Danica Jakubíková-Studenovská
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Czechoslovak Mathematical Journal, Vol. 53 (2003), No. 3, 655–668

Persistent URL: http://dml.cz/dmlcz/127831

Czechoslovak Mathematical Journal, 53 (128) (2003), 655–668

PARTIALLY-2-HOMOGENEOUS MONOUNARY ALGEBRAS

Danica Jakubíková-Studenovská, Košice

(Received October 13, 2000)

Abstract. This paper is a continuation of [5], where k-homogeneous and k-set-homogeneous algebras were defined. The definitions are analogous to those introduced by Fraïssé [2] and Droste, Giraudet, Macpherson, Sauer [1] for relational structures. In [5] we found all 2-homogeneous and all 2-set-homogeneous monounary algebras when the homogenity is considered with respect to subalgebras, to connected subalgebras and with respect to connected partial subalgebras, respectively. The results of [3], where all homogeneous monounary algebras were characterized, were applied in [4] for 1-homogeneity.

The aim of the present paper is to describe all monounary algebras which are 2-homogeneous and 2-set-homogeneous with respect to partial subalgebras, respectively; we will say that they are partially-2-homogeneous and partially-2-set-homogeneous.

Keywords: monounary algebra, 2-homogeneous, 2-set-homogeneous, partially-2-homogeneous, partially-2-set-homogeneous

MSC 2000: 08A60

1. Preliminaries

We will apply notions and definitions from [5]; let us recall some of them.

Let $A = (A, f)$ be a monounary algebra. Let $\emptyset \neq B \subseteq A$ and let $B = (B, f_B)$ be a partial monounary algebra such that whenever $b \in B$, then $b \in \operatorname{dom} f_B$ if and only if $f(b) \in B$, and then $f_B(b) = f(b)$. We will say that B is a partial subalgebra of A. The system of all 2-element partial subalgebras of A is denoted by the symbol $P_2(A)$.

The algebra A is said to be 2*-set-homogeneous with respect to partial subalgebras* or *partially-*2*-set-homogeneous* if, whenever $U, V \in P_2(A)$, $U \cong V$, then there is an automorphism φ of A with $\varphi(U) = V$. Also, A is called 2*-homogeneous with respect to partial subalgebras* or *partially-*2*-homogeneous* if every isomorphism between $U, V \in P_2(A)$ can be extended to an automorphism of A.

Supported by grant VEGA 1/7468/20.

Let us denote by $\mathscr{H}_2(P)$ the class of all monounary algebras which are partially-2-homogeneous and by $\mathscr{S}h_2(P)$ the class of all partially-2-set-homogeneous monounary algebras.

The following assertion is obvious:

1.1. Lemma. $\mathscr{H}_2(P) \subseteq \mathscr{S}h_2(P)$.

1.2. Notation. *Let λ, α be cardinals, $\lambda > 0$. We denote by $M_{\lambda\alpha} = (M_{\lambda\alpha}, f)$ a fixed monounary algebra such that*

(a) *there is $c \in M_{\lambda\alpha}$ with $f(c) = c$,*

(b) *if $x \in M_{\lambda\alpha}$, then $f^2(x) = c$,*

(c) $\operatorname{card} f^{-1}(c) - \{c\} = \lambda$,

(d) *if $a \in f^{-1}(c) - \{c\}$, then $\operatorname{card} f^{-1}(a) = \alpha$.*

We will write also M_λ instead of $M_{\lambda 0}$.

1.3. Notation. *For $\alpha \in \mathbb{N}$ let $Z_\alpha = (Z_\alpha, f)$ be a monounary algebra such that $Z_\alpha = \{0, 1, \ldots, \alpha - 1\}$, $f(i) \equiv i + 1 \pmod{\alpha}$ for each $i \in Z_\alpha$.*

2. The class $\mathscr{S}h_2(P)$—necessary conditions

In this section let $A = (A, f)$ be a monounary algebra belonging to $\mathscr{S}h_2(P)$.

2.1. Lemma. *There do not exist distinct elements $a, b, c, d \in A$ such that $f(a) = b$, $f(b) = c$, $f(c) = d$ and $f(d) \neq a \neq f^2(d)$.*

Proof. Assume that such elements exist. First suppose that $f(d) \neq b$. Take $U = \{b, d\}$, $V = \{a, d\}$. Then $U, V \in P_2(A)$, $U \cong V$, thus there is $\varphi \in \operatorname{Aut} A$ with $\varphi(U) = V$. If $\varphi(b) = a$, then

$$\varphi(d) = \varphi(f^2(b)) = f^2(\varphi(b)) = f^2(a) = c \neq d,$$

a contradiction. If $\varphi(b) = d$, then

$$a = \varphi(d) = \varphi(f^2(b)) = f^2(\varphi(b)) = f^2(a) = c \neq a,$$

which is a contradiction, too.

