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CONTINUOUS EXTENDIBILITY OF SOLUTIONS OF THE THIRD PROBLEM FOR THE LAPLACE EQUATION

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Abstract. A necessary and sufficient condition for the continuous extendibility of a solution of the third problem for the Laplace equation is given.

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For $x, y \in \mathbb{R}^m$, $m > 2$, denote

$$h_x(y) = \begin{cases} (m-2)^{-1} A^{-1} |x-y|^{2-m} & \text{for } x \neq y, \\ \infty & \text{for } x = y, \end{cases}$$

where A is the area of the unit sphere in $\mathbb{R}^m$. For a finite real Borel measure ν denote

$$\mathscr{U}\nu(x) = \int_{\mathbb{R}^m} h_x(y)\,\mathrm{d}\nu(y),$$

the single layer potential corresponding to ν, for each x for which this integral has sense.

Suppose that $G \subset \mathbb{R}^m$ $(m > 2)$ is an open set with a non-void compact boundary ∂G such that $\partial G = \partial(\mathbb{R}^m \setminus G)$. Suppose moreover that for each $x \in \partial G$ there exists

$$d_G(x) = \lim_{r \searrow 0} \frac{\mathcal{H}_m(G \cap \Omega_r(x))}{\mathcal{H}_m(\Omega_r(x))} > 0.$$

Here $\Omega_r(x)$ is the open ball with centre x and diameter r, and $\mathcal{H}_k$ is the k-dimensional Hausdorff measure normalized so that $\mathcal{H}_k$ is the Lebesgue measure in $\mathbb{R}^k$.

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Fix a nonnegative element λ of $\mathcal{C}'(\partial G)$ (= the Banach space of all finite signed Borel measures with support in ∂G, with the total variation as the norm) and suppose that the single layer potential $\mathscr{U}\lambda$ is finite and continuous on ∂G. It was shown in [23] that $\mathscr{U}\lambda$ is finite and continuous on ∂G if and only if

$$\lim_{r\to 0+} \sup_{y\in\partial G} \int_{\Omega_r(y)} h_y(x)\,\mathrm{d}\lambda(x) = 0.$$

According to [11], Lemma 2.18 this is true if there are constants $\alpha > m-2$ and $k > 0$ such that $\lambda(\Omega_r(x)) \leqslant kr^\alpha$ for all $x \in \mathbb{R}^m$ and all $r > 0$.

If h is a harmonic function on G such that

$$\int_H |\nabla h|\,\mathrm{d}\mathcal{H}_m < \infty$$

for all bounded open subsets H of G we define the weak normal derivative $N^G h$ of h as the distribution

$$\langle N^G h, \varphi\rangle = \int_G \nabla\varphi\cdot\nabla h\,\mathrm{d}\mathcal{H}_m$$

for $\varphi \in \mathscr{D}$ (= the space of all compactly supported infinitely differentiable functions in $\mathbb{R}^m$).

If $H \subset \mathbb{R}^m$ is an open set with a compact smooth boundary, $u \in \mathcal{C}^1(\mathrm{cl}\,H)$ is a harmonic function on H and

$$\frac{\partial u}{\partial n} + fu = g \ \text{ on } \ \partial H$$

where $f, g \in \mathcal{C}(\partial H)$ (= the space of all finite continuous functions on ∂H equipped with the maximum norm) and n is the exterior unit normal of H, then for $\varphi \in \mathscr{D}$ we have

$$\int_{\partial H} \varphi g\,\mathrm{d}\mathcal{H}_{m-1} = \int_H \nabla\varphi\cdot\nabla u\,\mathrm{d}\mathcal{H}_m + \int_{\partial H} \varphi f u\,\mathrm{d}\mathcal{H}_{m-1}. \tag{1}$$

If we denote by $\mathcal{H}$ the restriction of $\mathcal{H}_{m-1}$ to ∂H then (1) has the form

$$N^H u + uf\mathcal{H} = g\mathcal{H}. \tag{2}$$

The formula (2) motivates our definition of the solution of the third problem for the Laplace equation

$$\begin{aligned} \Delta u &= 0 \quad \text{in } G, \\ N^G u + u\lambda &= \mu, \end{aligned} \tag{3}$$

where $\mu \in \mathcal{C}'(\partial G)$ (compare [11], [22]).

Let $\mu \in \mathcal{C}'(\partial G)$. Now we formulate *the third problem for the Laplace equation* (3) as follows: *Find a function* $u \in L^1(\lambda)$ *on* $\operatorname{cl} G$*, the closure of* G*, harmonic on* G*, for which* $|\nabla u|$ *is integrable over all bounded open subsets of* G*, such that for* λ*-a.a.* $x \in \partial G$ *there is a set* H *with* $d_H(x) = 0$ *and*

$$\lim_{\substack{y \to x \\ y \in G \setminus H}} u(y) = u(x), \tag{4}$$

and such that $N^G u + u\lambda = \mu$*.*

Suppose in this paragraph that G has a locally Lipschitz boundary and $u \in W^{1,2}(G)$. It is well-known that we can even suppose that $u \in W^{1,2}(\mathbb{R}^m)$ (see [30], Remark 2.52). We can choose such a representation of u that u is approximately continuous at $\mathcal{H}_{m-1}$-a.a. points of $\mathbb{R}^m$ (see [30], Theorem 3.3.3, Theorem 2.6.16 and Remark 3.3.5). The restriction of u to ∂G is the trace of u (see [30], p. 190). If $\mathcal{H}$ denotes the restriction of $\mathcal{H}_{m-1}$ to ∂G, then $u \in L_2(\mathcal{H})$ (see [19], Theorem 1.2). If f is a nonnegative bounded Baire function on ∂G and $g \in L_2(\mathcal{H})$, then u is called a weak solution of the problem $\Delta u = 0$ in G, $\partial u/\partial n + fu = g$ on ∂G if

$$\int_{\partial G} vg \, \mathrm{d}\mathcal{H}_{m-1} = \int_G \nabla v \cdot \nabla u \, \mathrm{d}\mathcal{H}_m + \int_{\partial G} fvu \, \mathrm{d}\mathcal{H}_{m-1}$$

for each $v \in W^{1,2}(G)$ (compare [19], Example 2.12). Put $\lambda = f\mathcal{H}$, $\mu = g\mathcal{H}$. Using Hölder's inequality we see that $|\nabla u|$ is integrable over all bounded open subsets of G. Since u is approximately continuous at $\mathcal{H}_{m-1}$-a.a. points of $\mathbb{R}^m$ and λ is absolutely continuous with respect to $\mathcal{H}_{m-1}$, we obtain that for λ-a.a. $x \in \partial G$ there is a set H with $d_H(x) = 0$ such that (4) holds. Since $\mathscr{D} \subset W^{1,2}(G)$, u is a solution of (3). Therefore, our definition is a generalization of the weak solution of the third problem for the Laplace equation in the Sobolev space $W^{1,2}(G)$.

It is usual to look for a solution u in the form of the single layer potential $\mathscr{U}\nu$, where $\nu \in \mathcal{C}'(\partial G)$. It was shown in [16] that $\mathscr{U}\nu$ has all the properties of a solution of the third problem with some boundary condition, but our "continuity" on the boundary is replaced by the fine continuity at λ-a.a. points of the boundary. If $\mathscr{U}\nu$ is finely continuous at $x \in \partial G$ with respect to $\operatorname{cl} G$ then there is H with $d_H(x) = 0$ such that

$$\lim_{\substack{y \to x \\ y \in G \setminus H}} u(y) = u(x)$$

(see [10], Theorem 10.15, Corollary 10.5). If $\mathscr{U}\nu$ is a solution of the third problem in the sense of [16] then it is a solution of the third problem in our sense.

The operator $\tau\colon\ \nu \mapsto N^G \mathscr{U}\nu + (\mathscr{U}\nu)\lambda$ is a bounded linear operator on $\mathcal{C}'(\partial G)$ if and only if $V^G < \infty$, where

$$V^G = \sup_{x \in \partial G} v^G(x),$$

$$v^G(x) = \sup\left\{ \int_G \nabla\varphi \cdot \nabla h_x \,\mathrm{d}\mathcal{H}_m;\ \varphi \in \mathscr{D},\ |\varphi| \leqslant 1,\ \operatorname{spt}\varphi \subset \mathbb{R}^m - \{x\} \right\}$$

(see [11]). There are more geometrical characterizations of $v^G(x)$ in [11] which ensure that $V^G < \infty$ for G convex or for G with $\partial G \subset \bigcup_{i=1}^{k} L_i$, where L_i are $(m-1)$-dimensional Ljapunov surfaces, i.e., of class $C^{1+\alpha}$.

