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ON THE NORMALITY OF AN ALMOST CONTACT 3-STRUCTURE ON QR-SUBMANIFOLDS

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Abstract. We study n-dimensional QR-submanifolds of QR-dimension $(p-1)$ immersed in a quaternionic space form $QP^{(n+p)/4}(c)$, $c \geqslant 0$, and, in particular, determine such submanifolds with the induced normal almost contact 3-structure.

Keywords: quaternionic projective space, quaternionic number space, QR-submanifold, normal almost contact 3-structure

MSC 2000: 53C40

1. Introduction

Let M be an n-dimensional QR-submanifold of QR-dimension $(p-1)$ isometrically immersed in a quaternionic Kähler manifold $\overline{M}^{(n+p)/4}$. Denoting by $\{F, G, H\}$ the quaternionic Kähler structure of $\overline{M}^{(n+p)/4}$, it follows by definition (cf. [9]) that there exists a $(p-1)$-dimensional subbundle ν of the normal bundle $TM^{\perp}$ such that

$$\begin{cases} F\nu_x \subset \nu_x, \quad G\nu_x \subset \nu_x, \quad H\nu_x \subset \nu_x, \\ F\nu_x^{\perp} \subset T_xM, \quad G\nu_x^{\perp} \subset T_xM, \quad H\nu_x^{\perp} \subset T_xM \end{cases} \tag{1.1}$$

for each $x \in M$, where $\nu^{\perp}$ denotes the complementary orthogonal subbundle to ν in $TM^{\perp}$. Thus there is a naturally distinguished unit normal vector field ξ to M such that $\nu_x^{\perp} = \mathrm{Span}\{\xi\}$ for each $x \in M$, and the vector fields U, V, W defined by

$$U = -F\xi, \quad V = -G\xi, \quad W = -H\xi \tag{1.2}$$

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are tangent to M. On the other hand, each tangent space T_xM is decomposed as

$$T_xM = D_x \oplus D_x^{\perp},$$

where D_x is the maximal quaternionic invariant subspace of T_xM defined by

$$D_x = T_xM \cap FT_xM \cap GT_xM \cap HT_xM$$

and $D_x^{\perp}$ its orthogonal complement in T_xM. In our case, as already shown in [2], [9], $D_x^{\perp} = \mathrm{Span}\{U, V, W\}$ and so $D\colon\ x \mapsto D_x$ defines an $(n-3)$-dimensional distribution on M. But D cannot be a quaternionic CR-distribution in the sense of [1]. Further it is clear that

$$FT_xM,\ GT_xM,\ HT_xM \subset T_xM \oplus \mathrm{Span}\{\xi\}$$

and, consequently, for any tangent vector X to M, we have the following decomposition in tangential and normal components

$$\begin{cases} FX = \varphi X + u(X)\xi, \quad GX = \psi X + v(X)\xi, \\ HX = \theta X + w(X)\xi. \end{cases} \tag{1.3}$$

By means of the hermitian property of $\{F, G, H\}$ it can be easily shown that φ, ψ and θ are skew-symmetric endomorphisms acting on T_xM. Moreover it is known ([9], [10], [11]) that the aggregate $\{\varphi, \psi, \theta, u, v, w\}$ gives an almost contact 3-structure on the QR-submanifold M of QR-dimension $(p-1)$ in $\overline{M}^{(n+p)/4}$ (see also Proposition 2.1).

On the other hand the normality of an almost contact 3-structure was defined by one of the present authors ([13]) and by Yano, Ishihara and Konishi ([14]) in a different point of view. But, in this paper, it will be shown that the normalities of the induced almost contact 3-structure in the sense of [13] and [14] are equivalent to each other, and the submanifold with the induced normal almost contact 3-structure will be determined when the ambient manifold $\overline{M}$ is a quaternionic space form of constant Q-sectional curvature $c \geqslant 0$.

2. Fundamental formulas for QR-submanifolds

Let $\overline{M}^{(n+p)/4}$ be a real $(n+p)$-dimensional quaternionic Kähler manifold. Then, by definition, there is a 3-dimensional vector bundle V consisting of tensor fields of type (1,1) over $\overline{M}$ satisfying the following conditions (a), (b) and (c):

(a) In any coordinate neighborhood $\overline{\mathscr{U}}$, there is a local basis $\{F, G, H\}$ of V such that

$$\begin{cases} F^2 = -I,\ \ G^2 = -I,\ \ H^2 = -I, \\ FG = -GF = H,\ \ GH = -HG = F,\ \ HF = -FH = G. \end{cases} \tag{2.1}$$

(b) There is a Riemannian metric g which is hermitian with respect to all of F, G and H.

(c) For the Riemannian connection $\overline{\nabla}$ with respect to g

$$\begin{pmatrix} \overline{\nabla}F \\ \overline{\nabla}G \\ \overline{\nabla}H \end{pmatrix} = \begin{pmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{pmatrix} \begin{pmatrix} F \\ G \\ H \end{pmatrix} \tag{2.2}$$

where p, q and r are local 1-forms defined in $\overline{\mathscr{U}}$. Such a local basis $\{F, G, H\}$ is called a *canonical local basis* of the bundle V in $\overline{\mathscr{U}}$.

For canonical local bases $\{F, G, H\}$ and $\{'F, 'G, 'H\}$ of V in coordinate neighborhoods $\overline{\mathscr{U}}$ and $'\overline{\mathscr{U}}$, it follows that in $\overline{\mathscr{U}} \cap {}'\overline{\mathscr{U}}$

$$\begin{pmatrix} 'F \\ 'G \\ 'H \end{pmatrix} = (s_{xy}) \begin{pmatrix} F \\ G \\ H \end{pmatrix} \quad (x, y = 1, 2, 3) \tag{2.3}$$

with differentiable functions s_{xy}, where the matrix $S = (s_{xy})$ is contained in $SO(3)$ as a consequence of (2.1). As is well known [5], [6], every quaternionic Kähler manifold is orientable.