Now let $f(d) = b$. Then the partial monounary algebras defined on $\{d, b\}$ and on $\{a, b\}$ are isomorphic, but there is no automorphism ψ of A with $\psi(\{d, b\}) = \{a, b\}$, since if $\psi(b) = b$, then

$$a = \psi(d) = \psi(f^2(b)) = f^2(\psi(b)) = f^2(b) = d$$

and if $\psi(b) = a$, then

$$b = \psi(d) = \psi(f^2(b)) = f^2(a) = c.$$

$\square$

2.2. Corollary. *Each connected component of A contains a cycle and each cycle has at most 5 elements.*

2.3. Corollary. *If C is a cycle of A, $\operatorname{card} C > 2$, then $f^{-1}(C) - C = \emptyset$.*

2.4. Corollary. *If C is a cycle of A, $\operatorname{card} C = 2$, then $f^{-1}(f^{-1}(C) - C) = \emptyset$.*

2.5. Corollary. *If C is a cycle of A, $\operatorname{card} C = 1$, then $f^{-2}(f^{-1}(C) - C) = \emptyset$.*

2.6. Lemma. *If B is a connected component of A and a, b, c are distinct elements of B such that $f(a) = b$, $f(b) = c = f(c)$, then $B \cong M_{1\alpha}$ for some $\alpha \geqslant 1$.*

Proof. Let the assumption hold and suppose that B is not isomorphic to $M_{1\alpha}$ for any $\alpha \geqslant 1$. In view of 2.5 there is $d \in B - \{b, c\}$ such that $f(d) = c$. Take $U = \{b, d\}$, $V = \{a, d\}$. Then $U, V \in P_2(A)$, $U \cong V$, thus there is $\varphi \in \operatorname{Aut} A$ with $\varphi(U) = V$. Then either $\varphi(d) = a$ or $\varphi(b) = a$, which implies either

$$\varphi(c) = \varphi(f(d)) = f(\varphi(d)) = f(a) = b$$

or

$$\varphi(c) = \varphi(f(b)) = f(\varphi(b)) = f(a) = b,$$

i.e., $\varphi(c) = b$, which is a contradiction. $\square$

2.7. Lemma. *Let there be distinct elements $a, b, c \in A$ such that $f(a) = f(c) = b$, $f(b) = c$. Then $A = \{a, b, c\}$.*

Proof. Let $d \in A - \{a, b, c\}$. By 2.4, $f(d) \neq a$.

First suppose that $f(d) \neq d$. Put $U = \{a, d\}$, $V = \{a, c\}$. Then $U, V \in P_2(A)$, $U \cong V$ and there is $\varphi \in \operatorname{Aut} A$ such that either $\varphi(a) = a$, $\varphi(d) = c$ or $\varphi(a) = c$, $\varphi(d) = a$. In the first case,

$$\varphi(d) = c = f^2(a) = f^2(\varphi(a)) = \varphi(f^2(a)) = \varphi(c),$$

and in the second case,

$$\varphi(a) = c = f^2(a) = f^2(\varphi(a)) = \varphi(f^2(a)) = \varphi(c),$$

thus φ is not bijective, which is a contradiction.

Now suppose that $f(d) = d$. Take $U = \{b, d\}$, $V = \{a, d\}$. Then $U, V \in P_2(A)$, $U \cong V$, thus there is $\varphi \in \operatorname{Aut} A$ with $\varphi(U) = V$. Since b belongs to a 2-element cycle and d to a 1-element cycle, we obtain $\varphi(b) \neq d$. Hence $\varphi(b) = a$, which is a contradiction as well. □

2.8. Lemma. *Let C be a 3-element cycle of A. Further, let B be a connected component of A such that B has a cycle with less than 3 elements. Then* $\operatorname{card} B \leqslant 2$.

P r o o f. Suppose that $\operatorname{card} B > 2$. Then the cycle of B has only 1 element according to 2.7. Therefore there exist distinct elements $b_1, b_2 \in B$ such that either

$$b_1 \neq f(b_1) = f(b_2) \neq b_2 \tag{1}$$

or

$$f(b_1) = b_2, \quad f(b_2) \notin \{b_1, b_2\}. \tag{2}$$

Let $c \in C$. First let (1) hold. Take $U = \{c, b_1\}$, $V = \{b_1, b_2\}$. Then $U, V \in P_2(A)$, $U \cong V$, but there is no $\varphi \in \operatorname{Aut} A$ with $\varphi(c) \in \{b_1, b_2\}$, which is a contradiction, since a 3-element cycle would be mapped into a 1-element cycle.

Suppose that (2) is valid. Put $U = \{c, f(c)\}$, $V = \{b_1, b_2\}$. Then $U, V \in P_2(A)$, $U \cong V$ and there is $\varphi \in \operatorname{Aut} A$ with $\varphi(U) = V$. Thus $\varphi(c) \in \{b_1, b_2\}$, a contradiction. □

2.9. Lemma. *Let $a, b, c \in A$ be distinct, $f(a) = b$, $f(b) = c = f(c)$. Then A is connected.*

P r o o f. Suppose that A is not connected, i.e., there is $d \in A$ such that c and d do not belong to the same connected components of A.

First suppose that that $f(d) \neq d$. Take $U = \{d, c\}$, $V = \{a, c\}$. Then $U, V \in P_2(A)$, $U \cong V$ and there is $\varphi \in \operatorname{Aut} A$ with $\varphi(U) = V$. If $\varphi(d) = c$, then

$$\varphi(c) = c = f^2(a) = f^2(\varphi(d)) = \varphi(f^2(d)),$$

thus $c = f^2(d)$, a contradiction. The case $\varphi(d) = c$, $\varphi(c) = a$ yields a contradiction as well.

