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EXISTENCE OF SOLUTIONS FOR THE DIRICHLET PROBLEM WITH SUPERLINEAR NONLINEARITIES

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Abstract. In this paper we establish the existence of nontrivial solutions to

$$\frac{\mathrm{d}}{\mathrm{dt}}L_{x'}(t,x'(t))+V_x(t,x(t))=0,\quad x(0)=0=x(T),$$

with V_x superlinear in x.

Keywords: nonlinear Dirichlet problem, nontrivial solution, duality method, superlinear nonlinearity

MSC 2000: 34B15, 49J40

1. Introduction

We investigate the nonlinear Dirichlet problem

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{dt}}L_{x'}(t,x'(t))+V_x(t,x(t))=0,\quad \text{a.e. in } [0,T],\\ x(0)=0=x(T),\end{gathered}\tag{1.1}$$

where

(H) $T>0$ is arbitrary, $L,V\colon\ \mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ are convex, Gâteaux differentiable in the second variable and measurable in t functions.

We are looking for solutions of (1.1) being a pair (x,p) of absolutely continuous functions $x,p\colon\ [0,T]\longrightarrow\mathbb{R}^n$, $x(0)=0=x(T)$ such that

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{dt}}p(t)+V_x(t,x(t))=0,\\ p(t)=L_{x'}(t,x'(t)).\end{gathered}$$

Of course, if $L(t,x') = \frac{1}{2}|x'|^2$ or $t \to L_p^*(t,p(t))$ (L^* denotes the Fenchel conjugate of $L(t,\cdot)$) is an absolutely continuous function, then our solution of (1.1) belongs to the space $C^{1,+}([0,T],\mathbb{R}^n)$ of continuously differentiable functions x whose derivatives x' are absolutely continuous. In the sequel we assume that V_x is *superlinear*. It is clear that (1.1) is the Euler-Lagrange equation for the functional

$$J(x) = \int_0^T (-V(t,x(t)) + L(t,x'(t)))\,\mathrm{d}t \tag{1.2}$$

considered on the space $A_{0,0}$ of absolutely continuous functions $x\colon \mathbb{R} \to \mathbb{R}^n$, $x(0) = 0 = x(T)$.

The Dirichlet problem (1.1) was studied in the eighties by many authors in the sublinear case as well as in the superlinear one (see e.g. [6]). However, we believe that our paper may contribute some new look at this problem. This is because we propose to study (1.1) by duality methods in a way, to some extend, analogous to the methods developed for (1.1) in sublinear cases [6], [7]. Some cases of (1.1) for superlinear V_x were studied in [5], [6], [2], [9]. It is interesting that the method developed in [5] is based on the dual variational method for the problem, following the idea developed in [6]. Since the functional (1.2) is, in general, unbounded in $A_{0,0}$ (especially in the superlinear case), therefore it is obvious that we must look for critical points of J of "*minmax*" type. The main difficulties which appear here are the following what kind of sets we should choose over which we wish to calculate "*minmax*" of J and then to link this value with critical points of J. Of course, we have the mountain pass theorems, the saddle points theorems, the Morse theory, ... (see e.g. [8], [6]) but all these do not exhaust all critical points of J.

Our aim is to find a nonlinear subspace X of $A_{0,0}$ defined by the type of nonlinearity of V (and in fact also L). To be more precise let us formulate the basic hypothesis we need:

(H1) there exist $0 < \alpha_1,\ \alpha_2,\ \alpha_1 \leqslant \alpha_2$ and $d_1, d_2 \in \mathbb{R}$ such that for $x' \in L^2$

$$d_1 + \frac{\alpha_1}{2}\|x'\|_{L^2}^2 \leqslant \int_0^T L(t,x'(t))\,\mathrm{d}t \leqslant \frac{\alpha_2}{2}\|x'\|_{L^2}^2 + d_2, \tag{1.3}$$

$$L(t,\cdot) \text{ is strictly convex, } V_x(t,\cdot) \text{ is continuous, } t \in [0,T],$$

there exist $0 < \beta_1 < \beta_2$, $q_1 > 1$, $q > 2$, $k_1, k_2 \in \mathbb{R}$ such that for $x \in L^q$

$$k_1 + \frac{\beta_1}{q_1}\|x\|_{L^{q_1}}^{q_1} \leqslant \int_0^T V(t,x(t))\,\mathrm{d}t \leqslant \frac{\beta_2}{q}\|x\|_{L^q}^q + k_2. \tag{1.4}$$