From now on we consider a real n-dimensional QR-submanifold M of QR-dimension $(p-1)$ immersed in $\overline{M}^{(n+p)/4}$ and use the same notations as in Section 1. We now take a local orthonormal basis $\{\xi_\alpha;\ \alpha = 1, \ldots, p\}$ $(\xi_1 = \xi)$ of normal vectors to M and consider the following decompositions in tangential and normal components:

$$\begin{cases} F\xi_\alpha = -U_\alpha + P_1\xi_\alpha, \quad G\xi_\alpha = -V_\alpha + P_2\xi_\alpha, \\ H\xi_\alpha = -W_\alpha + P_3\xi_\alpha \end{cases} \tag{2.4}$$

$(\alpha = 1, \ldots, p)$. Then P_1, P_2 and P_3 are skew-symmetric endomorphisms acting on $T_xM^\perp$. Moreover, by means of (1.3), the hermitian property of $\{F, G, H\}$ and (2.4) imply

$$\begin{cases} g(X, \varphi U_\alpha) = -u(X)g(\xi_1, P_1\xi_\alpha), \\ g(X, \psi V_\alpha) = -v(X)g(\xi_1, P_2\xi_\alpha), \\ g(X, \theta W_\alpha) = -w(X)g(\xi_1, P_3\xi_\alpha), \quad \alpha = 1, \ldots, p, \end{cases} \tag{2.5}$$

$$\begin{cases} g(U_\alpha, U_\beta) = \delta_{\alpha\beta} - g(P_1\xi_\alpha, P_1\xi_\beta), \\ g(V_\alpha, V_\beta) = \delta_{\alpha\beta} - g(P_2\xi_\alpha, P_2\xi_\beta), \\ g(W_\alpha, W_\beta) = \delta_{\alpha\beta} - g(P_3\xi_\alpha, P_3\xi_\beta), \quad \alpha, \beta = 1, \ldots, p. \end{cases} \tag{2.6}$$

Also, from $g(FX,\xi_\alpha) = -g(X,F\xi_\alpha)$, $g(GX,\xi_\alpha) = -g(X,G\xi_\alpha)$ and $g(HX,\xi_\alpha) = -g(X,H\xi_\alpha)$, it follows that

$$g(X,U_\alpha) = u(X)\delta_{1\alpha}, \quad g(X,V_\alpha) = v(X)\delta_{1\alpha}, \quad g(X,W_\alpha) = w(X)\delta_{1\alpha}$$

and hence

$$\begin{gathered} g(U_1,X) = u(X), \quad g(V_1,X) = v(X), \quad g(W_1,X) = w(X), \\ U_\alpha = 0, \quad V_\alpha = 0, \quad W_\alpha = 0, \quad \alpha = 2,\dots,p. \end{gathered} \tag{2.7}$$

On the other hand, comparing (1.2) and (2.4) with $\alpha = 1$, we have $U_1 = U$, $V_1 = V$, $W_1 = W$, which together with (2.7) imply

$$g(U,X) = u(X), \quad g(V,X) = v(X), \quad g(W,X) = w(X). \tag{2.8}$$

In the sequel we shall use the notations U, V, W instead of U_1, V_1, W_1.

Next, applying F to the first equation of (1.3) and using (2.4), (2.7) and (2.8), we have

$$\varphi^2 X = -X + u(X)U, \quad u(X)P_1\xi = -u(\varphi X)\xi.$$

Similarly we have

$$\begin{cases} \varphi^2 X = -X + u(X)U, \quad \psi^2 X = -X + v(X)V, \\ \theta^2 X = -X + w(X)W, \end{cases} \tag{2.9}$$

$$\begin{cases} u(X)P_1\xi = -u(\varphi X)\xi, \quad v(X)P_2\xi = -v(\psi X)\xi, \\ w(X)P_3\xi = -w(\theta X)\xi, \end{cases} \tag{2.10}$$

from which, taking account of the skew-symmetry of P_1, P_2 and P_3 and using (2.5), we also have

$$\begin{cases} u(\varphi X) = 0, \quad v(\psi X) = 0, \quad w(\theta X) = 0, \\ \varphi U = 0, \quad \psi V = 0, \quad \theta W = 0, \\ P_1\xi = 0, \quad P_2\xi = 0, \quad P_3\xi = 0. \end{cases} \tag{2.11}$$

So (2.4) can be rewritten in the form

$$\begin{cases} F\xi = -U, \quad G\xi = -V, \quad H\xi = -W, \\ F\xi_\alpha = P_1\xi_\alpha, \quad G\xi_\alpha = P_2\xi_\alpha, \quad H\xi_\alpha = P_3\xi_\alpha, \end{cases} \tag{2.12}$$

where $\alpha = 2,\dots,p$.

Applying G and H to the first equation of (1.3) and using (1.3), (2.1) and (2.12), we have

$$\theta X + w(X)\xi = -\psi(\varphi X) - v(\varphi X)\xi + u(X)V,$$
$$\psi X + v(X)\xi = \theta(\varphi X) + w(\varphi X)\xi - u(X)W,$$

and consequently

$$\left\{\begin{array}{ll}\psi(\varphi X) = -\theta X + u(X)V, & v(\varphi X) = -w(X),\\ \theta(\varphi X) = \psi X + u(X)W, & w(\varphi X) = v(X).\end{array}\right. \tag{2.13}$$

From the other equations of (1.3) we have by a quite similar method

$$\left\{\begin{array}{ll}\varphi(\psi X) = \theta X + v(X)U, & u(\psi X) = w(X),\\ \theta(\psi X) = -\varphi X + v(X)W, & w(\psi X) = -u(X),\end{array}\right. \tag{2.14}$$

$$\left\{\begin{array}{ll}\varphi(\theta X) = -\psi X + w(X)U, & u(\theta X) = -v(X),\\ \psi(\theta X) = \varphi X + w(X)V, & v(\theta X) = u(X).\end{array}\right. \tag{2.15}$$

From the first three equations of (2.12), we also have

$$\left\{\begin{array}{llll}\psi U = -W, & v(U) = 0, & \theta U = V, & w(U) = 0,\\ \varphi V = W, & u(V) = 0, & \theta V = -U, & w(V) = 0,\\ \varphi W = -V, & u(W) = 0, & \psi W = U, & v(W) = 0.\end{array}\right. \tag{2.16}$$