If $z \in \mathbb{R}^m$ and θ is a unit vector such that the symmetric difference of G and the half-space $\{x \in \mathbb{R}^m;\ (x-z)\cdot\theta < 0\}$ has m-dimensional density zero at z then $n^G(z) = \theta$ is termed *the exterior normal* of G at z in Federer's sense. If there is no exterior normal of G at z in this sense, we denote by $n^G(z)$ the zero vector in $\mathbb{R}^m$. The set $\{y \in \mathbb{R}^m;\ |n^G(y)| > 0\}$ is called the reduced boundary of G and will be denoted by $\widehat{\partial} G$.

If G has a finite perimeter (which is fulfilled if $V^G < \infty$) then $\mathcal{H}_{m-1}(\widehat{\partial} G) < \infty$ and

$$v^G(x) = \int_{\widehat{\partial} G} |n^G(y) \cdot \nabla h_x(y)| \,\mathrm{d}\mathcal{H}_{m-1}(y)$$

for each $x \in \mathbb{R}^m$. Throughout the paper we shall assume that $V^G < \infty$.

If L is a bounded linear operator on a Banach space X we denote by $\|L\|_{\text{ess}}$ the essential norm of L, i.e. the distance of L from the space of all compact linear operators on X. The essential spectral radius of L is defined by

$$r_{\text{ess}} L = \lim_{n\to\infty} (\|L^n\|_{\text{ess}})^{1/n}.$$

Theorem 1. *Let $r_{\text{ess}}(\tau - \frac{1}{2}I) < \frac{1}{2}$, where I is the identity operator. Then G has finitely many components $G_1, \ldots, G_n$ and $\operatorname{cl} G_j \cap \operatorname{cl} G_k = \emptyset$ for $j \neq k$. If $\mu \in \mathcal{C}'(\partial G)$ then there is a harmonic function u on G, which is a solution of the third problem*

$$N^G u + u\lambda = \mu,$$

if and only if $\mu \in \mathcal{C}'_0(\partial G)$ (= the space of such $\nu \in \mathcal{C}'(\partial G)$ that $\nu(\partial G_k) = 0$ for each bounded G_k for which $\lambda(\partial G_k) = 0$). Moreover, if $\mu \in \mathcal{C}'_0(\partial G)$ then there is a solution of this problem in the form of the single layer potential $\mathscr{U}\nu$, where $\nu \in \mathcal{C}'_0(\partial G)$.

Proof. According to [18], Lemma 3 the set G has finitely many components $G_1, \ldots, G_n$ and $\operatorname{cl} G_j \cap \operatorname{cl} G_k = \emptyset$ for $j \neq k$. Let u be a solution of the third problem

$$N^G u + u\lambda = \mu.$$

If G_k is bounded and $\lambda(\partial G_k) = 0$ choose $\varphi \in \mathscr{D}$ such that $\varphi = 1$ on G_k and $\varphi = 0$ on $G \setminus G_k$. Then

$$\mu(\partial G_k) = \langle \mu, \varphi \rangle = \langle N^G u + u\lambda, \varphi \rangle = 0.$$

On the other hand, if $\mu \in \mathcal{C}'_0(\partial G)$ then [16], Theorem 1 yields that there is a solution of this problem in the form of the single layer potential $\mathscr{U}\nu$, where $\nu \in \mathcal{C}'_0(\partial G)$. □

Remark. It is well-known that the condition $r_{\mathrm{ess}}(\tau - \frac{1}{2}I) < \frac{1}{2}$ is fulfilled for sets with a smooth boundary (of class $C^{1+\alpha}$) (see [12]) and for convex sets (see [20]). R. S. Angell, R. E. Kleinman, J. Král and W. L. Wendland proved that rectangular domains (i.e. formed from rectangular parallelepipeds) in $\mathbb{R}^3$ have this property (see [3], [13]). A. Rathsfeld showed in [25], [26] that polyhedral cones in $\mathbb{R}^3$ have this property. (By a polyhedral cone in $\mathbb{R}^3$ we mean an open set Ω whose boundary is locally a hypersurface (i.e. every point of $\partial\Omega$ has a neighbourhood in $\partial\Omega$ which is homeomorphic to $\mathbb{R}^2$) and $\partial\Omega$ is formed by a finite number of plane angles. By a polyhedral open set with bounded boundary in $\mathbb{R}^3$ we mean an open set Ω whose boundary is locally a hypersurface and $\partial\Omega$ is formed by a finite number of polygons.) N. V. Grachev and V. G. Maz'ya obtained independently an analogous result for polyhedral open sets with bounded boundary in $\mathbb{R}^3$ (see [8]). (Let us note that there is a polyhedral set in $\mathbb{R}^3$ which has not a locally Lipschitz boundary.) In [15] it was shown that the condition $r_{\mathrm{ess}}(\tau - \frac{1}{2}I) < \frac{1}{2}$ has a local character. As a conclusion we obtain that this condition is fullfiled for $G \subset \mathbb{R}^3$ such that for each $x \in \partial G$ there are $r(x) > 0$, a domain D_x which is polyhedral or smooth or convex or a complement of a convex domain and a diffeomorphism $\psi_x \colon \mathscr{U}(x; r(x)) \to \mathbb{R}^3$ of class $C^{1+\alpha}$, where $\alpha > 0$, such that $\psi_x(G \cap \mathscr{U}(x; r(x))) = D_x \cap \psi_x(\mathscr{U}(x; r(x)))$. V. G. Maz'ya and N. V. Grachev proved this condition for several types of sets with "piecewise-smooth" boundary in the general Euclidean space (see [6], [7], [9]).

In the rest of paper we will suppose that $r_{\mathrm{ess}}(\tau - \frac{1}{2}I) < \frac{1}{2}$. Since $\tau - N^G\mathscr{U}$ is a compact operator (see [16], Remark 5), this condition is equivalent to the condition $r_{\mathrm{ess}}(N^G\mathscr{U} - \frac{1}{2}I) < \frac{1}{2}$. Denote by $\mathcal{H}$ the restriction of $\mathcal{H}_{m-1}$ onto ∂G. Then $\mathcal{H}(\mathbb{R}^m) < \infty$ (see [17], Lemma 2).

Notation. $\mathcal{C}'_c(\partial G)$ will stand for the subspace of those $\mu \in \mathcal{C}'(\partial G)$ for which there exists a finite continuous function $\mathscr{U}_c\mu$ on $\mathbb{R}^m$ coinciding with $\mathscr{U}\mu$ on $\mathbb{R}^m \setminus \partial G$. It was shown in [24] that if $\nu \in \mathcal{C}'(\partial G)$ and the restriction of $\mathscr{U}\nu$ to ∂G is finite and continuous then $\mathscr{U}\nu$ is finite and continuous in $\mathbb{R}^m$ and $\nu \in \mathcal{C}'_c(\partial G)$. If $\mu = f\mathcal{H}$, where $f \in L_p(\mathcal{H})$, $p > m - 1$ then $\mu \in \mathcal{C}'_c(\partial G)$ (see [16], Remark 6).

Remark. Let $\mu \in \mathcal{C}'(\partial G)$. According to [18], Theorem 1 the following assertions are equivalent:

1) $\mu \in \mathcal{C}'_c(\partial G)$.
2) There is a finite continuous extension of $\mathscr{U}\mu$ from G onto $\operatorname{cl} G$.
3) Put $K = \{x \in \partial G;\ \mathscr{U}|\mu|(x) = \infty\}$. Then there is a finite continuous function f on ∂G such that $\mathscr{U}\mu = f$ on $\partial G \setminus K$.