Having the type of nonlinearities of L and V fixed we are able to define nonlinear subspaces $\overline{X}$, $\tilde{X}$ and X as follows. First, for a given, arbitrary $k_3 \in \mathbb{R}$ we put

$$\overline{X} = \left\{ v \in A_{0,0}\colon \int_0^T V(t,v(t))\,\mathrm{d}t \leqslant \frac{1}{2}\int_0^T L(t,v'(t))\,\mathrm{d}t + k_3 \right\}.$$

We reduce the space $\overline{X}$ to the set

$$\tilde{X} = \{v \in \overline{X} \colon \ p(t) = L_{x'}(t, v'(t)), \ t \in [0,T] \text{ belongs to } A\},$$

where A is the space of absolutely continuous functions $v\colon \ [0,T] \to \mathbb{R}^n$ with $v' \in L^2$, and next to the set $X \subset \tilde{X}$ with the following property: for each $v \in X$, there exists (possible another) $\tilde{v} \in X$ such that $V_x(t, v(t)) = -\frac{\mathrm{d}}{\mathrm{d}t} L_{x'}(t, \tilde{v}'(t))$ for a.e. $t \in [0,T]$.

It is clear that, in general, the set X is much smaller than $\tilde{X}$ and that it depends strongly on the type of nonlinearities V and L. We easily see that X is not in general a closed set in A. As the dual set to X we shall consider the set

$$\begin{aligned} X^d = \{p \in A_T \colon \ &\text{there exist } \ v \in X \text{ and } d_p \in \mathbb{R}^n \\ &\text{such that } p(t) = L_{x'}(t, v'(t)) - d_p, \ t \in [0,T] \text{ a.e.}\}, \end{aligned}$$

where A_T denotes the space of absolutely continuous functions $v\colon \ [0,T] \to \mathbb{R}^n$ with $v' \in L^2$ and $v(T) = 0$.

The constant d_p from the specification of X^d possesses a very interesting property:

Lemma 1.1. *For any $p \in X^d$ the constant d_p from the specification of X^d is a minimizer of the functional*

$$d \longrightarrow \int_0^T L^*(t, p(t) + d)\,\mathrm{d}t.$$

P r o o f. From the definition of X^d we have $p(t) + d_p = L_{x'}(t, x'(t))$ a.e. in $[0,T]$ for some $x \in X$. This means that $x'(t) = L^*_p(t, p(t) + d_p)$ a.e. in $[0,T]$. Integrating this equality yields, since $x(0) = x(T) = 0$ and L^* is convex, the assertion of the lemma. □

Taking into account the structure of the set X we shall study the functional

$$J(x) = \int_0^T (-V(t, x(t)) + L(t, x'(t)))\,\mathrm{d}t$$

on X.

We shall look for a "min" of J over the set X, i.e.

$$\min_{x \in X} J(x).$$

To show that the element $\bar{x} \in X$ realizing "min" is a critical point of J we develop a duality theory between J and dual to it J_D, described in the next section. Just by virtue of the duality theory we are able to avoid in our proof of existence of critical points the deformation lemmas, the Ekeland variational principle or PS type conditions. One more advantage of our duality results is obtaining for the first time in the superlinear case a measure of the duality gap between the primal and the dual functional for approximate solutions to (1.1) (for the sublinear case see [7]).

The main result of our paper is the following:

Theorem (Main)**.** *Under hypotheses* (H) *and* (H1) *there exists a pair* $(\bar{x}, \bar{p}+d_{\bar{p}})$, $\bar{x} \in X$, $\bar{p} \in X^d$, $d_{\bar{p}} \in \mathbb{R}^n$ *which is a solution to* (1.1) *and such that*

$$J(\bar{x}) = \min_{x\in X} J(x) = \min_{p\in X^d} \max_{d\in\mathbb{R}^n} J_D(p,d) = J_D(\bar{p}, d_{\bar{p}}).$$

We see that our hypotheses on L and V concern only convexity of $L(t,\cdot)$ or $V(t,\cdot)$ and that the latter function is of the superquadratic type. We do not assume that $V(t,x) \geqslant 0$. However, we require that the above set X is nonempty, which we must check for each concrete type of equation. Some routine how to do that we show at the end of the paper for the equation

$$x'' + V_x(t,x) = 0.$$

2. Duality results

To obtain a duality principle we need a kind of perturbation of J. Thus define for each $x \in X$ the perturbation of J as

$$J_x(y) = \int_0^T (V(t, x(t)+y(t)) - L(t, x'(t)))\,\mathrm{d}t \tag{2.1}$$

for $y \in L^2$. Of course, $J_x(0) = -J(x)$. For $x \in X$ and $p \in X^d$, we define a type of conjugate of J by

$$J_x^{\#}(p) = \sup_{y\in L^2}\left\{\int_0^T \langle y(t), p'(t)\rangle\,\mathrm{d}t - \int_0^T V(t, x(t)+y(t))\,\mathrm{d}t\right\} + \int_0^T L(t, x'(t))\,\mathrm{d}t.$$