On the other hand, we may put

$$\left\{\begin{array}{ll}P_1\xi_\alpha = \sum_{\beta=2}^{p} P_{1\alpha\beta}\xi_\beta, & P_2\xi_\alpha = \sum_{\beta=2}^{p} P_{2\alpha\beta}\xi_\beta,\\ P_3\xi_\alpha = \sum_{\beta=2}^{p} P_{3\alpha\beta}\xi_\beta, & \alpha = 2,\dots,p,\end{array}\right. \tag{2.17}$$

from which, substituting into the last three equations of (2.12) and using the hermitian property of $\{F, G, H\}$, we have

$$\left\{\begin{array}{ll}\sum_\gamma P_{1\alpha\gamma}P_{1\gamma\beta} = -\delta_{\alpha\beta}, & \sum_\gamma P_{2\alpha\gamma}P_{2\gamma\beta} = -\delta_{\alpha\beta},\\ \sum_\gamma P_{3\alpha\gamma}P_{3\gamma\beta} = -\delta_{\alpha\beta}.\end{array}\right. \tag{2.18}$$

Also, from (2.1), (2.12) and (2.17), we have

$$\left\{\begin{array}{ll}\sum_\beta P_{1\alpha\beta}P_{2\beta\gamma} = -P_{3\alpha\gamma}, & \sum_\beta P_{1\alpha\beta}P_{3\beta\gamma} = P_{2\alpha\gamma},\\ \sum_\beta P_{2\alpha\beta}P_{3\beta\gamma} = -P_{1\alpha\gamma}, & \sum_\beta P_{2\alpha\beta}P_{1\beta\gamma} = P_{3\alpha\gamma},\\ \sum_\beta P_{3\alpha\beta}P_{1\beta\gamma} = -P_{2\alpha\gamma}, & \sum_\beta P_{3\alpha\beta}P_{2\beta\gamma} = P_{1\alpha\gamma}.\end{array}\right. \tag{2.19}$$

The equations (2.6)–(2.11) and (2.13)–(2.16) tell us

Proposition 2.1 ([9], [10], [11]). *An n-dimensional QR-submanifold of QR-dimension $(p-1)$ in a quaternionic Kähler manifold $\overline{M}^{(n+p)/4}$ admits an almost contact 3-structure.*

In general if the condition

$$[\varphi_i, \varphi_i] + du_i \otimes U_i = 0$$

is satisfied for some $1 \leqslant i \leqslant 3$, then the almost contact structure (φ_i, U_i, u_i) is said to be *normal*, where we put

$$\varphi_1 = \varphi, \quad \varphi_2 = \psi, \quad \varphi_3 = \theta,$$
$$U_1 = U, \quad U_2 = V, \quad U_3 = W; \qquad u_1 = u, \quad u_2 = v, \quad u_3 = w$$

and $[\varphi_i, \varphi_i]$ denotes the Nijenhuis tensor of φ_i. In their papers [8] and [14], Ishihara, Konishi, Kuo and Yano have proved

Lemma 2.2. *If, for an almost contact 3-structure$\{(\varphi_i, U_i, u_i);\ i = 1, 2, 3\}$, any two of the almost contact structures (φ_i, U_i, u_i) are normal, then so is the third.*

Moreoreover, in [14] the following lemma was proved.

Lemma 2.3. *For an almost contact 3-structure $\{(\varphi_i, U_i, u_i);\ i = 1, 2, 3\}$, a necessary and sufficient condition in order that the almost contact structures (φ_i, U_i, u_i) are all normal is that the condition*

$$\begin{cases} 2[\varphi_1, \varphi_2] + du_1 \otimes U_2 + du_2 \otimes U_1 = 0, \\ \mathscr{L}_{U_1}\varphi_2 + \mathscr{L}_{U_2}\varphi_1 = 0, \quad du_1 \overline{\wedge} \varphi_2 + du_2 \overline{\wedge} \varphi_1 = 0 \end{cases} \tag{2.20}$$

be valid, where $[\varphi_1, \varphi_2]$ denotes the Nijenhuis tensor of φ_1 and φ_2, $du_i \overline{\wedge} \varphi_j$ the 2-form defined by

$$(du_i \overline{\wedge} \varphi_j)(X, Y) = du_i(\varphi_j X, Y) + du_i(X, \varphi_j Y)$$

and $\mathscr{L}_{U_i}$ the Lie derivative with respect to U_i.

3. Further properties of the induced almost contact 3-structure

In this section we shall use the same notations and terminology as in the previous section.

Now let ∇ be the Levi-Cività connection on M and $\nabla^{\perp}$ the normal connection induced from $\overline{\nabla}$ in the normal bundle $TM^{\perp}$ of M. Then Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{3.1}$$

$$\overline{\nabla}_X \xi_\alpha = -A_\alpha X + \nabla^{\perp}_X \xi_\alpha, \quad \alpha = 1, \ldots, p \tag{3.2}$$

for X, Y tangent to M. Here h denotes the second fundamental form and A_α the shape operator corresponding to ξ_α. They are related by $h(X,Y) = \sum_{\alpha=1}^{p} g(A_\alpha X, Y)\xi_\alpha$. Furthermore, put

$$\nabla^{\perp}_X \xi_\alpha = \sum_{\beta=1}^{p} s_{\alpha\beta}(X)\xi_\beta, \tag{3.3}$$

where $(s_{\alpha\beta})$ is the skew-symmetric matrix of connection forms of $\nabla^{\perp}$.

Differentiating the first equation of (1.3) covariantly and using (1.3), (2.2), (2.4), (2.7), (3.1) and (3.2), we have

$$\begin{aligned} (\nabla_Y \varphi)X &= r(Y)\psi X - q(Y)\theta X + u(X)A_1 Y - g(A_1 Y, X)U, \\ (\nabla_Y u)X &= r(Y)v(X) - q(Y)w(X) + g(\varphi A_1 Y, X). \end{aligned} \tag{3.4}$$

From the other equations of (1.3) we also have

$$\begin{aligned} (\nabla_Y \psi)X &= -r(Y)\varphi X + p(Y)\theta X + v(X)A_1 Y - g(A_1 Y, X)V, \\ (\nabla_Y v)X &= -r(Y)u(X) + p(Y)w(X) + g(\psi A_1 Y, X), \end{aligned} \tag{3.5}$$

$$\begin{aligned} (\nabla_Y \theta)X &= q(Y)\varphi X - p(Y)\psi X + w(X)A_1 Y - g(A_1 Y, X)W, \\ (\nabla_Y w)X &= q(Y)u(X) - p(Y)v(X) + g(\theta A_1 Y, X). \end{aligned} \tag{3.6}$$