Lemma 1. *If H is a bounded component of G then there is $\nu \in \mathcal{C}'_c(\partial G)$ such that $\mathscr{U}\nu = 1$ on H and $\mathscr{U}\nu = 0$ on $G \setminus H$.*

Proof. Denote by $G_1, \ldots, G_n$ all bounded components of G. If $\sigma \in \operatorname{Ker} N^G\mathscr{U}$ then $\sigma \in \mathcal{C}'_c(\partial G)$ and $\mathscr{U}\sigma$ is locally constant on G by [17], Lemma 4, Lemma 12. Since $\mathscr{U}\sigma(x) \to 0$ as $|x| \to \infty$, the function $\mathscr{U}\sigma$ vanishes on the unbounded component of G. If $\mathscr{U}\sigma = 0$ in G then $\mathscr{U}_c\sigma$ is a harmonic function in $\mathbb{R}^m \setminus \partial G$ which vanishes on ∂G and converges to 0 at infinity, hence $\mathscr{U}\sigma = \mathscr{U}_c\sigma = 0$ in $\mathbb{R}^m \setminus \partial G$. Since $\mathcal{H}_m(\partial G) = 0$ (see [17], Lemma 2) we obtain $\sigma = 0$ by [14], Theorem 1.12. Since $N^G\mathscr{U}$ is a Fredholm operator with index 0 and the codimension of the range of $N^G\mathscr{U}$ is equal to n by [17], Theorem 1, the dimension of $\operatorname{Ker} N^G\mathscr{U}$ is equal to n. Therefore there is $\nu \in \operatorname{Ker} N^G\mathscr{U} \subset \mathcal{C}'_c(\partial G)$ such that $\mathscr{U}\nu = 1$ on H and $\mathscr{U}\nu = 0$ on $G \setminus H$. □

Lemma 2. *Let $K \subset \mathbb{R}^m$ be compact, u be a harmonic function on $\mathbb{R}^m \setminus K$, and $x_0 \in K$. Denote $U = \{(x - x_0)/|x - x_0|^2;\ x \in \mathbb{R}^m \setminus K\} \cup \{0\}$. Then there are a real number a, a function v harmonic on U with $v(0) = 0$ and a function w harmonic on $\mathbb{R}^m$ such that*

$$u(x) = w(x) + a h_{x_0} + |x - x_0|^{2-m} v((x - x_0)/|x - x_0|^2) \tag{5}$$

in $\mathbb{R}^m \setminus K$. This decomposition is unique.

Proof. We can suppose that $x_0 = 0$. According to [1], Corollary 2.3 there is a unique function w harmonic on $\mathbb{R}^m$ such that $u(x) - w(x) = O(|x|^{2-m})$ as $|x| \to \infty$. Denote

$$\tilde{v}(x) = |x|^{2-m}[u(x/|x|^2) - w(x/|x|^2)] \quad \text{for } x \in U \setminus \{0\}.$$

Then $\tilde{v}$, the Kelvin transfomation of the function $u - w$, is a harmonic function on $U \setminus \{0\}$ (see [5], Theorem B.15). Since U is a neighbourhood of 0, $\tilde{v}$ is bounded on $U \cap \Omega_r(0) \setminus \{0\}$ for some $r > 0$, so there is a harmonic extension $\hat{v}$ of $\tilde{v}$ onto U (see for example [2]). Put $a = \hat{v}(0)$, $v(x) = \hat{v}(x) - a$. An easy calculation yields (5). □

Notation. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a multiindex. Denote $|\alpha| = \alpha_1 + \ldots + \alpha_m$ the length of α. For a function w denote

$$D^\alpha w(x) = \frac{\partial^{|\alpha|} w(x)}{\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}}.$$

If n is a positive integer denote $\nabla^n w(x) = \{D^\alpha w(x);\ |\alpha| = n\}$,

$$|\nabla^n w(x)| = \left[\sum_{|\alpha|=n} |D^\alpha w(x)|^2\right]^{\frac{1}{2}}.$$

Further denote $\nabla^0 w = w$.

Lemma 3. *Let $x_0 \in K \subset \mathbb{R}^m$ be compact, u be a harmonic function on $\mathbb{R}^m \setminus K$. Let n be nonnegetive integer. Then the following assertions are equivalent:*

a) $u(x) = o(|x|^n)$ *as* $|x| \to \infty$.

b) $u(x) = P(x) + ah_{x_0} + |x - x_0|^{2-m} v((x - x_0)/|x - x_0|^2)$, *where a is a real number, v is a harmonic function on a neighbourhood of 0 with $v(0) = 0$, $P \equiv 0$ for $n = 0$ and P is a harmonic polynomial of degree smaller than n for $n > 0$.*

c) *There are $R > 0$, $1 \leqslant p < \infty$ such that $|\nabla^n u| \in L_p(\mathbb{R}^m \setminus \Omega_R(x_0))$.*

d) *There is $R > 0$ such that $|\nabla^k u| \in L_p(\mathbb{R}^m \setminus \Omega_R(x_0))$ for each integer $k \geqslant n$ and for each $p > m/(m + k - 2)$.*

Proof. The implications b) $\Rightarrow$ d) $\Rightarrow$ c), b) $\Rightarrow$ a) are evident.

a) $\Rightarrow$ b) Denote $U = \{(x - x_0)/|x - x_0|^2;\ x \in \mathbb{R}^m \setminus K\} \cup \{0\}$. Then there are a real number a, a function v harmonic on U with $v(0) = 0$ and a function w harmonic on $\mathbb{R}^m$ such that $u(x) = w(x) + ah_{x_0}(x) + |x - x_0|^{2-m} v((x - x_0)/|x - x_0|^2)$ in $\mathbb{R}^m \setminus K$. Then $w(x) = o(|x|^n)$ as $|x| \to \infty$. Therefore there is a constant c such that $|w(x)| \leqslant c|x|^n$ for each $x \in \mathbb{R}^m$. If $\alpha = (\alpha_1, \ldots, \alpha_m)$ is a multiindex with the length $|\alpha|$ greater than n then [5], Theorem B.9 yields that there is a positive constant c_α such that

$$\sup_{|x| \leqslant r} |D^\alpha w(x)| \leqslant c_\alpha r^{-|\alpha|} \sup_{|x| \leqslant 2r} |w(x)| \leqslant c_\alpha c 2^n r^{n - |\alpha|}$$

for each $r > 0$. Putting $r \to \infty$ we get $D^\alpha w \equiv 0$. Therefore w is a polynomial of degree at most n (see for example [28], Chapter IV, Theorem 2.16). Since $w(x) = o(|x|^n)$ as $|x| \to \infty$, w is a polynomial of degree smaller than n for $n > 0$ and $w \equiv 0$ for $n = 0$.

c) $\Rightarrow$ b) For $1 < p$ see [28], Chapter IV, Lemma 4.1, Lemma 4.2. Let now $p = 1$. Denote $U = \{(x - x_0)/|x - x_0|^2;\ x \in \mathbb{R}^m \setminus K\} \cup \{0\}$. Then there are a real number a, a function v harmonic on U with $v(0) = 0$ and a function w harmonic on $\mathbb{R}^m$ such that $u(x) = w(x) + ah_{x_0} + |x - x_0|^{2-m} v((x - x_0)/|x - x_0|^2)$ in $\mathbb{R}^m \setminus K$. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a multiindex with the length $|\alpha| = n$. We will show that $D^\alpha w \equiv 0$. Suppose that $|w(y)| > 0$. Fix $\varrho > 0$ such that $\Omega_R(0) \subset \Omega_\varrho(y)$. It is easy to see that there is a constant b such that

$$|D^\alpha [ah_{x_0}(x) + |x - x_0|^{2-m} v((x - x_0)/|x - x_0|^2)]| \leqslant b|x - y|^{2-m}$$

for $x \in \mathbb{R}^m \setminus \Omega_\varrho(y)$. Using mean-value property of the harmonic function w we get

$$\int_{\mathbb{R}^m \setminus \Omega_R(0)} |D^\alpha u| \, \mathrm{d}\mathcal{H}_m \geqslant \int_\varrho^\infty \left[\left| \int_{\partial \Omega_t(y)} D^\alpha w \, \mathrm{d}\mathcal{H}_{m-1} \right| - \int_{\partial \Omega_t(y)} b t^{2-m} \, \mathrm{d}\mathcal{H}_{m-1} \right] \mathrm{d}t$$
$$= \int_\varrho^\infty [|w(y)| t^{m-1} - bt] \mathcal{H}_{m-1}(\partial \Omega_1(0)) \, \mathrm{d}t = \infty,$$

which contradicts the fact that $D^\alpha u \in L_1(\mathbb{R}^m \setminus \Omega_R(0))$. Since $D^\alpha w \equiv 0$ for each multiindex α with $|\alpha| \geqslant n$, w is a polynomial of degree smaller than n for $n > 0$ (see [28], Chapter IV, Theorem 2.16) and $w \equiv 0$ for $n = 0$. □

Notation. For $p \geqslant 1$ denote by $W^{1,p}(G)$ the collection of all functions $f \in L_p(G)$ the distributional gradient of which belongs to $[L_p(G)]^m$.

