0$ in equation (E). Then Theorems 3.2 and 4.1 (Theorems 3.3 and 4.3) reduce to Theorem 2.1 (Theorem 2.2) of Zhang [5]

Remark 2. All our results here could easily be extended to the difference equation

$$\Delta(c_n\Delta(y_n + p_n y_{n-k})) + q_n f(y_{n+1-m}) = 0$$

where f is continuous, $uf(u) > 0$ for $u \neq 0$ and $f(u)/|u|^{\beta}\operatorname{sgn} u \geqslant M > 0$, with appropriate changes in the proofs given.

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Authors' address: Dept. of Mathematics, Peryiar Univ., Salem-636011, Tamilnadu, India, e-mail:ethandapani@yahoo.co.in.