Now suppose that $f(d) = d$. Let $U = \{b, d\}$, $V = \{a, d\}$. Then $U, V \in P_2(A)$, $U \cong V$, thus there is $\varphi \in \operatorname{Aut} A$ such that $\varphi(U) = V$. Obviously, $\varphi(d) \neq a$, therefore $\varphi(d) = d$, $\varphi(b) = a$, which is a contradiction. □

2.10. Lemma. *Let C be a cycle of A,* $\operatorname{card} C > 3$. *Then $f(x) = x$ for each $x \in A - C$.*

Proof. There exist distinct elements $a, b, c \in C$ with $f(a) = b$, $f(b) = c$. By 2.3, C is a connected component of A. Suppose that there is $d \in A - C$ such that $f(d) \neq d$. If we take $U = \{d, c\}$, $V = \{a, c\}$, then $U, V \in P_2(A)$, $U \cong V$ and there is $\varphi \in \operatorname{Aut} A$ with $\varphi(U) = V$. Thus $\varphi(d) \in C$ and $\varphi(C) = C$, therefore φ is not bijective, which is a contradiction. □

2.11. Lemma. *Let $a, b, c \in A$ be distinct, $f(a) = f(b) = f(c) = c$. If B is a connected component, $c \notin B$, then* $\operatorname{card} B = 1$.

Proof. Assume that $c \notin B$ and that there are $e, d \in B$, $e \neq d$ such that $f(e) = d$. Let $U = \{a, b\}$, $V = \{a, e\}$. Then $U, V \in P_2(A)$, $U \cong V$ and there is $\varphi \in \operatorname{Aut} A$ with $\varphi(U) = V$. If $\varphi(a) = a$, $\varphi(b) = e$, then

$$d = f(e) = f(\varphi(b)) = \varphi(f(b)) = \varphi(c) = \varphi(f(a)) = f(\varphi(a)) = f(a) = c,$$

which is a contradiction. If $\varphi(a) = e$, $\varphi(b) = a$, then

$$c = f(a) = f(\varphi(b)) = \varphi(f(b)) = \varphi(c) = \varphi(f(a)) = f(\varphi(a)) = f(e) = d,$$

a contradiction. □

2.12. Lemma. *Let B_1, B_2, B_3 be distinct connected components of A which have more than 1 element. Then $B_1 \cong B_2 \cong B_3$.*

Proof. There are $a \in B_1$, $b \in B_2$, $c \in B_3$ with $f(a) \neq a$, $f(b) \neq b$, $f(c) \neq c$. Suppose that e.g. B_1 is not isomorphic to B_2. Take $U = \{a, b\}$, $V = \{b, c\}$. Then $U, V \in P_2(A)$, $U \cong V$ and there is $\varphi \in \operatorname{Aut} A$ with $\varphi(U) = V$. Since B_1 is not isomorphic to B_2, $\varphi(a) \neq b$, thus $\varphi(a) = c$, $\varphi(b) = b$. The relation $\varphi(a) = c$ implies $B_1 \cong B_3$. Let $U' = \{a, b\}$, $V' = \{a, c\}$. Then $U', V' \in P_2(A)$, $U' \cong V'$. Hence there is $\psi \in \operatorname{Aut} A$ with $\psi(U) = V$. We have either $\psi(b) = a$ or $\psi(b) = c$, which yields that either $B_1 \cong B_2$ or $B_2 \cong B_3$. But $B_3 \cong B_1$, therefore $B_1 \cong B_2$, which is a contradiction. □

2.13. Lemma. *Let $a, b, c \in A$ be distinct, $f(a) = f(b) = f(c) = c$. If $p, q \in A$, $f(p) = p$, $f(q) = q$, then* $\operatorname{card}\{c, p, q\} \leqslant 2$.

Proof. Assume that c, p, q are distinct elements of A and that $f(p) = p$, $f(q) = q$. By 2.11, $\{p\}$ and $\{q\}$ are connected components of A. Consider $U = \{c, p\}$, $V = \{p, q\}$. Then $U, V \in P_2(A)$, $U \cong V$ and there is $\varphi \in \operatorname{Aut} A$ with $\varphi(U) = V$. We obtain $\varphi(c) \in \{p, q\}$, which yields a contradiction, since the connected component containing c has more than one element and cannot be embedded into a component $\{p\}$ or $\{q\}$. □

2.14. Lemma. *Let $a, b, c, d \in A$ be distinct and $f(a) = f(b) = b, f(d) = f(c) = c$. Then there is no one-element connected component of A.*

Proof. Suppose that there is $p \in A$ such that $\{p\}$ is a connected component of A. Let $U = \{p, c\}$, $V = \{b, c\}$. Then $U, V \in P_2(A)$, $U \cong V$ and there is $\varphi \in \operatorname{Aut} A$ with $\varphi(U) = V$, which implies $\varphi(p) \in \{c, b\}$, and this is a contradiction. □

2.15. Lemma. *Let c, d be distinct elements of A such that $f(d) = f(c) = c$. Then there is at most one 1-element connected component of A.*

Proof. Suppose that there are $a, b \in A$ such that $a \neq b$ and $\{a\}$, $\{b\}$ are 1-element connected components of A. If we take $U = \{a, c\}$, $V = \{a, b\}$, then $U, V \in P_2(A)$, $U \cong V$ and there is $\varphi \in \operatorname{Aut} A$ with $\varphi(c) \in \{a, b\}$, a contradiction. □