By direct calculation we obtain

$$\begin{aligned} J_x^{\#}(p) = &-\int_0^T \langle x(t), p'(t)\rangle\,\mathrm{d}t + \int_0^T L(t,x'(t))\,\mathrm{d}t + \int_0^T V^*(t,p'(t))\,\mathrm{d}t \\ = &\int_0^T \langle x'(t), p(t)+d\rangle\,\mathrm{d}t + \int_0^T L(t,x'(t))\,\mathrm{d}t \\ &+ \int_0^T V^*(t,p'(t))\,\mathrm{d}t \quad \text{for each } d\in\mathbb{R}^n. \end{aligned} \tag{2.2}$$

Now we take "min" from $J_x^{\#}(p)$ with respect to $x \in X$ and calculate it. Because X is not a linear space we need some trick to avoid calculation of the conjugate with respect to a nonlinear space. To this effect we use the special structure of the set X^d. First we observe that for each $p \in X^d$ and appropriate d_p there exists $x_p \in X$ such that

$$p(t) + d_p = L_{x'}(t, x_p'(t))$$

and, by the classical convex analysis argument

$$x_p'(t) = L_p^*(t, p(t) + d_p),$$

where L^* is the Fenchel conjugate to L. Therefore for the above d_p we have

$$\int_0^T \left\langle x_p'(t), p(t) + d_p \right\rangle \, \mathrm{d}t - \int_0^T L(t, x_p'(t)) \, \mathrm{d}t = \int_0^T L^*(t, p(t) + d_p) \, \mathrm{d}t.$$

Next let us note that, on the other hand,

$$\begin{aligned}
&\int_0^T \left\langle x_p'(t), p(t) + d_p \right\rangle \, \mathrm{d}t - \int_0^T L(t, x_p'(t)) \, \mathrm{d}t \\
&\leqslant \sup_{x \in X} \left\{ \int_0^T \langle x'(t), p(t) + d_p \rangle \, \mathrm{d}t - \int_0^T L(t, x'(t)) \, \mathrm{d}t \right\} \\
&\leqslant \sup_{x' \in L^2} \left\{ \int_0^T \langle x'(t), p(t) + d_p \rangle \, \mathrm{d}t - \int_0^T L(t, x'(t)) \, \mathrm{d}t \right\} \\
&= \int_0^T L^*(t, p(t) + d_p) \, \mathrm{d}t
\end{aligned}$$

and actually all inequalities above are equalities. Therefore we can calculate for $p \in X^d$ and an appropriate d_p

$$\begin{aligned}
\text{(2.3)} \qquad \sup_{x \in X} -J_x^{\#}(-p) &= \sup_{x \in X} \left\{ \int_0^T \langle x'(t), p(t) + d_p \rangle \, \mathrm{d}t - \int_0^T L(t, x'(t)) \, \mathrm{d}t \right\} \\
&\quad - \int_0^T V^*(t, -p'(t)) \, \mathrm{d}t \\
&= \int_0^T L^*(t, p(t) + d_p) \, \mathrm{d}t - \int_0^T V^*(t, -p'(t)) \, \mathrm{d}t.
\end{aligned}$$

For $p \in X^d$ and each $d \in \mathbb{R}^n$ let us put

$$J_D(p, d) = - \int_0^T L^*(t, p(t) + d) \, \mathrm{d}t + \int_0^T V^*(t, -p'(t)) \, \mathrm{d}t.$$

From (2.3) we infer for $p \in X^d$ that

$$\sup_{x\in X} -J_x^{\#}(-p) = -J_D(p, d_p). \tag{2.4}$$

We can also define a type of the second conjugate of J: for $y \in L^2$, $x \in X$, put

$$\begin{aligned} J_x^{\#\#}(y) = \sup_{p\in X^d} \Big\{ &\int_0^T \langle y(t), -p'(t)\rangle \,\mathrm{d}t + \int_0^T \langle x(t), -p'(t)\rangle \,\mathrm{d}t \\ &- \int_0^T L(t, x'(t))\,\mathrm{d}t - \int_0^T V^*(t, -p'(t))\,\mathrm{d}t \Big\}. \end{aligned}$$