Next, differentiating the first equation of (2.12) covariantly and comparing the tangential and normal parts, we have

$$\begin{cases} \nabla_Y U = r(Y)V - q(Y)W + \varphi A_1 Y, \\ g(A_\alpha U, Y) = -\sum_{\beta=2}^{p} s_{1\beta}(Y)P_{1\beta\alpha}, \quad \alpha = 2, \ldots, p. \end{cases} \tag{3.7}$$

From the other equations of (2.12), we have similarly

$$\tag{3.8} \begin{cases} \nabla_Y V = -r(Y)U + p(Y)W + \psi A_1 Y, \\ g(A_\alpha V, Y) = -\sum_{\beta=2}^{p} s_{1\beta}(Y) P_{2\beta\alpha}, \quad \alpha = 2, \dots, p, \end{cases}$$

$$\tag{3.9} \begin{cases} \nabla_Y W = q(Y)U - p(Y)V + \theta A_1 Y, \\ g(A_\alpha W, Y) = -\sum_{\beta=2}^{p} s_{1\beta}(Y) P_{3\beta\alpha}, \quad \alpha = 2, \dots, p. \end{cases}$$

In what follows *we assume that the distinguished normal vector field* ξ *is parallel with respect to the normal connection, that is,* $\nabla^\perp_X \xi = 0$. Hence it follows from (3.3) that $s_{\beta 1} = 0$, $\beta = 2, \dots, p$, and, consequently,

$$A_\alpha U = 0, \quad A_\alpha V = 0, \quad A_\alpha W = 0, \quad \alpha = 2, \dots, p$$

because of (3.7)–(3.9).

In particular when the ambient manifold is a quaternionic space form $\overline{M}^{(n+p)/4}(c)$, that is, a quaternionic Kähler manifold of constant Q-sectional curvature c, the curvature tensor $\overline{R}$ of $\overline{M}^{(n+p)/4}(c)$ has the form

$$\begin{aligned} \overline{R}_{\overline{X}\overline{Y}}\overline{Z} = \frac{c}{4}\{ & g(\overline{Y}, \overline{Z})\overline{X} - g(\overline{X}, \overline{Z})\overline{Y} \\ & + g(F\overline{Y}, \overline{Z})F\overline{X} - g(F\overline{X}, \overline{Z})F\overline{Y} - 2g(F\overline{X}, \overline{Y})F\overline{Z} \\ & + g(G\overline{Y}, \overline{Z})G\overline{X} - g(G\overline{X}, \overline{Z})G\overline{Y} - 2g(G\overline{X}, \overline{Y})G\overline{Z} \\ & + g(H\overline{Y}, \overline{Z})H\overline{X} - g(H\overline{X}, \overline{Z})H\overline{Y} - 2g(H\overline{X}, \overline{Y})H\overline{Z}\} \end{aligned}$$

for $\overline{X}$, $\overline{Y}$, $\overline{Z}$ tangent to $\overline{M}^{(n+p)/4}(c)$ (cf. [5], [6]). So the above assumption implies that the equation of Codazzi and Ricci is of the form

$$\tag{3.10} \begin{aligned} & g((\nabla_X A_1)Y - (\nabla_Y A_1)X, Z) \\ & = \frac{c}{4}\{g(\varphi Y, Z)u(X) - g(\varphi X, Z)u(Y) - 2g(\varphi X, Y)u(Z) \\ & \quad + g(\psi Y, Z)v(X) - g(\psi X, Z)v(Y) - 2g(\psi X, Y)v(Z) \\ & \quad + g(\theta Y, Z)w(X) - g(\theta X, Z)w(Y) - 2g(\theta X, Y)w(Z)\}, \end{aligned}$$

$$\tag{3.11} g(\overline{R}(X, Y)\xi_\alpha, \xi_\beta) = g(R^\perp(X, Y)\xi_\alpha, \xi_\beta) + g([A_\beta, A_\alpha]X, Y)$$

for any X, Y, Z tangent to M, where R and $R^\perp$ denote the curvature tensor of ∇ and $\nabla^\perp$, respectively (cf. [3], [9], [10], [11]).

Finally we introduce a theorem due to Kwon and one of the present authors ([9]) for later use.

Theorem K-P. *Let M be an n-dimensional QR-submanifold of QR-dimension $(p-1)$ in a quaternionic projective space $QP^{(n+p)/4}(4)$ and let the normal vector field ξ be parallel with respect to the normal connection. If*

$$A_1\varphi = \varphi A_1, \quad A_1\psi = \psi A_1, \quad A_1\theta = \theta A_1$$

on M, then $\pi^{-1}(M)$ is locally a product of $M_1 \times M_2$ where M_1 and M_2 lie on some $(4n_1+3)$- and $(4n_2+3)$-spheres, respectively, and A_1 denotes the shape operator corresponding to ξ (π is the Hopf fibration $S^{n+p+3}(1) \to QP^{(n+p)/4}(4)$).

4. The submanifolds with the induced normal almost contact 3-structure

In this section we introduce the notion of the normality of almost contact 3-structure in the sense of [13].

From now on we put in each coordinate neighborhood $\mathscr{U}$ of M

$$\text{(4.1)} \qquad \begin{pmatrix} \overset{\circ}{\nabla}\varphi \\ \overset{\circ}{\nabla}\psi \\ \overset{\circ}{\nabla}\theta \end{pmatrix} = \begin{pmatrix} \nabla\varphi \\ \nabla\psi \\ \nabla\theta \end{pmatrix} + \begin{pmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \\ \theta \end{pmatrix},$$

$$\text{(4.2)} \qquad \begin{pmatrix} \overset{\circ}{\nabla}U \\ \overset{\circ}{\nabla}V \\ \overset{\circ}{\nabla}W \end{pmatrix} = \begin{pmatrix} \nabla U \\ \nabla V \\ \nabla W \end{pmatrix} + \begin{pmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \end{pmatrix}.$$