2.16. Lemma. *Let $a, b, c, d \in A$ be distinct and $f(a) = f(b) = b, f(d) = c$, $f(c) = d$. Then there is no one-element connected component of A.*

Proof. Suppose that $\{p\}$ is a connected component and put $U = \{p, a\}, V = \{p, c\}$. Then $U \cong V$. If $\varphi \in \operatorname{Aut} A$, then $\varphi(a) \neq c$. Further, the relation $\varphi(a) = p$ implies $\varphi(b) = p = \varphi(a)$, a contradiction. □

In 2.17 and 2.18 we can repeat the steps of the proof of 2.14; therefore we have:

2.17. Lemma. *Let a, b, c, d, e be distinct elements of A, $f(a) = b$, $f(b) = d$, $f(d) = a$, $f(c) = e$, $f(e) = c$. Then there is no one-element connected component of A.*

2.18. Lemma. *Let a, b, c, d, e be distinct elements of A, $f(a) = b$, $f(b) = d$, $f(d) = a$, $f(c) = f(e) = e$. Then there is no one-element connected component of A.*

3. The class $\mathscr{H}_2(P)$—auxiliary results

In this section we will give some sufficient conditions under which a monounary algebra belongs to the class $\mathscr{H}_2(P)$.

Let $A = (A, f)$ be a monounary algebra.

3.1.1. Lemma. *Let A be a cycle with 4 elements. Then $A \in \mathscr{H}_2(P)$.*

P r o o f. Assume that $A = \{c_1, c_2, c_3, c_4\}$, $f(c_1) = c_2$, …, $f(c_4) = c_1$. Consider $U, V \in P_2(A)$ such that $U \cong V$. Without loss of generality, one of the following conditions is satisfied:

(1) $U = \{c_1, c_3\}$, $V = \{c_2, c_4\}$,

(2) $U = \{c_1, c_2\}$, $V = \{c_2, c_3\}$,

(3) $U = \{c_1, c_2\}$, $V = \{c_3, c_4\}$,

(4) $U = \{c_1, c_3\} = V$,

(5) $U = \{c_1, c_2\} = V$.

Let φ be an isomorphism of U onto V, $\varphi \neq \mathrm{id}_U$. Then (5) fails to hold.

First let (1) be valid. If $\varphi(c_1) = c_2$, $\varphi(c_3) = c_4$, then $\overline{\varphi} = f$ is an extension of φ and $\overline{\varphi} \in \operatorname{Aut} A$. If $\varphi(c_1) = c_4$, $\varphi(c_3) = c_2$, then we can take $\overline{\varphi} = f^3$; then $\overline{\varphi} \in \operatorname{Aut} A$ and $\overline{\varphi}$ is an extension of φ.

Assume that (2) is satisfied. Then $\varphi(c_1) = c_2$, $\varphi(c_2) = c_4$ and φ can be extended by putting $\overline{\varphi} = f$. If (3) holds, then $\varphi(c_1) = c_3$, $\varphi(c_2) = c_4$ and we can put $\overline{\varphi} = f^2$. Let (4) be valid. Then $\varphi(c_1) = c_3$, $\varphi(c_3) = c_1$ and $\overline{\varphi} = f^2 \in \operatorname{Aut} A$ is an extension of φ. Therefore $A \in \mathscr{H}_2(P)$. □

3.1.2. Lemma. *Let C be a cycle of A such that $\operatorname{card} C = 4$ and $f(x) = x$ for each $x \in A - C$. Then $A \in \mathscr{H}_2(P)$.*

P r o o f. Assume that $C = \{c_1, c_2, c_3, c_4\}$, $f(c_1) = c_2, \ldots, f(c_4) = c_1$. Further suppose that U, V are elements of $P_2(A)$ such that $U \cong V$. One of the following cases occurs:

(1) $U, V \subseteq C$,

(2) $U, V \subseteq A - C$,

(3) $U = \{a, c_i\}$, $V = \{b, c_j\}$, where $a, b \in A - C$, $c_i, c_j \in C$.

Let φ be an isomorphism of U onto V, $\varphi \neq \mathrm{id}_U$. If (1) is valid, then φ can be extended analogously as in 3.1.1. Let (2) hold. Then $U = \{u_1, u_2\}$, $V = \{v_1, v_2\}$ and $\varphi(u_1) = v_1$, $\varphi(u_2) = v_2$. If $u_1 = v_1$, then $\varphi \neq \mathrm{id}_U$ implies $u_2 \neq v_2 \neq v_1$; put

$$\overline{\varphi}(x) = \begin{cases} v_2 & \text{if } x = u_2, \\ u_2 & \text{if } x = v_2, \\ x & \text{otherwise.} \end{cases}$$

Then $\overline{\varphi}$ is an extension of φ and $\overline{\varphi} \in \operatorname{Aut} A$. The case $u_1 \neq v_1$, $u_2 = v_2$ is analogous. If $v_2 = u_1$, $v_1 = u_2$, then it is obvious that we can define $\overline{\varphi}$ as above. If u_1, u_2, v_1,

v_2 are mutually distinct, then we set

$$\overline{\varphi}(x) = \begin{cases} v_1 & \text{if } x = u_1, \\ u_1 & \text{if } x = v_1, \\ v_2 & \text{if } x = u_2, \\ u_2 & \text{if } x = v_2, \\ x & \text{otherwise} \end{cases}$$

and we obtain an extension $\overline{\varphi}$ of φ such that $\overline{\varphi} \in \operatorname{Aut} A$.