Theorem 2. *Denote by $G_1, \ldots, G_k$ all components of G such that $\lambda(\partial G_j) = 0$. If $\mu \in \mathcal{C}'_0(\partial G)$ then there is a solution of the third problem*

$$N^G u + u\lambda = \mu,$$

which is finite and continuous up to the boundary, if and only if $\mu \in \mathcal{C}'_c(\partial G)$. If G is bounded then the general form of this solution is

$$u = \mathscr{U}\nu + \sum_{j=1}^k c_j \chi_{G_j}, \tag{6}$$

where

$$\nu = \sum_{n=0}^\infty \Big(-\frac{\tau - \alpha I}{\alpha} \Big)^n \frac{\mu}{\alpha}, \tag{7}$$
$$\alpha > \frac{1}{2} \Big(V^G + 1 + \sup_{x \in \partial G} \mathscr{U}\lambda(x) \Big),$$

χ_{G_j} are characteristic functions of G_j, and c_j are arbitrary constants. If G is unbounded and G_j are bounded for $j = 1, \ldots, k$ then (6) is a general form of solutions continuously extendible to the closure of G for which there are $R > 0$, $p \geqslant 1$ such that $u \in L_p(G \setminus \Omega_R(0))$. If G is unbounded and there is $j \in \{1, \ldots, k\}$ such that G_j is unbounded, then (6) is a general form of solutions continuously extendible to the closure of G for which there are $R > 0$, $p \geqslant 1$ such that $|\nabla u| \in L_p(G \setminus \Omega_R(0))$.

P r o o f. If $\mu \in \mathcal{C}'_c(\partial G)$ then [16], Theorem 1, Theorem 2 yield that the function u given by (6) is a solution of the third problem (3), which is finite and continuous up to the boundary. If G is unbounded then $|\nabla u| \in L_q(G \setminus \Omega_R(0))$ for $q \geqslant 2$; if moreover G_j are bounded for $j = 1, \dots, k$ then $u \in W^{1,q}(G \setminus \Omega_R(0))$ for $q \geqslant 4$.

Let now v be a solution of the third problem (3), which is finite and continuous up to the boundary. Then v is a solution of the Neumann problem in the sense of distributions with the boundary condition $\mu - v\lambda$. Since $\mu - v\lambda \in \mathcal{C}'_c(\partial G)$ by [18], Theorem 2 and $v\lambda = v^+\lambda - v^-\lambda \in \mathcal{C}'_c(\partial G)$ by [22], Proposition 6, we have $\mu \in \mathcal{C}'_c(\partial G)$.

If G is unbounded and there is $j \in \{1, \dots, k\}$ such that G_j is unbounded, suppose that there are $R > 0$, $p \geqslant 1$ such that $|\nabla v| \in L_p(G \setminus \Omega_R(0))$. According to Lemma 3 we have $|\nabla v| \in L_q(G \setminus \Omega_R(0))$ for all $q \geqslant 2$. If G is unbounded and G_j are bounded for $j = 1, \dots, k$ suppose that there are $R > 0$, $p \geqslant 1$ such that $v \in L_p(G \setminus \Omega_R(0))$. According to Lemma 3 we have $v \in W^{1,q}(G \setminus \Omega_R(0))$ for all $q \geqslant 4$.

Put $w = u - v$. Then w is a solution of the Neumann problem in the sense of distributions with the boundary condition $-w\lambda$, which is continuous up to the boundary. Let $G_1, \dots, G_n$ be all components of G. According to [18], Theorem 2, Theorem 1 there are $\varrho \in \mathcal{C}'_c(\partial G)$ and constants $d_1, \dots, d_n$ such that

$$w = \mathscr{U}\varrho + \sum_{j=1}^{n} d_j \chi_{G_j}.$$

If $j > k$ and G_j is unbounded then $d_j = 0$, because $w \in W^{1,4}(G \setminus \Omega_R(0))$. If $G_1, \dots, G_k$ are bounded then there is $\sigma \in \mathcal{C}'_c(\partial G)$ such that $w = \mathscr{U}\sigma$ by Lemma 1. Since $\tau\sigma = 0$, w is locally constant on G and $w = 0$ on G_j for $j > k$ by [16], Lemma 11.

Suppose now that there is $i \leqslant k$ such that G_i is unbounded. Put $H = G \setminus G_i$. Since w is a solution of the third problem $N^H w + w\lambda = 0$ on H, which is continuously extendible to $\operatorname{cl} H$, w is locally constant on H and $w = 0$ on G_j for $j > k$. Since w is a solution of the Neumann problem on G_i with the zero boundary condition (in the sense of distributions), which is continuously extendible to $\operatorname{cl} G_i$, w is constant on G_i by [18], Theorem 2. □

Remark. Put $G = \mathbb{R}^m \setminus \operatorname{cl} \Omega_1(0)$, $\lambda = \mathcal{H}$, $u(x) = |x|^{2-m} + m - 3$. Then u is a nonconstant harmonic function in G, continuous on the closure of G, $|\nabla u| \in L_2(G)$ (compare Lemma 3) and $N^G u - u\lambda = 0$. Therefore we see that the condition $u \in L_p(G \setminus \Omega_R(0))$ in Theorem 2 cannot be substituted by the condition $|\nabla u| \in L_p(G \setminus \Omega_R(0))$ (compare [18], Theorem 2).

Corollary 1. *Let $\mu \in \mathcal{C}'(\partial G)$ and let v be a solution of the third problem for the Laplace equation in the sense of distributions with the boundary condition μ. Suppose that v is continuously extendible to the closure of G. If $|\nabla v| \in L_p(G \setminus \Omega_R(0))$ for some $R > 0$, $p \geqslant 1$ then $|\nabla v| \in L_2(G)$. If $v \in L_p(G \setminus \Omega_R(0))$ for some $R > 0$,*

$p \geqslant 1$ and $m > 4$ then $v \in W^{1,2}(G)$. If $v \in L_p(G \setminus \Omega_R(0))$ for some $R > 0$, $p \geqslant 1$, $m \leqslant 4$ and λ does not charge the unbounded component of $\operatorname{cl} G$ then $v \in W^{1,2}(G)$ if and only if $\mu(\partial H) = 0$ for the unbounded component H of $\operatorname{cl} G$.

P r o o f. If G is bounded then this assertion is a consequence of Theorem 2 and [18], Lemma 8. Suppose now that G is unbounded. Let u is given by (6). According to Lemma 3 we have $|\nabla u|, |\nabla v| \in L_q(G \setminus \Omega_R(0))$ for all $q \geqslant 2$. Put $w = v - u$. Then w is a solution of the Neumann problem $N^G w = -w\lambda$, which is continuously extendible to the closure of G. Let $G_1, \dots, G_n$ be all components of G. According to [18], Theorem 2, Theorem 1 there are $\varrho \in \mathcal{C}'_c(\partial G)$ and constants $d_1, \dots, d_n$ such that

$$w = \mathscr{U} \varrho + \sum_{j=1}^{n} d_j \chi_{G_j}.$$

Since $|\nabla u|, |\nabla w| \in L_2(G)$ by [16], Theorem 1, Theorem 2, [18], Lemma 7, we have $|\nabla v| \in L_2(G)$. Suppose now that $v \in L_p(G \setminus \Omega_R(0))$ for some $R > 0$, $p \geqslant 1$. Since v is continuous on $\operatorname{cl} G$, $v \in L_2(G_j)$ for each bounded component G_j of G. Denote by $\tilde{G}$ the unbounded component of G, $\tilde{\lambda}$ the restriction of λ to $\operatorname{cl} \tilde{G}$, $\tilde{\mu}$ the restriction of μ to $\operatorname{cl} \tilde{G}$. Then $N^{\tilde{G}} v + v\tilde{\lambda} = \tilde{\mu}$. Since $V^{\tilde{G}} < \infty$, $r_{\text{ess}}(N^{\tilde{G}} \mathscr{U} - \frac{1}{2}) < \frac{1}{2}$ (see [15], Theorem 2.3), Theorem 2 yields that $v = \mathscr{U} \tilde{\nu}$ on $\tilde{G}$, where $\tilde{\nu} \in \mathcal{C}'_c(\partial \tilde{G})$. Since v is continuous on the closure of G, we have $v \in L_2(G)$ for $m > 4$. Let now $\tilde{\lambda} = 0$. According to [17], Theorem 1 we can choose

$$\tilde{\nu} = \tilde{\mu} + \sum_{j=0}^{\infty} (I - 2N^{\tilde{G}} \mathscr{U})^j (I - N^{\tilde{G}} \mathscr{U}) 2\tilde{\mu}.$$

Since $\tilde{\nu}(\mathbb{R}^m) = 0$ if and only if $\tilde{\mu}(\mathbb{R}^m) = 0$ (see [17], Lemma 9), $v \in W^{1,2}(\tilde{G})$ if and only if $\tilde{\mu}(\mathbb{R}^m) = 0$ by [18], Lemma 8. $\square$

Theorem 3. *Let G be an unbounded domain, $\mu \in \mathcal{C}'_c(\partial G) \cap \mathcal{C}'_0(\partial G)$. Then the general form of a solution of the third problem (3), which is finite and continuous up to the boundary, is*

$$u = \mathscr{U} \nu + w, \tag{8}$$

where w is a harmonic function in $\mathbb{R}^m$ and

$$\nu = \sum_{n=0}^{\infty} \Big(- \frac{\tau - \alpha I}{\alpha} \Big)^n \frac{1}{\alpha} \Big(\mu - \frac{\partial w}{\partial n} \mathcal{H} - w\lambda \Big), \tag{9}$$

$$\alpha > \frac{1}{2} \Big(V^G + 1 + \sup_{x \in \partial G} \mathscr{U} \lambda(x) \Big).$$

Let k be a positive integer. Then u is a solution of the third problem (3), *which is finite and continuous up to the boundary and $u(x) = O(|x|^{k-1})$ as $|x| \to \infty$, if and only if u is given by* (8), *where ν is given by* (9) *and w is a harmonic polynomial of degree smaller than k.*

Proof. If u is given by (8) then u is a solution of the third problem (3), which is finite and continuous up to the boundary (see Theorem 2). If w is a harmonic polynomial of degree smaller than k then $u(x) = O(|x|^{k-1})$ as $|x| \to \infty$ by Lemma 3.