Then it follows from (2.3) that

$$\text{(4.3)} \qquad \begin{pmatrix} \overset{\circ}{\nabla}{}'\varphi \\ \overset{\circ}{\nabla}{}'\psi \\ \overset{\circ}{\nabla}{}'\theta \end{pmatrix} = (s_{xy}) \begin{pmatrix} \overset{\circ}{\nabla}\varphi \\ \overset{\circ}{\nabla}\psi \\ \overset{\circ}{\nabla}\theta \end{pmatrix}, \qquad \begin{pmatrix} \overset{\circ}{\nabla}{}'U \\ \overset{\circ}{\nabla}{}'V \\ \overset{\circ}{\nabla}{}'W \end{pmatrix} = (s_{xy}) \begin{pmatrix} \overset{\circ}{\nabla}U \\ \overset{\circ}{\nabla}V \\ \overset{\circ}{\nabla}W \end{pmatrix}$$

in $\mathscr{U} \cap {}'\mathscr{U}$. Now, in each coordinate neighborhood $\mathscr{U}$, we consider local tensor fields $S(\varphi_i, \varphi_j)$ $(i, j = 1, 2, 3)$ of type $(1, 2)$ such that

$$\begin{aligned} \text{(4.4)} \quad & S(\varphi_i, \varphi_j)(X, Y) \\ & = (\overset{\circ}{\nabla}_{\varphi_i X}\varphi_j)Y - (\overset{\circ}{\nabla}_{\varphi_i Y}\varphi_j)X + (\overset{\circ}{\nabla}_{\varphi_j X}\varphi_i)Y - (\overset{\circ}{\nabla}_{\varphi_j Y}\varphi_i)X \\ & \quad + \varphi_i\{(\overset{\circ}{\nabla}_Y\varphi_j)X - (\overset{\circ}{\nabla}_X\varphi_j)Y\} + \varphi_j\{(\overset{\circ}{\nabla}_Y\varphi_i)X - (\overset{\circ}{\nabla}_X\varphi_i)Y\} \\ & \quad + \{(\overset{\circ}{\nabla}_X u_i)Y - (\overset{\circ}{\nabla}_Y u_i)X\}U_j + \{(\overset{\circ}{\nabla}_X u_j)Y - (\overset{\circ}{\nabla}_Y u_j)X\}U_i \end{aligned}$$

where we again put

$$\varphi_1 = \varphi,\ \ \varphi_2 = \psi,\ \ \varphi_3 = \theta,\ \ U_1 = U,\ \ U_2 = V,\ \ U_3 = W$$

and

$$(\overset{\circ}{\nabla}_X u_i)Y = g(\overset{\circ}{\nabla}_X U_i, Y), \quad i = 1, 2, 3. \tag{4.5}$$

Then a simple computation using (4.3) implies that

$$S('\varphi_i, '\varphi_j) = (s_{xy})(S(\varphi_i, \varphi_j))(s_{xy})^{-1}$$

in $\mathscr{U} \cap {}'\mathscr{U}$. Hence we have the global tensor fields Σ_1 and Σ_2 on M defined by

$$\Sigma_1 = S(\varphi_1, \varphi_1) + S(\varphi_2, \varphi_2) + S(\varphi_3, \varphi_3), \tag{4.6}$$

$$\begin{aligned} \Sigma_2 = {} & S(\varphi_1, \varphi_1) \otimes S(\varphi_2, \varphi_2) + S(\varphi_2, \varphi_2) \otimes S(\varphi_3, \varphi_3) \\ & + S(\varphi_3, \varphi_3) \otimes S(\varphi_1, \varphi_1) - S(\varphi_1, \varphi_2) \otimes S(\varphi_2, \varphi_1) \\ & - S(\varphi_2, \varphi_3) \otimes S(\varphi_3, \varphi_2) - S(\varphi_3, \varphi_1) \otimes S(\varphi_1, \varphi_3) \end{aligned} \tag{4.7}$$

up to a sign. It is said that the induced almost contact 3-structure is *normal* if $\Sigma_1 = 0$ and $\Sigma_2 = 0$ (for details see [13]).

Remark 4.1 ([13])**.** A necessary and sufficient condition in order for the almost contact 3-structure to be normal is

$$S(\varphi_i, \varphi_j) = 0, \quad i, j = 1, 2, 3.$$

We next consider the traceless part of δ-decomposition of the global tensor field Σ_1 in the sense of Krupka ([7]). Since Σ_1 is of type (1,2) and $n \geqslant 2$, using (3.4)–(3.6) and (4.4)–(4.6) we can easily verify that the traceless part $\overset{\circ}{\Sigma}_1$ of Σ_1 is given by

$$\begin{aligned} \overset{\circ}{\Sigma}_1(X, Y) = {} & \Sigma_1(X, Y) - \frac{1}{2(n-1)}\{u(A_1\varphi Y)X - u(A_1\varphi X)Y \\ & + v(A_1\psi Y)X - v(A_1\psi X)Y + w(A_1\theta Y)X - w(A_1\theta X)Y\}, \end{aligned} \tag{4.8}$$

or equivalently

$$\begin{aligned} 2\overset{\circ}{\Sigma}_1(X, Y) = {} & u(Y)(A_1\varphi - \varphi A_1)X - u(X)(A_1\varphi - \varphi A_1)Y \\ & + v(Y)(A_1\psi - \psi A_1)X - v(X)(A_1\psi - \psi A_1)Y \\ & + w(Y)(A_1\theta - \theta A_1)X - w(X)(A_1\theta - \theta A_1)Y \\ & - \frac{1}{n-1}\{u(A_1\varphi Y)X - u(A_1\varphi X)Y + v(A_1\psi Y)X \\ & - v(A_1\psi X)Y + w(A_1\theta Y)X - w(A_1\theta X)Y\}. \end{aligned} \tag{4.8$'$}$$

From now on we assume that $\overset{\circ}{\Sigma}_1 = 0$ identically on M. Putting $Y = U$ in (4.8)$'$ with $\overset{\circ}{\Sigma}_1 = 0$ and using (2.13)–(2.16), we obtain

$$
\begin{aligned}
(4.9) \qquad 0 = & (A_1\varphi - \varphi A_1)X + u(X)\varphi A_1 U + v(X)\{A_1 W + \psi A_1 U\} \\
& - w(X)\{A_1 V - \theta A_1 U\} \\
& + \frac{1}{n-1}\{u(A_1\varphi X) + v(A_1\psi X) + w(A_1\theta X)\}U,
\end{aligned}
$$

from which, taking the inner product with U, it follows that

$$
(4.10) \qquad \frac{1}{n-1}(n\varphi A_1 U + \psi A_1 V + \theta A_1 W) = 2\{u(A_1 W)V - u(A_1 V)W\}.
$$