Now suppose that (3) is valid. Then clearly $\varphi(a) \neq c_j$, whence $\varphi(a) = b$, $\varphi(c_i) = c_j$. Put

$$\overline{\varphi}(x) = \begin{cases} a & \text{if } x = b, \\ b & \text{if } x = a, \\ f^k(c_j) & \text{if } x = f^k(c_i), \ \ k \in \mathbb{N}, \\ x & \text{otherwise.} \end{cases}$$

Then $\overline{\varphi}$ is an extension of φ and $\overline{\varphi} \in \operatorname{Aut} A$. Thus we have proved that $A \in \mathscr{H}_2(P)$. □

3.2.1. Lemma. *If A is connected and* $\operatorname{card} A \leqslant 3$, *then* $A \in \mathscr{S}h_2(P)$.

P r o o f. Let A be connected. The assertion is obvious if $\operatorname{card} A = 2$, thus assume that card $A = 3$. Then either A is a 3-element cycle or A contains a cycle with less than 3 elements. Let $U, V \in P_2(A)$ and let $\varphi \neq \mathrm{id}_U$ be an isomorphism of U onto V. Then A is a 3-element cycle and there is $u \in A$ such that $U = \{u, f(u)\}$, $V = \{f(u), f^2(u)\}$ or $U = \{u, f(u)\}$, $V = \{f^2(u), u\}$. Then either $\overline{\varphi} = f$ or $\overline{\varphi} = f^2$ is an automorphism of A which is an extension of φ. Therefore $A \in \mathscr{S}h_2(P)$. □

3.2.2. Lemma. *Let A consist of k 2-element cycles and of m 1-element cycles, $(k, m) \neq (0, 0)$, $k \geqslant 0$, $m \geqslant 0$. Then $A \in \mathscr{H}_2(P)$.*

P r o o f. Consider $U, V \in P_2(A)$ such that $U \cong V$. One of the following conditions is satisfied:

(1) U, V are 2-element cycles,

(2) $U = \{u_1, u_2\}$, $V = \{v_1, v_2\}$, where u_1, u_2, v_1, v_2 are 1-element cycles,

(3) $U = \{a, u\}$, $V = \{b, v\}$, where $f(a) \neq a$, $f(u) = u$, $f(b) \neq b$, $f(v) = v$.

Let $\varphi \neq \mathrm{id}_U$ be an isomorphism of U onto V. First assume that (1) is valid. Then $\overline{\varphi}$ defined by the formula

$$\overline{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } x \in U, \\ \varphi^{-1}(x) & \text{if } x \in V, \\ x & \text{otherwise} \end{cases}$$

belongs to $\operatorname{Aut} A$ and it is an extension of φ. If (2) is valid, then we proceed analogously as in 3.1.2, case (2). Let (3) hold. Then $\varphi(a)=b$, $\varphi(u)=v$; let us put

$$\overline{\varphi}(x)=\begin{cases} f^i(b) & \text{if } x=f^i(a),\ \ i\in\{0,1\},\\ f^i(a) & \text{if } x=f^i(b),\ \ i\in\{0,1\},\\ u & \text{if } x=v,\\ v & \text{if } x=u,\\ x & \text{otherwise.}\end{cases}$$

Then $\overline{\varphi}$ is an extension of φ and $\overline{\varphi}\in\operatorname{Aut} A$. Therefore $A\in\mathscr{H}_2(P)$. □

3.2.3. Lemma. *Let A consist of k 3-element cycles and of m 1-element cycles, $k>0$, $m\geqslant 0$. Then $A\in\mathscr{H}_2(P)$.*

Proof. Let $U,V\in P_2(A)$, $U\cong V$. One of the following cases occurs:
(1) U, V are subsets of one 3-element cycle,
(2) $U=\{a,f(a)\}$, $V=\{b,f(b)\}$, a,b belong to distinct 3-element cycles,
(3) $U=\{u_1,u_2\}$, $V=\{v_1,v_2\}$, where u_1, u_2, v_1, v_2 are 1-element cycles,
(4) $U=\{a,u\}$, $V=\{b,v\}$, where $f(a)\neq a$, $f(u)=u$, $f(b)\neq b$, $f(v)=v$.
Let $\varphi\neq\mathrm{id}_U$ be an isomorphism of U onto V. If (1) is valid, then φ can be extended analogously as in 3.2.1. If (2), (3) or (4) holds, then φ can be extended analogously as in 3.2.2, cases (1), (2) or (3), respectively. Thus we obtain that $A\in\mathscr{H}_2(P)$. □