We assert that $J_x^{\#\#}(0) = -J(x)$. To prove that, we use the special structure of X. First we observe that for each $x \in X$ there exists $\bar{p} \in X^d$ such that $\bar{p}'(\cdot) = -V_x(\cdot, x(\cdot))$ and therefore

$$\int_0^T \langle -\bar{p}'(t), x(t)\rangle \,\mathrm{d}t - \int_0^T V^*(t, -\bar{p}'(t))\,\mathrm{d}t = \int_0^T V(t, x(t))\,\mathrm{d}t.$$

Next let us note that

$$\begin{aligned} &\int_0^T \langle -\bar{p}'(t), x(t)\rangle \,\mathrm{d}t - \int_0^T V^*(t, -\bar{p}'(t))\,\mathrm{d}t \\ &\qquad \leqslant \sup_{p\in X^d} \Big\{ \int_0^T \langle -p'(t), x(t)\rangle \,\mathrm{d}t - \int_0^T V^*(t, -p'(t))\,\mathrm{d}t \Big\} \\ &\qquad \leqslant \sup_{p'\in L^2} \Big\{ \int_0^T \langle -p'(t), x(t)\rangle \,\mathrm{d}t - \int_0^T V^*(t, -p'(t))\,\mathrm{d}t \Big\} \\ &\qquad = \int_0^T V(t, x(t))\,\mathrm{d}t. \end{aligned}$$

Hence we see that, for $x \in X$,

$$J_x^{\#\#}(0) = -\int_0^T (-V(t, x(t)) + L(t, x'(t)))\,\mathrm{d}t = -J(x). \tag{2.5}$$

We easily compute (see (2.4))

$$\begin{aligned} \sup_{x\in X} J_x^{\#\#}(0) &= \sup_{x\in X} \sup_{p\in X^d} -J_x^{\#}(-p) = \sup_{p\in X^d} \sup_{x\in X} -J_x^{\#}(-p) \\ &= \sup_{p\in X^d} -J_D(p, d_p) = \sup_{p\in X^d} \inf_{d} -J_D(p, d) \end{aligned} \tag{2.6}$$

where the last equality is a consequence of Lemma 1.1.

Hence, from the above and (2.6) we obtain the following duality principle:

Theorem 2.1. *For functionals J and J_D we have the duality relation*

$$\inf_{x\in X} J(x) = \inf_{p\in X^d} \sup_{d} J_D(p,d). \tag{2.7}$$

Denote by $\partial J_x(y)$ the subdifferential of J_x. In particular, if q' is such that $1/q' + 1/q = 1$ then

$$\partial J_{\bar{x}}(0) = \left\{ p' \in L^{q'} : \int_0^T V^*(t, p'(t))\, \mathrm{d}t + \int_0^T V(t, \bar{x}(t))\, \mathrm{d}t = \int_0^T \langle p'(t), \bar{x}(t)\rangle \, \mathrm{d}t \right\}.$$

The next result formulates a variational principle for "minmax" arguments.

Theorem 2.2. *Let $\bar{x} \in X$ be such that*

$$+\infty > J(\bar{x}) = \inf_{x\in X} J(x) > -\infty$$

and let the set $\partial J_{\bar{x}}(0)$ be nonempty. Then there exists $-\bar{p}' \in \partial J_{\bar{x}}(0)$ with $\bar{p}(t) = -\int_t^T \bar{p}'(s)\, \mathrm{d}s$ belonging to X^d, such that $\bar{p}$ together with $d_{\bar{p}}$ satisfies

$$J_D(\bar{p}, d_{\bar{p}}) = \inf_{p\in X^d} \sup_{d} J_D(p,d).$$

Furthermore,

$$J_{\bar{x}}(0) + J_{\bar{x}}^{\#}(-\bar{p}) = 0, \tag{2.8}$$

$$J_D(\bar{p}, d_{\bar{p}}) - J_{\bar{x}}^{\#}(-\bar{p}) = 0. \tag{2.9}$$

Proof. By Theorem 2.1 to prove the first assertion it suffices to show that $J(\bar{x}) \geqslant J_D(\bar{p}, d_{\bar{p}})$. Let us observe that $-\bar{p}' \in \partial J_{\bar{x}}(0)$ means, in fact, that $-\bar{p}'(t) = V_x(t, \bar{x}(t))$ for a.e. $t \in [0,T]$ and therefore we have

$$\begin{aligned} -J(\bar{x}) &= \int_0^T (V(t,\bar{x}(t)) - L(t, \bar{x}'(t)))\, \mathrm{d}t \\ &= \int_0^T (-V^*(t, -\bar{p}'(t)) - L(t, \bar{x}'(t)))\, \mathrm{d}t + \int_0^T \langle \bar{x}(t), -\bar{p}'(t)\rangle \, \mathrm{d}t \\ &\leqslant \int_0^T (-V^*(t, -\bar{p}'(t)) + L^*(t, \bar{p}(t) + d_{\bar{p}}))\, \mathrm{d}t = -J_D(\bar{p}, d_{\bar{p}}). \end{aligned}$$