Taking the inner product of (5.3) with V and W, respectively, and using (2.13)–(2.16), we have

$$
u(A_1 W) = u(A_1 V) = 0,
$$

which together with (4.10) yields

$$
n\varphi A_1 U + \psi A_1 V + \theta A_1 W = 0.
$$

Similarly we have

$$
\begin{aligned}
n\varphi A_1 U + \psi A_1 V + \theta A_1 W &= 0, \\
\varphi A_1 U + n\psi A_1 V + \theta A_1 W &= 0, \\
\varphi A_1 U + \psi A_1 V + n\theta A_1 W &= 0
\end{aligned}
$$

and, consequently,

$$
\varphi A_1 U = \psi A_1 V = \theta A_1 W = 0.
$$

Moreover, the last equations imply

$$
A_1 U = u(A_1 U)U, \quad A_1 V = v(A_1 V)V, \quad A_1 W = w(A_1 W)W,
$$

which together with (4.8) gives the following implication:

$$
\overset{\circ}{\Sigma}_1 = 0 \implies \Sigma_1 = 0.
$$

Since the converse is trivial, we have

Lemma 4.1. *Let M be an n-dimensional QR-submanifold of QR-dimension $(p-1)$ in a quaternionic Kähler manifold $\overline{M}^{(n+p)/4}$ and let the normal vector field ξ be parallel with respect to the normal connection. Then we have*

$$\overset{\circ}{\Sigma}_1 = 0 \iff \Sigma_1 = 0.$$

By means of Lemma 4.1 we have

Theorem 1. *Let M be as in Lemma 4.1. Then the following are equivalent to each other:*

(a) *The almost contact 3-structure is normal.*
(b) *The global tensor field Σ_1 defined by (4.6) vanishes.*
(c) *The traceless part $\overset{\circ}{\Sigma}_1$ of Σ_1 vanishes.*
(d) *The relation given by (2.20) is valid.*
(e) $A_1\varphi = \varphi A_1, \quad A_1\psi = \psi A_1, \quad A_1\theta = \theta A_1.$

Proof. Substituting (3.4)–(3.9) into (4.4), we can easily obtain that

$$\begin{aligned} S(\varphi,\varphi)(X,Y) &= 2\{u(Y)(A_1\varphi-\varphi A_1)X - u(X)(A_1\varphi-\varphi A_1)Y\},\\ S(\psi,\psi)(X,Y) &= 2\{v(Y)(A_1\psi-\psi A_1)X - v(X)(A_1\psi-\psi A_1)Y\},\\ S(\theta,\theta)(X,Y) &= 2\{w(Y)(A_1\theta-\theta A_1)X - w(X)(A_1\theta-\theta A_1)Y\}, \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} S(\varphi,\psi)(X,Y) &= v(Y)(A_1\varphi-\varphi A_1)X - v(X)(A_1\varphi-\varphi A_1)Y\\ &\quad + u(Y)(A_1\psi-\psi A_1)X - u(X)(A_1\psi-\psi A_1)Y,\\ S(\psi,\theta)(X,Y) &= w(Y)(A_1\psi-\psi A_1)X - w(X)(A_1\psi-\psi A_1)Y\\ &\quad + v(Y)(A_1\theta-\theta A_1)X - v(X)(A_1\theta-\theta A_1)Y,\\ S(\theta,\varphi)(X,Y) &= u(Y)(A_1\theta-\theta A_1)X - u(X)(A_1\theta-\theta A_1)Y\\ &\quad + w(Y)(A_1\varphi-\varphi A_1)X - w(X)(A_1\varphi-\varphi A_1)Y, \end{aligned} \tag{4.12}$$

which together with Lemmas 2.2, 2.3 and Remark 4.1 yields the implications

$$\text{(e)} \Longrightarrow \text{(a)}, \quad \text{(e)} \Longrightarrow \text{(b)}, \quad \text{(e)} \Longrightarrow \text{(d)}.$$

In order to prove that the other implications are valid, it suffices to show the implication (b) $\Longrightarrow$ (e). Now we assume that (b) is valid. Then (4.11) implies

$$\begin{aligned} &u(Y)(A_1\varphi-\varphi A_1)X - u(X)(A_1\varphi-\varphi A_1)Y\\ &\quad + v(Y)(A_1\psi-\psi A_1)X - v(X)(A_1\psi-\psi A_1)Y\\ &\quad + w(Y)(A_1\theta-\theta A_1)X - w(X)(A_1\theta-\theta A_1)Y = 0. \end{aligned} \tag{4.13}$$

Putting $Y = U$ in (4.13) and using (2.11) and (2.16), we have

$$(A_1\varphi - \varphi A_1)X - u(X)\varphi A_1 U + v(X)(A_1 W + \psi A_1 U) - w(X)(A_1 V - \theta A_1 U) = 0, \tag{4.14}$$

from which, taking the inner product with U, it follows that

$$g(\varphi A_1 U, X) = 2u(A_1 W)v(X) - 2u(A_1 V)w(X)$$

and, consequently,

$$\varphi A_1 U = 0, \quad u(A_1 W) = 0, \quad u(A_1 V) = 0.$$

Similarly we have

$$A_1 U = u(A_1 U)U, \quad A_1 V = v(A_1 V)V, \quad A_1 W = w(A_1 W)W, \tag{4.15}$$

$$u(A_1 V) = v(A_1 U) = u(A_1 W) = w(A_1 U) = v(A_1 W) = w(A_1 V) = 0. \tag{4.16}$$

Substituting (4.15) into (4.14) and using (2.16), we have

$$(A_1\varphi - \varphi A_1)X + v(X)\{w(A_1 W) - u(A_1 U)\}W - w(X)\{v(A_1 V) - u(A_1 U)\}V = 0, \tag{4.17}$$

from which, taking the symmetric part,

$$2g((A_1\varphi - \varphi A_1)X, Y) + \{w(A_1 W) - v(A_1 V)\} \times \{v(X)w(Y) + v(Y)w(X)\} = 0.$$