3.3. Lemma. *Let $A\cong M_\alpha$, $\alpha\geqslant 1$. Then $A\in\mathscr{H}_2(P)$.*

Proof. We assume that there is $c\in A$ with $f(x)=c$ for each $x\in A$, $\operatorname{card} A\geqslant 2$. Let $U,V\in P_2(A)$ be such that $U\cong V$. One of the following two conditions is satisfied:
(1) $U=\{a,c\}$, $V=\{b,c\}$ for some $a,b\in A-\{c\}$,
(2) $U=\{u_1,u_2\}$, $V=\{v_1,v_2\}$, $u_1,u_2,v_1,v_2\in A-\{c\}$.
Let $\varphi\neq\mathrm{id}_U$ be an isomorphism of U onto V. If (1) is valid, then put

$$\overline{\varphi}(x)=\begin{cases} b & \text{if } x=a,\\ a & \text{if } x=b,\\ x & \text{otherwise;}\end{cases}$$

we obtain that $\overline{\varphi}$ is an extension of φ and $\overline{\varphi}\in\operatorname{Aut} A$. If (2) is satisfied, then we proceed analogously as in the proof of 3.1.2, case (2). Therefore $A\in\mathscr{H}_2(P)$. □

3.4. Lemma. *Suppose that $A \cong M_{1\alpha}$ for some $\alpha \geqslant 1$. Then $A \in \mathscr{H}_2(P)$.*

Proof. By the assumption, there are distinct $b, c \in A$ with $f(b) = f(c) = c$ and $f(x) = b$ for each $x \in A - \{b, c\}$. Let $U, V \in P_2(A)$, $U \cong V$. Then we have one of the following possibilities:

(1) $U = \{a, b\}$, $V = \{d, b\}$, $a, d \in A - \{b, c\}$,
(2) $U = \{a, c\}$, $V = \{d, c\}$, $a, d \in A - \{b, c\}$,
(3) $U = \{u_1, u_2\}$, $V = \{v_1, v_2\}$, $\{u_1, u_2, v_1, v_2\} \subseteq A - \{b, c\}$.

Then each isomorphism φ of U onto V can be extended to $\overline{\varphi} \in \operatorname{Aut} A$, thus $A \in \mathscr{H}_2(P)$. $\square$

3.5. Lemma. *Suppose that each connected component of A has 2 elements and it is not a cycle. Then $A \in \mathscr{H}_2(P)$.*

Proof. Let $U, V \in P_2(A)$, $U \cong V$. Let C be the set of all $x \in A$ with $f(x) = x$, $B = A - C$. One of the following conditions is satisfied:

(1) $U = \{a, f(a)\}$, $V = \{b, f(b)\}$, $\{a, b\} \subseteq B$,
(2) $U = \{u_1, u_2\}$, $V = \{v_1, v_2\}$ and
either $\{u_1, u_2, v_1, v_2\} \subseteq B$ or $\{u_1, u_2, v_1, v_2\} \subseteq C$,
(3) $U = \{a_1, c_1\}$, $V = \{a_2, c_2\}$, $\{a_1, a_2\} \subseteq B$, $\{c_1, c_2\} \subseteq C$, $f(a_1) \neq c_1$, $f(a_2) \neq c_2$.

Let $\varphi \neq \mathrm{id}_U$ be an isomorphism of U onto V. If (1) is valid, then it is obvious that φ can be extended to $\overline{\varphi} \in \operatorname{Aut} A$. In the case (2) we denote by u_1', u_2', v_1', v_2' the elements of the connected components of A which contain the elements u_1, u_2, v_1, v_2, respectively, such that $u_1' \neq u_1$, $u_2' \neq u_2$, $v_1' \neq v_1$, $v_2' \neq v_2$. Let $\varphi(u_1) = v_1$, $\varphi(u_2) = v_2$. Then we proceed analogously as in 3.1.2, e.g., if $u_1 = v_1$, $u_2 \neq v_2$, then we can put

$$\overline{\varphi}(x) = \begin{cases} u_2 & \text{if } x = v_2, \\ u_2' & \text{if } x = v_2', \\ v_2 & \text{if } x = u_2, \\ v_2' & \text{if } x = u_2', \\ x & \text{otherwise;} \end{cases}$$

then $\overline{\varphi}$ is an extension of φ and $\overline{\varphi} \in \operatorname{Aut} A$.

Suppose that (3) holds. Then $\varphi(a_1) = a_2$, $\varphi(c_1) = c_2$. If either $a_1 = a_2$ or $c_1 = c_2$, then it is obvious that φ can be extended to $\overline{\varphi} \in \operatorname{Aut} A$. Let $a_1 \neq a_2$, $c_1 \neq c_2$. Denote by $b_1, b_2 \in A$ such that $f(b_1) = c_1$, $f(b_2) = c_2$. Let us define the mapping $\overline{\varphi}$ as follows:

a) Let $b_1 = a_2$, $b_2 = a_1$. We put $a_1 \to a_2 \to a_1$, $c_1 \to c_2 \to c_1$ and for the other elements, $x \to x$.

b) Let $b_1 \neq a_2$, $b_2 = a_1$. Then we put $a_2 \to b_1 \to a_1 \to a_2$, $f(a_2) \to c_1 \to c_2 \to f(a_2)$ and for the other elements, $x \to x$.

c) Let $b_1 = a_2$, $b_2 \neq a_1$. Then we put $a_2 \to b_2 \to a_1 \to a_2$, $c_1 \to c_2 \to f(a_1) \to c_1$, $x \to x$ otherwise.

d) Let $b_1 \neq a_2$, $b_2 \neq a_1$. Then put $a_1 \to a_2 \to a_1$, $c_1 \to c_2 \to c_1$, $b_1 \to b_2 \to b_1$, $x \to x$ otherwise.