Hence $J(\bar{x}) \geqslant J_D(\bar{p}, d_{\bar{p}})$ and so $J(\bar{x}) = J_D(\bar{p}, d_{\bar{p}}) = \inf_{p\in X^d} \sup_{d} J_D(p,d)$. The first assertion will be proved if we show that $\bar{p} \in X^d$.

The second assertion is a simple consequence of two facts: $J_{\bar{x}}(0) = -J(\bar{x})$ so $J_{\bar{x}}(0) + J(\bar{x}) = 0$ and $-\bar{p}' \in \partial J_{\bar{x}}(0)$ i.e. $J_{\bar{x}}(0) + J_{\bar{x}}^{\#}(-\bar{p}) = 0$.

Then equality (2.9) implies that

$$\int_0^T (L^*(t, \bar{p}(t) + d_{\bar{p}})) + L(t, \bar{x}'(t)))\,\mathrm{d}t = \int_0^T \langle \bar{x}'(t), \bar{p}(t) + d_{\bar{p}} \rangle\,\mathrm{d}t$$

and so $\bar{p}(t) + d_{\bar{p}} = L_{x'}(t, \bar{x}'(t))$. By the definition of $\bar{p}$ we also have $\bar{p}(T) = 0$ and therefore $\bar{p} \in X^d$. $\square$

From equations (2.8), (2.9) we are able to derive a dual to the Euler-Lagrange equations (1.1).

Corollary 2.1. *Let $\bar{x} \in X$ be such that*

$$+\infty > J(\bar{x}) = \inf_{x \in X} J(x) > -\infty.$$

Then there exists $\bar{p} \in X^d$ such that the pair $(\bar{x}, \bar{p})$ satisfies the relations

$$-\bar{p}'(t) = V_x(t, \bar{x}(t)), \tag{2.10}$$

$$\bar{p}(t) + d_{\bar{p}} = L_{x'}(t, \bar{x}'(t)), \tag{2.11}$$

$$J_D(\bar{p}, d_{\bar{p}}) = \inf_{p \in X^d} \sup_d J_D(p, d) = \inf_{x \in X} J(x) = J(\bar{x}). \tag{2.12}$$

P r o o f. By the assumptions on V we see that $y \to \int_0^T V(t, y(t))\,\mathrm{d}t$ is finite in L^q, convex and lower semicontinuous. Therefore $J_{\bar{x}}(y)$ is continuous in L^q. Hence $\partial J_{\bar{x}}(0)$ is nonempty and so the existence of $\bar{p}'$ in Theorem 2.2 is now obvious. Equations (2.8) and (2.9) imply

$$\int_0^T V(t, \bar{x}(t))\,\mathrm{d}t + \int_0^T V^*(t, -\bar{p}'(t))\,\mathrm{d}t - \int_0^T \langle \bar{x}(t), -\bar{p}'(t) \rangle\,\mathrm{d}t = 0,$$
$$\int_0^T L^*(t, \bar{p}(t) + d_{\bar{p}})\,\mathrm{d}t + \int_0^T L(t, \bar{x}'(t))\,\mathrm{d}t - \int_0^T \langle \bar{x}'(t), \bar{p}(t) + d_{\bar{p}} \rangle\,\mathrm{d}t = 0,$$

and then (2.10), (2.11). Relations (2.12) are a direct consequence of Theorem 2.1 and Theorem 2.2. $\square$

As a direct consequence of the above corollary and the definition of X^d we have

Corollary 2.2. *Under the same assumptions as in Corollary* 2.1 *there exists a pair* $(\bar{x}, \bar{p}) \in X \times X^d$ *satisfying together with* $d_{\bar{p}}$ *relations* (2.12), *and the pair* $(\bar{x}, \bar{p} + d_{\bar{p}})$ *is a solution to* (1.1). *Conversely, each pair* $(\bar{x}, \bar{p})$ *satisfying, together with* $d_{\bar{p}}$, *relations* (2.12) *satisfies also equations* (2.10), (2.11).