Putting $X = V$ and $Y = W$ in the last equation and using (2.16) and (4.15), we obtain

$$v(A_1 V) = w(A_1 W).$$

Similarly we have

$$u(A_1 U) = v(A_1 V) = w(A_1 W),$$

which together with (4.17) gives

$$A_1\varphi = \varphi A_1.$$

By the quite similar method we have

$$A_1\varphi = \varphi A_1, \quad A_1\psi = \psi A_1, \quad A_1\theta = \theta A_1,$$

which yields the implication (b) $\Longrightarrow$ (e). □

Combining Theorem 1 with Theorem K-P, we have

Theorem 2. *Let M be an n-dimensional QR-submanifold of QR-dimension $(p-1)$ in $QP^{(n+p)/4}(4)$ and let the normal vector field ξ be parallel with respect to the normal connection. If one of the conditions* (a)–(e) *stated in Theorem* 1 *is valid on M, then $\pi^{-1}(M)$ is locally a product $M_1 \times M_2$ where M_1 and M_2 lie on some $(4n_1+3)$- and $(4n_2+3)$-dimensional spheres, respectively (π is the Hopf fibration $S^{n+p+3}(1) \to QP^{(n+p)/4}(4)$).*

5. The special case of an ambient quaternionic Kähler manifold

In this section we specify the ambient manifold $\overline{M}$ as a quaternionic space form $\overline{M}^{(n+p)/4}(c)$ with $c=0$ and assume that one of the conditions (a)–(e) stated in Theorem 1 is valid on M. Then Theorem 1 implies

$$A_1\varphi = \varphi A_1, \;\; A_1\psi = \psi A_1, \;\; A_1\theta = \theta A_1, \tag{5.1}$$

from which, taking account of (2.9) and (2.11), we have

$$A_1U = \lambda U, \;\; A_1V = \mu V, \;\; A_1W = \nu W,$$

where $\lambda = u(A_1U)$, $\mu = v(A_1V)$, $\nu = w(A_1W)$. But, applying ψ to the first equation of (5.1) and using (2.13) and (5.1) itself, we have

$$u(X)A_1V = u(A_1X)V,$$

from which, putting $X=U$, it follows that

$$A_1V = \lambda V$$

and, consequently, $\lambda = \mu$. Similarly we $\lambda = \mu = \nu$ which yields

$$A_1U = \lambda U, \;\; A_1V = \lambda V, \;\; A_1W = \lambda W. \tag{5.2}$$

Differentiating the first equation of (5.2) covariantly and using (3.7), (5.1) and (5.2) itself, we have

$$g((\nabla_X A_1)Y, U) + g(\varphi A_1^2 X, Y) = (X\lambda)u(Y) + \lambda g(\varphi A_1 X, Y),$$

from which, taking the skew-symmetric part and making use of (3.10) with $c=0$ and (5.1), it follows that

$$2g(\varphi A_1^2 X, Y) = (X\lambda)u(Y) - (Y\lambda)u(X) + 2\lambda g(\varphi A_1 X, Y). \tag{5.3}$$

Now we put $Y = U$ in (5.3). Then the skew-symmetry of φ and (2.11) imply $X\lambda = (U\lambda)u(X)$. Similary we have

$$X\lambda = (U\lambda)u(X) = (V\lambda)v(X) = (W\lambda)w(X)$$

and consequently $U\lambda = V\lambda = W\lambda = 0$ which yield that λ is constant. Combining this fact with (5.3) gives $\varphi(A_1^2X - \lambda A_1X) = 0$, from which, applying φ and using (2.9) and (5.2), we obtain $A_1^2 = \lambda A_1$. Thus we have

Lemma 5.1. *Let M be an n-dimensional QR-submanifold of QR-dimension $(p-1)$ in a quaternionic space form $\overline{M}^{(n+p)/4}(c)$ with $c = 0$ such that the distinguished normal vector field ξ is parallel with respect to the normal connection. If one of the conditions* (a)–(e) *stated in Theorem* 1 *is valid on M, then*

$$A_1^2 = \lambda A_1 \tag{5.4}$$

and λ is constant.

In particular, we can prove

Lemma 5.2. *Let M be as in Lemma* 5.1. *Then*

$$\nabla A_1 = 0, \tag{5.5}$$

provided $\lambda \neq 0$.

Proof. Differentiating (5.4) covariantly and using the fact that λ is constant, we have

$$(\nabla_Y A_1)A_1X + A_1(\nabla_Y A_1)X = \lambda(\nabla_Y A_1)X, \tag{5.6}$$

from which, taking the skew-symmetric part and using (3.10) with $c = 0$, we find

$$(\nabla_Y A_1)A_1X = (\nabla_X A_1)A_1Y$$

and, consequently,

$$g((\nabla_Y A_1)A_1X, Z) = g((\nabla_X A_1)A_1Y, Z) = g(A_1(\nabla_X A_1)Z, Y).$$

On the other hand

$$g((\nabla_Y A_1)A_1X, Z) = g((\nabla_Z A_1)A_1X, Y),$$

which together with the last equation gives

$$g((\nabla_Y A_1)A_1X, Z) = g(A_1(\nabla_X A_1)Y, Z),$$

that is, $(\nabla_Y A_1)A_1X = A_1(\nabla_Y A_1)X$. Hence (5.6) reduces to

$$2A_1(\nabla_Y A_1)X = \lambda(\nabla_Y A_1)X,$$

from which, applying A_1 and using (5.4), it is clear that

$$\lambda A_1(\nabla_Y A_1)X = 0$$

and therefore $\lambda(\nabla_Y A_1)X = 0$. Thus we complete the proof. □

Remark 5.1. When the ambient space is a quaternionic projective space $QP^{(n+p)/4}$, the assumptions stated in Lemma 5.1 yield that the shape operator A_1 is cyclic-parallel, that is,

$$g(\nabla_X A_1)Y, Z) + g(\nabla_Y A_1)Z, X) + g(\nabla_Z A_1)X, Y) = 0.$$

But, in this case we don't need the hypothesis $\lambda \neq 0$. (For details, see [9].)