In each of these cases, $\overline{\varphi} \in \operatorname{Aut} A$ and $\overline{\varphi}$ is an extension of φ. Therefore $A \in \mathscr{H}_2(P)$. □

4. Characterization of the classes $\mathscr{S}h_2(P)$ and $\mathscr{H}_2(P)$

The aim of this section is to prove necessary and sufficient conditions under which a monounary algebra belongs to $\mathscr{S}h_2(P)$ or to $\mathscr{H}_2(P)$, respectively.

4.1. Lemma. *Let $\alpha \geqslant 1$. Then $M_\alpha + Z_1 \notin \mathscr{H}_2(P)$.*

Proof. Let $A = M_\alpha + Z_1$ and let $c \in M_\alpha$ be such that $f(c) = c$. We have $Z_1 = \{0\}$. Take $U = \{c, 0\} = V$, $\varphi(c) = 0$, $\varphi(0) = c$. Then $U, V \in P_2(A)$, φ is an isomorphism of U onto V, but φ cannot be extended to an automorphism of A. Therefore $A \notin \mathscr{H}_2(P)$. □

4.2. Lemma. *Let $\alpha \geqslant 1$. Then $M_\alpha + Z_1 \in \mathscr{S}h_2(P)$.*

Proof. Let A, c, 0 be as in the previous proof. Take $U, V \in P_2(A)$ such that $U \cong V$, $U \neq V$. We obtain one of the following cases:

(1) $U = \{a, c\}$, $V = \{b, c\}$ for some $a, b \in f^{-1}(c) - \{c\}$,

(2) $U = \{u_1, u_2\}$, $V = \{v_1, v_2\}$, $u_1, u_2, v_1, v_2 \in f^{-1}(c) - \{c\}$,

(3) $U = \{a, 0\}$, $V = \{b, 0\}$ for some $a, b \in f^{-1}(c) - \{c\}$.

It is easy to see that in each of the cases there exists an automorphism φ of A with $\varphi(U) = V$. Hence $A \in \mathscr{S}h_2(P)$. □

It is easy to show

4.3.1. Lemma. *The algebras $Z_3 + Z_2$, $Z_3 + M_1$, $Z_2 + M_1$ belong to $\mathscr{S}h_2(P)$.*

4.3.2. Lemma. *The algebras $Z_3 + Z_2$, $Z_3 + M_1$, $Z_2 + M_1$ do not belong to $\mathscr{H}_2(P)$.*

Proof. Let us show e.g., that $Z_3 + Z_2 \notin \mathscr{H}_2(P)$. Let $A = \{a, b, c, d, e\}$, where $\{a, b, c\}$, $\{d, e\}$ are 3-, 2-element cycles, respectively. Put $U = \{a, d\}, V = \{d, a\}$, $\varphi(a) = d, \varphi(d) = a$. Then φ is an isomorphism of U onto V, thus φ can be extended to an automorphism ψ of A. For $\psi \in \operatorname{Aut} A$ we have $\psi(a) \in \{a, b, c\}$, which is a contradiction. □

4.4.1. Lemma. *If* $m \geqslant 0$, *then* $Z_5 + m \cdot Z_1 \notin \mathscr{H}_2(P)$.

P r o o f. Take $U = \{0,2\}$, $V = \{0,3\}$, $\varphi(0) = 0$, $\varphi(2) = 3$. Then φ is an isomorphism of U onto V, but it cannot be extended to an automorphism of $Z_5 + m \cdot Z_1$. □

4.4.2. Lemma. *If* $m \geqslant 0$, *then* $Z_5 + m \cdot Z_1 \in \mathscr{S}h_2(P)$.

P r o o f. Denote $A = Z_5 + m \cdot Z_1$, $B = m \cdot Z_1$. Let $U, V \in P_2(A)$, $U \cong V$, $U \neq V$. Without loss of generality we obtain one of the following cases:
(1) $U \subseteq B$, $V \subseteq B$,
(2) $U \cap B \neq \emptyset \neq U \cap Z_5$, $V \cap B \neq \emptyset \neq V \cap Z_5$,
(3) $U = \{0,1\}$, $V = \{v, f(v)\}$, $v \in Z_5$,
(4) $U = \{0,2\}$, $V = \{v, f^2(v)\}$, $v \in Z_5$.
It is obvious that in each of these cases we can find $\varphi \in \operatorname{Aut} A$ with $\varphi(U) = V$; therefore $A \in \mathscr{S}h_2(P)$. □

4.5. Lemma. *If a monounary algebra* A *belongs to* $\mathscr{S}h_2(P)$, *then* A *is isomorphic to some of the following algebras:*
(1) $Z_5 + m \cdot Z_1$, $m \geqslant 0$,
(2) $Z_4 + m \cdot Z_1$, $m \geqslant 0$,
(3) $Z_3 + Z_2$,
(4) $Z_3 + M_1$,
(5) $k \cdot Z_3 + m \cdot Z_1$, $k > 0$, $m \geqslant 0$,
(6) *connected* 3-*element monounary algebra with a* 2-*element cycle,*
(7) $m \cdot Z_2 + k \cdot Z_1$, $m, k \geqslant 0$, $(m,k) \neq (0,0)$,
(8) $Z_2 + M_1$,
(9) $M_{1\alpha}$, $\alpha > 0$,
(10) $M_\alpha + Z_1$, $\alpha > 0$,
(11) M_α, $\alpha > 0$,
(12) $m \cdot M_1$, $m > 0$.