3. Variational principles and the duality gap for minimizing sequences

In this section we show that a statement similar to Theorem 2.2 is true for a minimizing sequence of J.

Theorem 3.1. *Let* $\{x_j\}$, $x_j \in X$, $j = 1, 2, \ldots$, *be a minimizing sequence for* J *and let*

$$+\infty > \inf_j J(x_j) = a > -\infty.$$

Then there exist $-p_j' \in \partial J_{x_j}(0)$ *with* $p_j \in X^d$, *such that* $\{(p_j, d_{p_j})\}$ *is a minimizing sequence for* J_D, *i.e.*

$$\inf_{x \in X} J(x) = \inf_{x_j \in X} J(x_j) = \inf_{p_j \in X^d} \sup_{d \in \mathbb{R}^n} J_D(p_j, d) = \inf_{p_j \in X^d} J_D(p_j, d_{p_j}).$$

Furthermore,

$$J_{x_j}(0) + J_{x_j}^{\#}(-p_j) = 0, \quad J_D(p_j, d_{p_j}) - J_{x_j}^{\#}(-p_j) \leqslant \varepsilon, \quad 0 \leqslant J(x_j) - J_D(p_j, d_{p_j}) \leqslant \varepsilon$$

for a given $\varepsilon > 0$ *and sufficiently large* j.

P r o o f. We have $\infty > \inf_{x_j \in X} J(x_j) = a > -\infty$, and therefore for a given $\varepsilon > 0$ there exists j_0 such that $J(x_j) - a < \varepsilon$ for all $j \geqslant j_0$. Further, the proof is similar to that of Theorem 2.2, so we only sketch it. First, as in the proof of Corollary 2.1 we observe that $\partial J_{x_j}(0)$ is nonempty for $j \geqslant j_0$ and take $-p_j' \in \partial J_{x_j}(0)$. In accordance with to the definition of X^d let us take as a primitive of p_j' such p_j that $p_j(T) = 0$. Therefore, for all $d \in \mathbb{R}^n$ we also have

$$\begin{aligned} -J(x_j) &= \int_0^T (V(t, x_j(t)) - L(t, x_j'(t)))\,\mathrm{d}t \\ &= \int_0^T (-V^*(t, -p_j'(t)) - L(t, x_j'(t)))\,\mathrm{d}t + \int_0^T \langle x_j(t), -p_j'(t)\rangle\,\mathrm{d}t \\ &\leqslant \int_0^T (-V^*(t, -p_j'(t)) + L^*(t, p_j(t) + d))\,\mathrm{d}t = \\ &= -J_D(p_j, d). \end{aligned}$$

Hence, due to Theorem 2.1,

$$a+\varepsilon \geqslant \sup_{d\in\mathbb{R}^n} J_D(p_j,d) = J_D(p_j,d_{p_j}) \geqslant a \quad \text{for } j \geqslant j_0.$$

The second assertion is a simple consequence of two facts: $J_{x_j}(0) = -J(x_j)$ so $J_{x_j}(0) + J(x_j) = 0$ and $-p_j' \in \partial J_{x_j}(0)$ i.e. $J_{x_j}(0) + J^{\#}_{x_j}(-p_j) = 0$. $\square$

The following corollary is a direct consequence of this theorem.

Corollary 3.1. *Let $\{x_j\}$, $x_j \in X$, $j = 1,2,\ldots$, be a minimizing sequence for J and let*

$$+\infty > \inf_j J(x_j) = a > -\infty.$$

If

$$-p_j'(t) = V_x(t, x_j(t))$$

then $p_j(t) = -\int_t^T p_j'(s)\,\mathrm{d}s$ belongs to X^d, and $\{(p_j, d_{p_j})\}$ is a minimizing sequence for J_D, i.e.

$$\inf_{x\in X} J(x) = \inf_{x_j\in X} J(x_j) = \inf_{p_j\in X^d}\sup_{d\in\mathbb{R}^n} J_D(p_j,d) = \inf_{p_j\in X^d} J_D(p_j,d_{p_j}).$$

Furthermore,

$$\begin{aligned} &J_D(p_j,d_{p_j}) - J^{\#}_{x_j}(-p_j) \leqslant \varepsilon,\\ &0 \leqslant J(x_j) - J_D(p_j,d_{p_j}) \leqslant \varepsilon \end{aligned} \tag{3.1}$$

for a given $\varepsilon > 0$ and sufficiently large j.

4. Existence of a minimum of J

The last problem which we have to solve is to prove the existence of $\overline{x} \in X$ such that

$$J(\overline{x}) = \min_{x\in X} J(x).$$

To obtain this it is enough to use hypothesis (H1), the results of the former section and known compactness theorems.