Let now u be a solution of the third problem (3) which is finite and continuous up to the boundary. According to Lemma 2 there are a function v harmonic on G and a function w harmonic on $\mathbb{R}^m$ such that $u = w + v$, $v(x) = o(1)$ as $|x| \to \infty$. According to Lemma 3 there are $p \geqslant 1$ and $R > 0$ such that $v \in L_p(\mathbb{R}^m \setminus \Omega_R(0))$. Since v is a solution of the third problem in the sense of distributions with the boundary condition $\mu - (\partial w/\partial n)\mathcal{H} - w\lambda$, which is finite and continuous up to the boundary, Theorem 2 yields that $v = \mathscr{U}\nu$, where ν is given by (9). If $u(x) = O(|x|^{k-1})$ as $|x| \to \infty$ then $w(x) = O(|x|^{k-1})$ as $|x| \to \infty$ and w is a harmonic polynomial of degree smaller than k by Lemma 3 and Lemma 2. □

Definition. Suppose that G has a locally Lipschitz boundary. Let $f \in L_\infty(\mathcal{H})$ be a nonnegative function. Let L be a bounded linear functional on $W^{1,2}(G)$ such that $L(\varphi) = 0$ for each $\varphi \in \mathscr{D}(G) = \{\varphi \in \mathscr{D};\ \operatorname{spt}\varphi \subset G\}$. We say that $u \in W^{1,2}(G)$ is a weak solution of the third problem

$$\Delta u = 0 \quad \text{on } G,$$
$$\frac{\partial u}{\partial n} + uf = L \quad \text{on } \partial G, \tag{10}$$

if

$$\int_G \nabla u \cdot \nabla v \,\mathrm{d}\mathcal{H}_m + \int_{\partial G} ufv \,\mathrm{d}\mathcal{H} = L(v) \tag{11}$$

for each $v \in W^{1,2}(G)$.

Lemma 4. *Suppose that G has a locally Lipschitz boundary, $\mu \in \mathcal{C}_c'(\partial G)$. Then there is a unique bounded linear functional L_μ on $W^{1,2}(G)$ such that*

$$L_\mu(\varphi) = \int_{\partial G} \varphi \,\mathrm{d}\mu \tag{12}$$

for each $\varphi \in \mathscr{D}$.

P r o o f. Fix a real number c such that $\mu(\partial G) - c\mathcal{H}(\partial G) = 0$. Since $c\mathcal{H} \in \mathcal{C}'_c(\partial G)$ there is a bounded linear functional L on $W^{1,2}(G)$ such that

$$L(\varphi) = \int_{\partial G} \varphi \, \mathrm{d}(\mu - c\mathcal{H})$$

for each $\varphi \in \mathscr{D}$ (see [18], Lemma 9). If we define $L_\mu(v) = L(v) + c\int v \, \mathrm{d}\mathcal{H}$ for $v \in W^{1,2}(G)$, then L_μ is a bounded linear operator on $W^{1,2}(G)$ satisfying (12). Since $\mathscr{D}$ is dense in $W^{1,2}(G)$, the bounded operator L_μ on $W^{1,2}(G)$ satisfying (12) is unique. □

Theorem 4. *Suppose that G has a locally Lipschitz boundary. Let $f \in L_\infty(\mathcal{H})$ be a nonnegative function. Let $\mu \in \mathcal{C}'_0(\partial G) \cap \mathcal{C}'_c(\partial G)$. If G is unbounded and $m \leqslant 4$ suppose moreover that $\mu(\partial H) = 0$ and $f = 0$ on ∂H, where H is the unbounded component of G. Then there is $u \in W^{1,2}(G)$ a weak solution of the third problem for the Laplace equation (10) with the boundary condition $L \equiv L_\mu$. Put $\lambda = f\mathcal{H}$. If $G_1, \ldots, G_k$ are all components of G such that $\lambda(\partial G_j) = 0$, then the general solution of this problem has the form (6), where ν is given by (7) and $c_j = 0$ for G_j unbounded and c_j is an arbitrary constant for G_j bounded.*

P r o o f. Let ν be given by (7). Then $N^G \mathscr{U}\nu + \mathscr{U}\nu\lambda = \mu$ and $\nu \in \mathcal{C}'_c(\partial G)$ by Theorem 2 and [18], Theorem 1. According to Corollary 1 we have $\mathscr{U}\nu \in W^{1,2}(G)$. For fixed $v \in W^{1,2}(G)$ choose $\varphi_n \in \mathscr{D}$ such that $\varphi_n \to v$ in $W^{1,2}(G)$ as $n \to \infty$. Then

$$\begin{aligned} L_\mu(v) &= \lim_{n\to\infty} \int \varphi_n \, \mathrm{d}\mu = \lim_{n\to\infty} \left[\int_G \nabla\varphi_n \cdot \nabla\mathscr{U}\nu \, \mathrm{d}\mathcal{H}_m + \int_{\partial G} \varphi_n f \mathscr{U}_c \nu \, \mathrm{d}\mathcal{H} \right] \\ &= \int_G \nabla v \cdot \nabla\mathscr{U}\nu \, \mathrm{d}\mathcal{H}_m + \int_{\partial G} v f \mathscr{U}_c \nu \, \mathrm{d}\mathcal{H}. \end{aligned}$$

$\mathscr{U}\nu$ is a weak solution of the third problem (10) with the boundary condition $L \equiv L_\mu$. If u has a form (6), where $c_j = 0$ for G_j unbounded, then u is a weak solution of this third problem.

Let $u \in W^{1,2}(G)$ be a weak solution of the third problem (10) with the boundary condition $L \equiv L_\mu$. Since $u - \mathscr{U}\nu \in W^{1,2}(G)$ we have

$$\begin{aligned} 0 &= \int_G \nabla u \cdot \nabla(u - \mathscr{U}\nu) \, \mathrm{d}\mathcal{H}_m + \int_{\partial G} f u (u - \mathscr{U}\nu) \, \mathrm{d}\mathcal{H} - \int_G \nabla\mathscr{U}\nu \cdot \nabla(u - \mathscr{U}\nu) \, \mathrm{d}\mathcal{H}_m \\ &\quad - \int_{\partial G} f \mathscr{U}\nu (u - \mathscr{U}\nu) \, \mathrm{d}\mathcal{H} \\ &= \int_G |\nabla(u - \mathscr{U}\nu)|^2 \, \mathrm{d}\mathcal{H}_m + \int_{\partial G} f (u - \mathscr{U}\nu)^2 \, \mathrm{d}\mathcal{H}. \end{aligned}$$

Since $\int |\nabla(u - \mathscr{U}\nu)|^2 \, \mathrm{d}\mathcal{H}_m \geqslant 0$, $\int f(u - \mathscr{U}\nu)^2 \, \mathrm{d}\mathcal{H} \geqslant 0$, we have $\int |\nabla(u - \mathscr{U}\nu)|^2 \, \mathrm{d}\mathcal{H}_m = 0$. Since $(u - \mathscr{U}\nu)$ is locally constant on G, u has the form (6). □

Theorem 5. *Suppose that G has a locally Lipschitz boundary. Let $f \in L_\infty(\mathcal{H})$ be a nonnegative function. Let L be a bounded linear functional on $W^{1,2}(G)$ and $\mu \in \mathcal{C}'(\partial G)$ be such that $L(\varphi) = \int \varphi \, d\mu$ for each $\varphi \in \mathscr{D}$. If $u \in W^{1,2}(G)$ is a weak solution of the third problem for the Laplace equation* (10) *then u is continuously extendible to the closure of G if and only if $\mu \in \mathcal{C}'_c(\partial G)$.*