P r o o f. Let $A \in \mathscr{S}h_2(P)$. By 2.2, each connected component of A contains a cycle with at most 5 elements. If there is a cycle with 5 or with 4 elements, then 2.10 yields that A is isomorphic either to (1) or to (2). Thus suppose that each cycle of A has at most 3 elements.

a) Assume that there exists a connected component containing a cycle C such that $\operatorname{card} C = 3$. By 2.3, C is a connected component of A. Further, in view of 2.8 we obtain that if D is a connected component of A, then either $D \cong C$ or $\operatorname{card} D \leqslant 2$. Thus either A is isomorphic to (5) or there is a connected component D of A with card $D=2$. If such D exists, then 2.12 implies that $f(x)=x$ for each $x\in A-(C\cup D)$ and 2.17 yields that A is isomorphic either to (3) or to (4).

b) Now suppose that each connected component of A contains a cycle with at most 2 elements. First assume that there is a cycle C_0 of A with $\operatorname{card} C_0=2$. If C_0 does not form a connected component, then we obtain according to 2.7 that A is isomorphic to (6). Thus let each connected component containing a 2-element cycle be a cycle. If there are two 2-element cycles in A, then A is isomorphic to (7) in view of 2.12. Suppose that A is not isomorphic to (7). Therefore there is a connected component D with $\operatorname{card} D>1$ and such that D contains a 1-element cycle. By 2.12, $f(x)=x$ for each $x\in A-(C_0\cup D)$, but by 2.16, there is no 1-element connected component of A. Thus $A=C_0\cup D$. Further, 2.9 yields that $\operatorname{card} D=2$, thus we obtain that A is isomorphic to (8).

c) Assume that each connected component of A contains a cycle with one element. If there is a cycle $\{c\}$ such that $f^{-2}(c)-\{c\}\neq\emptyset$, then 2.9 implies that A is connected and by 2.6 we get that A is isomorphic to (9). Let $f^{-2}(c)-\{c\}=\emptyset$ for each cycle $\{c\}$ of A. First let there exist a connected component C and distinct elements $a,b,c\in C$ with $f(a)=f(b)=f(c)=c$. By 2.11, $f(x)=x$ for each $x\in A-C$ and by 2.13, $\operatorname{card}(A-C)\leqslant 1$. Then $A\cong M_\alpha+Z_1$ or $A\cong M_\alpha$ (i.e., (10) or (11)). Now suppose that such C does not exist. If a connected component of A has more than one element, then it is isomorphic to M_1. If there are at least two connected components isomorphic to M_1, then 2.14 implies that A is isomorphic to (12). If there is only one connected component isomorphic to M_1, then $A\cong M_1+k\cdot Z_1$, $k\geqslant 0$ and we obtain in view of 2.15 that $A\cong M_1+Z_1$ or $A\cong M_1$, i.e., A is isomorphic either to (10) or to (11). If there are only one-element connected components in A, then A is isomorphic to (7) for $m=0$. □

4.6. Lemma. *If A is isomorphic to some of the algebras* (2), (5), (7), (9), (11), (12), *then $A\in\mathscr{H}_2(P)$.*

P r o o f. If A is isomorphic to (2), then $A\in\mathscr{H}_2(P)$ according to 3.12. Similarly, we will write the reasons why $A\in\mathscr{H}_2(P)$ in the remaining cases: 3.2.3—(5); 3.2.2—(7); 3.4—(9); 3.3—(11); 3.5—(12). □

Now we can conclude with a characterization of the monounary algebras belonging to the classes $\mathscr{S}h_2(P)$ and $\mathscr{H}_2(P)$, as follows:

4.7. Theorem. *A monounary algebra A belongs to $\mathscr{S}h_2(P)$ if and only if A is isomorphic to some of the algebras* (1)–(12).

P r o o f. If A is isomorphic to (1), then $A\in\mathscr{S}h_2(P)$ in view of 4.4.2. Analogously as above $A\in\mathscr{S}h_2(P)$ in the following cases: 4.3.1—(3), (4), (8); 3.2.1—(6);

4.2—(10). In the remaining cases (2), (5), (7), (9), (11) and (12) we obtain by 4.6 that $A \in \mathscr{H}_2(P)$, thus $A \in \mathscr{S}h_2(P)$.

The converse implication was proved in 4.5. □

4.8. Theorem. *A monounary algebra A belongs to $\mathscr{H}_2(P)$ if and only if A is isomorphic to some of the algebras* (2), (5), (7), (9), (11), (12).

P r o o f. Let $A \in \mathscr{H}_2(P)$. Then A is not isomorphic to (1) by 4.4.1, to (3), (4) or (8) by 4.3.2, to (6) immediately, to (10) by 4.1. Since 1.1 yields that $A \in \mathscr{S}h_2(P)$, we have according to 4.5 that A is isomorphic to some of the algebras (2), (5), (7), (9), (11) and (12). Then 4.6 completes the proof. □

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Author's address: Department of Geometry and Algebra, Šafárik University, Jesenná 5, 041 54 Košice, Slovakia, e-mail: `studenovska@duro.science.upjs.sk`.