Theorem 4.1. *Under hypothesis* (H1) *there exists* $\bar{x} \in X$ *such that* $J(\bar{x}) = \min_{x \in X} J(x)$.

P r o o f. Let us observe that by (H1), $J(x)$ is bounded below on X. By (1.3), (1.4) we obtain

$$
\begin{aligned}
(4.1) \qquad J(x) &\geqslant \int_0^T L(t, x'(t))\,\mathrm{d}t - \frac{1}{2}\int_0^T L(t, x'(t))\,\mathrm{d}t - k_3 \\
&\geqslant \frac{\alpha_1}{4}\|x'\|^2 + d_1 - k_3.
\end{aligned}
$$

From (4.1) we infer the boundedness below of J on X as well as that the sets $S_b = \{x \in X\colon\ J(x) \leqslant b\}$, $b \in \mathbb{R}$ are nonempty for sufficiently large b and bounded with respect to the norm $\|x'\|_{L^2}$. The last means that S_b, $b \in \mathbb{R}$ are relatively weakly compact in $A_{0,0}$. It is a well known fact that the functional J is weakly lower semicontinuous in $A_{0,0}$ and thus also in X. Therefore there exists a sequence $\{x_n\}$, $x_n \in X$, such that $x_n \rightharpoonup \bar{x}$ weakly in $A_{0,0}$ with $\bar{x} \in A_{0,0}$ and $\liminf_{n\to\infty} J(x_n) \geqslant J(\bar{x})$. Moreover, we know that $\{x_n\}$ is uniformly convergent to $\bar{x}$. In order to complete the proof we must only show that $\bar{x} \in X$.

To prove that we apply the duality results of Section 3. To this effect let us recall from Corollary 3.1 that for

$$
(4.2) \qquad p_n'(t) = -V_x(t, x_n(t))
$$

$p_n(t) = -\int_t^T p_n'(s)\,\mathrm{d}s$ belongs to X^d, and take d_{p_n} such that $\max_{d\in\mathbb{R}^n} J_D(p_n, d) = J_D(p_n, d_{p_n})$. Then $\{(p_n, d_{p_n})\}$ is a minimizing sequence for J_D. We easily check that $\{d_{p_n}\}$ is a bounded sequence and therefore we may assume (up to a subsequence) that it is convergent. From (4.2) we infer that $\{p_n'\}$ is a bounded sequence in the L^2 norm and that it is pointwise convergent to

$$
\bar{p}'(t) = -V_x(t, \bar{x}(t))
$$

and so $\{p_n\}$ is uniformly convergent to $\bar{p}$ where $\bar{p}(t) = -\int_t^T \bar{p}'(s)\,\mathrm{d}s$. We can choose $d_{\bar{p}}$ satisfying the equality $\max_{d\in\mathbb{R}^n} J_D(\bar{p}, d) = J_D(\bar{p}, d_{\bar{p}})$.

By Corollary 3.1 (see (3.1)) we also have (taking into account (4.2)) that for $\varepsilon_n \to 0$ $(n \to \infty)$

$$
0 \leqslant \int_0^T (L^*(t, p_n(t) + d_{p_n}) + L(t, x_n'(t)))\,\mathrm{d}t - \int_0^T \langle x_n'(t), p_n(t) + d_{p_n}\rangle\,\mathrm{d}t \leqslant \varepsilon_n
$$

and so, passing to the limit we obtain

$$
0 = \int_0^T L^*(t, \bar{p}(t) + d_{\bar{p}})\,\mathrm{d}t + \lim_{n\to\infty}\int_0^T L(t, x_n'(t))\,\mathrm{d}t - \int_0^T \langle \bar{x}'(t), \bar{p}(t) + d_{\bar{p}}\rangle\,\mathrm{d}t
$$

and next, in view of the Fenchel inequality,

$$0 = \int_0^T L^*(t, \bar{p}(t) + d_{\bar{p}})\,\mathrm{d}t + \int_0^T L(t, \bar{x}'(t)))\,\mathrm{d}t - \int_0^T \langle \bar{x}'(t), \bar{p}(t) + d_{\bar{p}}\rangle\,\mathrm{d}t.$$

Hence $\bar{x} \in \overline{X}$. We have also

$$\bar{p}(t) + d_{\bar{p}} = L_{x'}(t, \bar{x}'(t)).$$

Thus $\bar{x} \in X$ and so the proof is completed. □

The following main theorem is a direct consequence of Theorem 4.1 and Corollary 2.1.