Proof. Put $\lambda = f\mathcal{H}$. Since $N^G u + u\lambda = \mu$, [16], Theorem 1 yields that $\mu \in \mathcal{C}'_0(\partial G)$. If u is continuously extendible to the closure of G then $\mu \in \mathcal{C}'_c(\partial G)$ by Theorem 2. Suppose now that $\mu \in \mathcal{C}'_c(\partial G)$. If G is bounded put $\tilde{G} = G$, $\tilde{\mu} = \mu$. If G is unbounded fix $R > 0$ such that $\partial G \subset \Omega_R(0)$ and put $\tilde{G} = G \cap \Omega_R(0)$, $\tilde{\mu} = \mu + \frac{\partial u}{\partial n}(\mathcal{H}_{m-1}/\partial\Omega_R(0))$, $f = 0$ on $\partial\Omega_R(0)$. Since $V^G < \infty$ we have $V^{\tilde{G}} < \infty$. Since $r_{\mathrm{ess}}(N^G\mathscr{U} - \frac{1}{2}I) < \frac{1}{2}$ and $(N^H\mathscr{U} - \frac{1}{2}I)$ is compact for each bounded open set H with a smooth boundary (see [11], Theorem 4.1, Proposition 2.20, [29], Theorem 4.1), [15], Theorem 2.3 yields that $r_{\mathrm{ess}}(N^{\tilde{G}}\mathscr{U} - \frac{1}{2}I) < \frac{1}{2}$. Since $N^{\tilde{G}}u + u\lambda = \tilde{\mu}$, [16], Theorem 1 yields that $\tilde{\mu} \in \mathcal{C}'_0(\partial G)$. If G is unbounded then $(\partial u \partial n)(\mathcal{H}_{m-1}/\partial\Omega_R(0)) \in \mathcal{C}'_c(\partial\tilde{G})$ by [16], Remark 6 and therefore $\tilde{\mu} \in \mathcal{C}'_c(\partial\tilde{G})$. Since u is a weak solution of the third problem for the Laplace equation on $\tilde{G}$ with the boundary condition $L_{\tilde{\mu}}$

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \tilde{G}, \\ \frac{\partial u}{\partial n} + fu &= L_{\tilde{\mu}} \text{ on } \partial\tilde{G}, \end{aligned}$$

Theorem 4 and Theorem 2 yield that u is continuously extendible to the closure of $\tilde{G}$. □

Definition. Suppose that G has a locally Lipschitz boundary. Let $f \in L_\infty(\mathcal{H})$ be a nonnegative function. Let $g \in L_2(G)$ and let L be a bounded linear functional on $W^{1,2}(G)$ such that $L(\varphi) = 0$ for each $\varphi \in \mathscr{D}(G)$. We say that $u \in W^{1,2}(G)$ is a weak solution of the third problem for the Poisson equation

$$\begin{aligned} \Delta u &= g \quad \text{on } G, \\ \frac{\partial u}{\partial n} + uf &= L \quad \text{on } \partial G, \end{aligned} \tag{13}$$

if

$$\int_G \nabla u \cdot \nabla v \, d\mathcal{H}_m + \int_{\partial G} ufv \, d\mathcal{H} = L(v) - \int_G gv \, d\mathcal{H}_m \tag{14}$$

for each $v \in W^{1,2}(G)$.

Lemma 5. *Suppose that G has a locally Lipschitz boundary. Let $f \in L_\infty(\mathcal{H})$ be a nonnegative function. Let $g \in L_p(\mathbb{R}^m)$, where $p > m$, be a compactly supported function. If G is unbounded and $m \leqslant 4$ suppose moreover that*

$$\int_{\mathbb{R}^m} g \,\mathrm{d}\mathcal{H}_m = 0.$$

Then $\mathscr{U}(g\mathcal{H}_m) \in \mathcal{C}^1(\mathbb{R}^m) \cap W^{1,2}(\mathbb{R}^m)$. Put $\varrho \equiv [n^G \cdot \nabla \mathscr{U}(g\mathcal{H}_m) + \mathscr{U}(g\mathcal{H}_m) f]\mathcal{H}$. Then $\varrho \in \mathcal{C}_c'(\partial G)$ and $\mathscr{U}(g\mathcal{H}_m)$ is a weak solution solution of the third problem for the Poisson equation

$$\begin{aligned} \Delta u &= -g \quad \text{on } G, \\ \frac{\partial u}{\partial n} + uf &= L_\varrho \quad \text{on } \partial G. \end{aligned} \tag{15}$$

Proof. $\mathscr{U}(g\mathcal{H}_m) \in \mathcal{C}^1(\mathbb{R}^m)$ by [5], Theorem A.6 and Theorem A.11. An easy calculation yields that $\mathscr{U}(g\mathcal{H}_m) \in W^{1,2}(\mathbb{R}^m)$. Since $[n^G \cdot \nabla \mathscr{U}(g\mathcal{H}_m)] \in L_\infty(\mathcal{H})$, we have $[n^G \cdot \nabla \mathscr{U}(g\mathcal{H}_m]\mathcal{H} \in \mathcal{C}_c'(\partial G)$. Since $\mathscr{U}(g\mathcal{H}_m)\lambda \in \mathcal{C}_c'(\partial G)$ (see [22], Proposition 9), we have $\varrho \in \mathcal{C}_c'(\partial G)$.

Put

$$\varphi(x) = \begin{cases} C \exp[-1/(1-|x|^2)] & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geqslant 1, \end{cases}$$

where C is chosen so that $\int \varphi = 1$. For $\varepsilon > 0$ put $\varphi_\varepsilon(x) = \varepsilon^{-m}\varphi(x\varepsilon)$. Then $\varphi_\varepsilon * \mathscr{U}(g\mathcal{H}_m) \to \mathscr{U}(g\mathcal{H}_m)$, $\varphi_\varepsilon * \nabla\mathscr{U}(g\mathcal{H}_m) \to \nabla\mathscr{U}(g\mathcal{H}_m)$ locally uniformly as $\varepsilon \searrow 0$ (see [30], Theorem 1.6.1, [27], §12). If $v \in \mathscr{D}$ then the Divergence Theorem (see [11], p. 49) and [5], Theorem A.16 yield

$$\begin{aligned} \int_G \nabla\mathscr{U}(g\mathcal{H}_m) \cdot \nabla v \,\mathrm{d}\mathcal{H}_m + \int_{\partial G} \mathscr{U}(g\mathcal{H}_m) f v \,\mathrm{d}\mathcal{H} = \lim_{\varepsilon\to 0_+} \int_G \varphi_\varepsilon * \nabla(g * h_0) \cdot \nabla v \,\mathrm{d}\mathcal{H}_m \\ + \int_{\partial G} \mathscr{U}(g\mathcal{H}_m) f v \,\mathrm{d}\mathcal{H} = \lim_{\varepsilon\to 0_+} \int_G \nabla(\varphi_\varepsilon * g * h_0) \cdot \nabla v \,\mathrm{d}\mathcal{H}_m + \int_{\partial G} \mathscr{U}(g\mathcal{H}_m) f v \,\mathrm{d}\mathcal{H} \\ = \lim_{\varepsilon\to 0_+} \left\{ \int_{\partial G} n^G \cdot \nabla(\varphi_\varepsilon * g * h_0) v \,\mathrm{d}\mathcal{H} - \int_G \Delta(\varphi_\varepsilon * g * h_0) v \,\mathrm{d}\mathcal{H}_m \right\} + \int_{\partial G} \mathscr{U}(g\mathcal{H}_m) f v \,\mathrm{d}\mathcal{H} \\ = \lim_{\varepsilon\to 0_+} \left\{ \int_{\partial G} v n^G \cdot [\varphi_\varepsilon * \nabla(h_0 * g)] \,\mathrm{d}\mathcal{H} + \int_G (\varphi_\varepsilon * g) v \,\mathrm{d}\mathcal{H}_m \right\} + \int_{\partial G} \mathscr{U}(g\mathcal{H}_m) f v \,\mathrm{d}\mathcal{H} \\ = \int_G v g \,\mathrm{d}\mathcal{H}_m + L_\varrho(v). \end{aligned}$$

Since $\mathscr{D}$ is dense in $W^{1,2}(G)$, $\mathscr{U}(g\mathcal{H}_m)$ is a weak solution of the third problem for the Poisson equation (15). $\square$