6. The main results when $\overline{M} = Q^{(n+p)/4}$

In this section we specialize to the case of an ambient quaternionic number space $Q^{(n+p)/4}$. In this case, as already shown in Lemma 5.1, the eigenvalues κ of the shape operator A_1 satisfy

$$\kappa(\kappa - \lambda) = 0.$$

Moreover it is clear from (5.1) and (5.2) that the multiplicity of λ must be $4m+3$ for some integer m at each point in M. Since λ is constant and trace A_1 is continuous, the multiplicity r of λ is constant. Hence it suffices to consider the following three cases

$$\text{(i)}\ \ r = 0, \quad \text{(ii)}\ \ r = n, \quad \text{(iii)}\ \ 3 \leqslant r < n.$$

We will start with the first case (i). In this case $A_1 = 0$. Since, by assumption, the normal vector field ξ is parallel with respect to the normal connection, Erbacher's reduction theorem ([4]) yields that there exists a totally geodesic hypersurface R^{n+p-1} in $Q^{(n+p)/4}$ which contains M.

Next, we consider the case (ii). In this case $A_1 = \lambda I$. Let $\overline{x}$ be the position vector of M and put $\overline{p} := \overline{x} + \lambda^{-1}\xi$. Then

$$\overline{\nabla}_X \overline{p} = \overline{\nabla}_X(\overline{x} + \lambda^{-1}\xi) = X - \lambda^{-1}(A_1 X - \nabla^{\perp}_X \xi) = 0,$$

which means that $\overline{p}$ is a fixed point in $Q^{(n+p)/4}$. Moreover, it is clear that $\|\overline{x} - \overline{p}\| = |\lambda|^{-1}$ and consequently M is contained in the hypersphere $S^{n+p-1}(|\lambda|^{-1})$ of radius $|\lambda|^{-1}$ centered at $\overline{p}$.

Finally we consider the case (iii). Since the multiplicity r of λ is constant, the eigenspaces corresponding to λ and 0 determine distributions of dimension r and $n-r$, which will be denoted by D_λ and D_0, respectively. Furthermore, by means of Lemma 5.2, $\nabla A_1 = 0$ and consequently it is easily verified that D_λ and D_0 are both involutive and that D_λ is parallel along D_0 and vice versa. Denoting by M_λ and M_0 the integral submanifolds of D_λ and D_0, respectively, we can see that M is locally the Riemannian product $M_\lambda \times M_0$.

From now on we shall study M_λ and M_0 in more detail and start with M_λ. Let $Z_1, \ldots, Z_{n-r}$ be orthonormal vector fields belonging to D_0. Since M_λ is totally geodesic in M, the shape operators $A'_1, \ldots, A'_{n-r}$ corresponding to those normal vectors vanish. On the other hand we may consider M_λ as a submanifold of $Q^{(n+p)/4}$. Then the vector fields $Z_1, \ldots, Z_{n-r}, \xi_1, \ldots, \xi_p$ form an orthonormal set of local vector fields normal to M_λ. In this case the shape operators corresponding to $Z_1, \ldots, Z_{n-r}$ also vanish. Hence it is clear from (3.11) that

$$'R^{\perp}_{X,Y} Z_i = 0, \quad i = 1, \ldots, n-r \tag{6.1}$$

and moreover $[A_1, A_\alpha] = 0$, where $'R^{\perp}$ denotes the curvature tensor of the normal connection $'\nabla^{\perp}$ of M_λ in $Q^{(n+p)/4}$. On the other hand, we can easily see that for any $X \in D_\lambda$

$$g('\nabla^{\perp}_X Z_i, \xi_\beta) = g(Z_i, A_\beta X), \quad \beta = 1, \ldots, p.$$

But, since $[A_1, A_\beta] = 0$, $\beta = 1, \ldots, p$, which is a direct consequence of (3.11) and $\nabla^{\perp}\xi_1 = 0$, we have $A_\beta X \in D_\lambda$ and, consequently,

$$g('\nabla^{\perp}_X Z_i, \xi_\beta) = 0, \quad \beta = 1, \ldots, p,$$

that is, $'\nabla^{\perp}_X Z_i \in D_0$. Thus, by the same method as in the proof of Proposition 1.1 in [3, p. 99], we may prove that (6.1) yields the existence of the normal vector fields $Z_1, \ldots, Z_{n-r}$ such that

$$'\nabla^{\perp}_X Z_i = 0, \quad i = 1, \ldots, n-r \tag{6.2}$$

for any tangent vector field X to M_λ.

Now let $\overline{x}$ be the position vector of M_λ in $Q^{(n+p)/4}$ and $X \in D_\lambda$. Then, by using (6.2) and $A'_i = 0$, $i = 1, \ldots, n-r$, we have

$$Xg(\overline{x}, Z_i) = g(X, Z_i) = 0, \quad i = 1, \ldots, n-r,$$

that is,

$$g(\overline{x}, Z_i) = c_i, \quad i = 1, \ldots, n-r, \tag{6.3}$$

where c_i is constant. Moreover, putting $\overline{p} := \overline{x} + \lambda^{-1}\xi$, we can see that

$$\overline{\nabla}_X \overline{p} = X - \lambda^{-1} A_1 X = 0$$

and $\|\overline{x} - \overline{p}\| = |\lambda|^{-1}$. Therefore M_λ belongs to the intersection of the hypersphere of radius $|\lambda|^{-1}$ centered at $\overline{p}$ and the $n-r$ hyperplanes defined by (6.3). We notice that $\overline{p}$ is contained in the $n-r$ hyperplanes.

In a similar way it can be shown that M_0 belongs to the intersection of the $r+1$ hyperplanes given by

$$g(\overline{x}, \xi) = c, \quad g(\overline{x}, Z_s) = c_s, \quad s = n-r+1, \ldots, n.$$

Summing up, we may conclude

Theorem 2. *Let M be an n-dimensional QR-submanifold of QR-dimension $(p-1)$ in $Q^{(n+p)/4}$ which satisfies one of the conditions stated in Theorem* 1. *If the distinguished normal vector field ξ is parallel with respect to the normal connection, then we have one of the following cases:*

(a) *M is contained in a hyperplane orthogonal to ξ.*
(b) *M is contained in a hypersphere orthogonal to ξ.*
(c) *M is locally a Riemannian product $M_\lambda \times M_0$, where M_λ is contained in a $(p+r-1)$-dimensional sphere $S^{(p+r-1)}$ and M_0 is contained in an $(n+p-r-1)$-dimensional subspace $R^{(n+p-r-1)}$.*

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