Theorem 4.2. *Under hypotheses* (H) *and* (H1) *there exists a pair* $(\bar{x}, \bar{p} + d_{\bar{p}})$ *which is a solution to* (1.1) *and such that*

$$J(\bar{x}) = \min_{x\in X} J(x) = \min_{p\in X^d} \max_{d\in\mathbb{R}^n} J_D(p,d) = J_D(\bar{p}, d_{\bar{p}}).$$

5. Example

Consider the problem

$$x''(t) + W_x(t, x(t)) = 0, \quad \text{a.e. in } [0,T], \qquad (5.1)$$
$$x(0) = 0 = x(T)$$

where $W(\cdot, x)$ is a measurable function in $[0,T]$, $x \in \mathbb{R}^n$, $W(t,\cdot)$, $t \in [0,T]$, is a convex, Frechet continuously differentiable function satisfying the following growth condition:

$$\text{there exist } 0 < \beta_1 < \beta_2,\ q_1 > 1,\ q > 2, \text{ such that for } x \in \mathbb{R}^n$$
$$\frac{\beta_1}{q_1}|x|^{q_1} \leqslant W(t,x) \leqslant \frac{\beta_2}{q}|x|^q.$$

In the notation of the paper we have $L(t,x') = \frac{1}{2}|x'|^2$ and $V(t,x) = W(t,x)$. It is easily seen that assumptions (H) and (H1) are satisfied. Therefore what we have to do is to construct a nonempty set X defined in Section 1. To this effect let us take any $k > 0$ and let $\overline{X}$ denote the same as in Section 1 with the new L and V. We assume the hypothesis

$$T^2\beta_2\Big(\frac{q}{q-1}\Big)^{q-1}k^{q-1} \leqslant k. \qquad (\mathrm{H1}')$$

Let us observe that hypothesis (H1′) asserts the following: if T or β_2 is large then k must be small and conversely, if we admit k large then T or β_2 must be small.

We shall show that the set

$$X = \tilde{X} = \{v \in \overline{X}\colon\ 0 < \|v\|_{L^\infty} \leqslant k,\ v' \in A\}$$

is a set X which we are looking for. That means: we must prove that for each function $x \in \tilde{X}$ the function

$$w\colon\ t \to \int_0^t \int_0^s W_x(\tau, x(\tau))\,\mathrm{d}\tau + at = w_0(t) + at \tag{5.2}$$

belongs to $\tilde{X}$ for $a = -\frac{1}{T} w_0(T)$. First note that in view of our assumption on W we have the estimate

$$\|W_x(\cdot, x(\cdot))\|_{L^\infty} \leqslant \beta_2 \Big(\frac{q}{q-1}\Big)^{q-1} \|x(\cdot)\|_{L^\infty}^{q-1}.$$

Therefore

$$\|w_0\|_{L^\infty} \leqslant \frac{T^2}{2} \beta_2 \Big(\frac{q}{q-1}\Big)^{q-1} \|x(\cdot)\|_{L^\infty}^{q-1}.$$

Hence, as $x \in \tilde{X}$, we have

$$\|w_0\|_{L^\infty} \leqslant \frac{T^2}{2} \beta_2 \Big(\frac{q}{q-1}\Big)^{q-1} k^{q-1}$$

and, by (H1′), $\|w\|_{L^\infty} \leqslant \|w_0\|_{L^\infty} + |w_0(T)| \leqslant k$. Since $0 \notin \tilde{X}$, it is clear that w is not identically zero. Thus

$$0 < \|w\|_{L^\infty} \leqslant k. \tag{5.3}$$

It is obvious that if we take k_3 sufficiently large then

$$\int_0^T W(t, z(t))\,\mathrm{d}t \leqslant \frac{1}{4} \int_0^T |z'(t)|^2\,\mathrm{d}t + k_3$$

for all z satisfying (5.3).

Therefore $w \in \tilde{X}$, and we can put $X = \tilde{X}$. It is also clear that the set $X = \tilde{X}$ is nonempty. Thus all assumptions of Theorem 4.2 are satisfied, so we come to the following theorem with $L = 1/2|x'|^2$.

Theorem 5.1. *There exists a pair* $(\bar{x}, \bar{p} + d_{\bar{p}})$ *which is a solution to* (5.1) *such that* $\bar{x} \neq 0$ *and*

$$J(\bar{x}) = \min_{x \in X} J(x) = \min_{p \in X^d} \max_{d \in \mathbb{R}^n} J_D(p, d) = J_D(\bar{p}, d_{\bar{p}}).$$

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