Theorem 6. *Suppose that G has a locally Lipschitz boundary. Let $f \in L_\infty(\mathcal{H})$ be a nonnegative function. Let $g \in L_p(\mathbb{R}^m)$, where $p > m$, be a compactly supported function. Put $\lambda = f\mathcal{H}$. Denote by $G_1, \dots, G_k$ all bounded components of G such that $\lambda(\partial G_j) = 0$. Let $\mu \in \mathcal{C}'_c(\partial G)$ be such that*

$$\mu(\partial G_j) = \int_{G_j} g \, \mathrm{d}\mathcal{H}_m$$

for $j = 1, \dots, k$. If G is unbounded and $m \leqslant 4$ suppose moreover that

$$\int_{\mathbb{R}^m} g \, \mathrm{d}\mathcal{H}_m = 0,$$

$$\mu(\partial H) = \int_H g \, \mathrm{d}\mathcal{H}_m,$$

$\lambda(\partial H) = 0$ for the unbounded component H of G. Then there is $u \in W^{1,2}(G)$, a weak solution of the third problem for the Poisson equation (13) *with the boundary condition $L \equiv L_\mu$. The general form of this solution is*

$$u = \mathcal{U}\nu - \mathcal{U}(g\mathcal{H}_m) + \sum_{j=1}^{k} c_j \chi_{G_j}, \tag{16}$$

where

$$\nu = \sum_{n=0}^{\infty} \Big(- \frac{\tau - \alpha I}{\alpha} \Big)^n \frac{\tilde{\mu}}{\alpha}, \tag{17}$$

$$\tilde{\mu} = \mu + [n^G \cdot \nabla \mathcal{U}(g\mathcal{H}_m)]\mathcal{H} + \mathcal{U}(g\mathcal{H}_m)\lambda, \tag{18}$$

$$\alpha > \frac{1}{2}\Big(V^G + 1 + \sup_{x \in \partial G} \mathcal{U}\lambda(x) \Big).$$

Proof. Put

$$\varphi(x) = \begin{cases} C \exp[-1/(1 - |x|^2)] & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geqslant 1, \end{cases}$$

where C is chosen so that $\int \varphi = 1$. For $\varepsilon > 0$ put $\varphi_\varepsilon(x) = \varepsilon^{-m}\varphi(x\varepsilon)$. Since $\mathcal{U}(g\mathcal{H}_m) \in \mathcal{C}^1(\mathbb{R}^m)$ (see [5], Theorem A.6, Theorem A.11), $\varphi_\varepsilon * \mathcal{U}(g\mathcal{H}_m) \to \mathcal{U}(g\mathcal{H}_m)$, $\varphi_\varepsilon * \nabla\mathcal{U}(g\mathcal{H}_m) \to \nabla\mathcal{U}(g\mathcal{H}_m)$ locally uniformly as $\varepsilon \searrow 0$ (see [30], Theorem 1.6.1, [27], §12). The Divergence Theorem (see [11], p. 49) and [5], Theorem A.16

yield for $j \in \{1, \dots, k\}$

$$
\begin{aligned}
\tilde{\mu}(\partial G_j) &= \mu(\partial G_j) + \int_{\partial G_j} n^G(y) \cdot \nabla \mathscr{U}(g\mathcal{H}_m)(y) \, \mathrm{d}\mathcal{H}(y) \\
&= \mu(\partial G_j) + \lim_{\varepsilon \to 0_+} \int_{\partial G_j} n^G(y) \cdot (\varphi_\varepsilon * \nabla \mathscr{U}(g\mathcal{H}_m))(y) \, \mathrm{d}\mathcal{H}(y) \\
&= \mu(\partial G_j) + \lim_{\varepsilon \to 0_+} \int_{\partial G_j} n^G(y) \cdot \nabla[\varphi_\varepsilon * (h_0 * g)](y) \, \mathrm{d}\mathcal{H}(y) \\
&= \mu(\partial G_j) + \lim_{\varepsilon \to 0_+} \int_{\partial G_j} n^G(y) \cdot \nabla[h_0 * (\varphi_\varepsilon * g)](y) \, \mathrm{d}\mathcal{H}(y) \\
&= \mu(\partial G_j) + \lim_{\varepsilon \to 0_+} \int_{G_j} \Delta \mathscr{U}[(\varphi_\varepsilon * g)\mathcal{H}_m] \, \mathrm{d}\mathcal{H}_m \\
&= \mu(\partial G_j) - \lim_{\varepsilon \to 0_+} \int_{G_j} (\varphi_\varepsilon * g) \, \mathrm{d}\mathcal{H}_m \\
&= \mu(\partial G_j) - \int_{G_j} g \, \mathrm{d}\mathcal{H}_m = 0.
\end{aligned}
$$

If G is unbounded and $m \leqslant 4$ then [5], Theorem A.16 and the Divergence Theorem (see [11], p. 49) yield

$$
\begin{aligned}
\tilde{\mu}(\partial H) = & \lim_{R \to \infty} \Big\{ \lim_{\varepsilon \to 0_+} \int_{\partial(H \cap \Omega_R(0))} n^{H \cap \Omega_R(0)} \cdot [\varphi_\varepsilon * \nabla \mathscr{U}(g\mathcal{H}_m)] \, \mathrm{d}\mathcal{H}_{m-1} \\
& - \int_{\partial \Omega_R(0)} n^{\Omega_R(0)}(y) \cdot \nabla \mathscr{U}(g\mathcal{H}_m)(y) \, \mathrm{d}\mathcal{H}_{m-1}(y) \Big\} + \mu(\partial H) \\
= & \lim_{R \to \infty} \lim_{\varepsilon \to 0_+} \int_{\partial(H \cap \Omega_R(0))} n^{H \cap \Omega_R(0)} \cdot \nabla[h_0 * (\varphi_\varepsilon * g)] \, \mathrm{d}\mathcal{H}_{m-1} + \mu(\partial H) \\
= & \lim_{R \to \infty} \lim_{\varepsilon \to 0_+} \int_{H \cap \Omega_R(0)} \Delta \mathscr{U}[(\varphi_\varepsilon * g)\mathcal{H}_m] \, \mathrm{d}\mathcal{H}_m + \mu(\partial H) \\
= & - \lim_{R \to \infty} \lim_{\varepsilon \to 0_+} \int_{H \cap \Omega_R(0)} (\varphi_\varepsilon * g) \, \mathrm{d}\mathcal{H}_m + \mu(\partial H) \\
= & - \int_H g \, \mathrm{d}\mathcal{H}_m + \mu(\partial H) = 0.
\end{aligned}
$$

According to Theorem 4,

$$
\mathscr{U}\nu + \sum_{j=1}^{k} c_j \chi_{G_j}
$$

is a weak solution of the third problem for the Laplace equation (10) with the boundary condition $L \equiv L_{\tilde{\mu}}$. If u has the form (16) then Lemma 5 yields that u is a weak solution of the third problem for the Poisson equation (13) with the boundary condition $L \equiv L_\mu$.

Let now $u \in W^{1,2}(G)$ be a weak solution of the third problem for the Poisson equation (13) with the boundary condition $L \equiv L_\mu$. Then

$$w = u - \mathscr{U}\nu + \mathscr{U}(g\mathcal{H}_m)$$

is a weak solution of the third problem for the Laplace equation with the zero boundary condition. According to Theorem 4 the function w is locally constant and vanishes on $G \setminus (G_1 \cup \ldots \cup G_k)$. □

Theorem 7. *Suppose that G has a locally Lipschitz boundary. Let $f \in L_\infty(\mathcal{H})$ be a nonnegative function. Let $g \in L_2(G) \cap L_{p,\mathrm{loc}}(\mathbb{R}^m)$, where $p > m$. Let L be a bounded linear functional on $W^{1,2}(G)$ and $\mu \in \mathcal{C}'(\partial G)$ be such that $L(\varphi) = \int \varphi \, \mathrm{d}\mu$ for each $\varphi \in \mathscr{D}$. If $u \in W^{1,2}(G)$ is a weak solution of the third problem for the Poisson equation* (13) *then u is continuously extendible to the closure of G if and only if $\mu \in \mathcal{C}'_c(\partial G)$.*

P r o o f. Suppose first that G is bounded. Put $\lambda = f\mathcal{H}$. If H is a component of G such that $\lambda(\partial H) = 0$ fix $\varphi \in \mathscr{D}$ such that $\varphi = 1$ on H and $\varphi = 0$ on $G \setminus H$. Since u is a weak solution of (13), we have

$$\mu(\partial H) = L(\varphi) = \int_H g \, \mathrm{d}\mathcal{H}_m.$$

If $\mu \in \mathcal{C}'_c(\partial G)$ then u has the form (16) by Theorem 6. Since $\tilde{\mu}$ given by (18) is an element of $\mathcal{C}'_c(\partial G)$ (see Lemma 5), Theorem 2 and [18], Theorem 1 yield that ν given by (17) is an element of $\mathcal{C}'_c(\partial G)$, too. Since $\mathscr{U}(g\mathcal{H}_m) \in \mathcal{C}^1(\mathbb{R}^m)$ by [5], Theorem A.6 and Theorem A.11, u is continuously extendible to the closure of G.

Suppose now that u is continuously extendible to the closure of G. Put $\varrho \equiv -[n^G \cdot \nabla \mathscr{U}(g\mathcal{H}_m)]\mathcal{H} - \mathscr{U}(g\mathcal{H}_m)\lambda$. Lemma 5 yields that $u + \mathscr{U}(g\mathcal{H}_m)$ is a weak solution of the Neumann problem for the Laplace equation with the boundary condition $L - L_\varrho$, which is continuosly extendible to the closure of G. Since $(\mu - \varrho) \in \mathcal{C}'_c(\partial G)$ by Theorem 5 and $\varrho \in \mathcal{C}'_c(\partial G)$ by Lemma 5, we get $\mu \in \mathcal{C}'_c(\partial G)$.

Suppose now that G is unbounded. Fix $R > 0$ such that $\Omega_R(0) \cap \partial G = \emptyset$. Fix $z \in \mathbb{R}^m \setminus \mathrm{cl}\, G$, $r > 0$ such that $\Omega_{2r}(z) \cap G = \emptyset$. Put

$$\tilde{g}(x) = \begin{cases} g(x) & \text{for } x \in G \cap \Omega_{2R}(0), \\ -\dfrac{1}{\mathcal{H}_m(\Omega_r(z))} \displaystyle\int_{G \cap \Omega_{2R}(0)} g \, \mathrm{d}\mathcal{H}_m & \text{for } x \in \Omega_r(z), \\ 0 & \text{elsewhere.} \end{cases}$$

Put $\tilde{G} = G \cap \Omega_R(0)$. Define $f = 0$ on $\partial\Omega_R(0)$. Put $\varrho \equiv [n^G \cdot \nabla \mathscr{U}(\tilde{g}\mathcal{H}_m) + \mathscr{U}(\tilde{g}\mathcal{H}_m) f]\mathcal{H}$, $\tilde{\varrho} \equiv [n^{\tilde{G}} \cdot \nabla \mathscr{U}(\tilde{g}\mathcal{H}_m) + \mathscr{U}(\tilde{g}\mathcal{H}_m) f][\mathcal{H}_{m-1}/\partial\tilde{G}]$. Lemma 5 yields that

$\mathscr{U}(\tilde{g}\mathcal{H}_m) \in \mathcal{C}^1(\mathbb{R}^m) \cap W^{1,2}(\mathbb{R}^m)$ is a weak solution solution of the third problems for the Poisson equation

$$\begin{aligned} \Delta w &= -\tilde{g} \quad \text{on } G, \\ \frac{\partial w}{\partial n} + wf &= L_\varrho \quad \text{on } \partial G \end{aligned}$$

and

$$\begin{aligned} \Delta w &= -\tilde{g} \quad \text{on } \tilde{G}, \\ \frac{\partial w}{\partial n} + wf &= L_{\tilde{\varrho}} \quad \text{on } \partial \tilde{G}. \end{aligned}$$

Choose $\tilde{\varphi} \in \mathscr{D}$ so that $\tilde{\varphi} = 1$ on a neighbourhood of ∂G, $\operatorname{spt} \tilde{\varphi} \subset \Omega_R(0)$. For $v \in W^{1,2}(\tilde{G})$ define

$$\tilde{v}(x) = \begin{cases} v(x)\tilde{\varphi}(x) & \text{for } x \in \tilde{G}, \\ 0 & \text{for } x \in G \setminus \tilde{G}, \end{cases}$$

$$\tilde{L}(v) = L(\tilde{v}) - L_{\tilde{\varrho}}(v) + L_\varrho(\tilde{v}) + \int_{\partial \Omega_R(0)} v(y) \frac{y}{R} \cdot \nabla u(y) \, \mathrm{d}\mathcal{H}_{m-1}(y).$$

Choose $\varphi \in \mathscr{D}$ so that $\varphi = 1$ on a neighbourhood of $\operatorname{cl} \Omega_R(0)$, $\operatorname{spt} \varphi \subset \Omega_{2R}(0)$. Since $u + \mathscr{U}(\tilde{g}\mathcal{H}_m)$ is harmonic on $G \cap \Omega_{2R}(0)]$ we have for $v \in \mathscr{D}$

$$\begin{aligned} \int_{\tilde{G}} \nabla u \cdot \nabla v \, \mathrm{d}\mathcal{H}_m + \int_{\partial \tilde{G}} ufv \, \mathrm{d}\mathcal{H} &= -\int_{\tilde{G}} \nabla \mathscr{U}(\tilde{g}\mathcal{H}_m) \cdot \nabla v \, \mathrm{d}\mathcal{H}_m - \int_{\partial \tilde{G}} \mathscr{U}(\tilde{g}\mathcal{H}_m) fv \, \mathrm{d}\mathcal{H} \\ &+ \int_G \nabla[u + \mathscr{U}(\tilde{g}\mathcal{H}_m)] \cdot \nabla(\varphi v) \, \mathrm{d}\mathcal{H}_m - \int_{\Omega_{2R}(0) \setminus \Omega_R(0)} \nabla[u + \mathscr{U}(\tilde{g}\mathcal{H}_m)] \cdot \nabla(\varphi v) \, \mathrm{d}\mathcal{H}_m \\ &+ \int_{\partial G} [u + \mathscr{U}(\tilde{g}\mathcal{H}_m)] f \varphi v \, \mathrm{d}\mathcal{H} = -L_{\tilde{\varrho}}(v) - \int_{\tilde{G}} gv \, \mathrm{d}\mathcal{H}_m + L_\varrho(\varphi v) + L(\varphi v) \\ &+ \int_{\partial \Omega_R(0)} v(y) \frac{y}{R} \cdot \nabla[u + \mathscr{U}(\tilde{g}\mathcal{H}_m)](y) \, \mathrm{d}\mathcal{H}_{m-1}(y) = \tilde{L}(v) - \int_{\tilde{G}} gv \, \mathrm{d}\mathcal{H}_m. \end{aligned}$$

Since $\mathscr{D}$ is dense in $W^{1,2}(\tilde{G})$, u is a weak solution of the third problem for the Poisson equation

$$\begin{aligned} \Delta u &= g \quad \text{on } \tilde{G}, \\ \frac{\partial u}{\partial n} + uf &= \tilde{L} \quad \text{on } \partial \tilde{G}. \end{aligned}$$

If u is continuously extendible to $\operatorname{cl} G$ then $[yR^{-1} \cdot \nabla u(y)][\mathcal{H}_{m-1}/\partial \Omega_R(0)] + \mu - \tilde{\varrho} + \varrho \in \mathcal{C}'_c(\partial \tilde{G})$. Since $yR^{-1} \cdot \nabla u(y) \in L_\infty(\mathcal{H}_{m-1}/\partial \Omega_R(0))$ we have $[yR^{-1} \cdot$

$\nabla u(y)][\mathcal{H}_{m-1}/\partial\Omega_R(0)] \in \mathcal{C}'_c(\partial\Omega_R(0))$. Therefore $\mu \in \mathcal{C}'_c(\partial G)$, because $\varrho \in \mathcal{C}'_c(\partial G)$, $\tilde{\varrho} \in \mathcal{C}'_c(\partial(\tilde{G}))$ by Lemma 5.

Let now $\mu \in \mathcal{C}'_c(\partial G)$. According to Lemma 5 we have $\varrho \in \mathcal{C}'_c(\partial G)$, $\tilde{\varrho} \in \mathcal{C}'_c(\partial(\tilde{G}))$. Since $yR^{-1} \cdot \nabla u(y) \in L_\infty(\mathcal{H}_{m-1}/\partial\Omega_R(0))$ we have $\mu - \tilde{\varrho} + \varrho + [yR^{-1} \cdot \nabla u(y)][\mathcal{H}_{m-1}/\partial\Omega_R(0)] \in \mathcal{C}'_c(\partial\tilde{G})$. Therefore u is continuously extendible to the closure of $\tilde{G}$. Since $R > \operatorname{dist}(0, \partial G)$ was arbitrary, u is continuously extendible to the closure of G. □

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