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Oleg Palumbíny

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ON OSCILLATION CRITERIA OF FOURTH ORDER LINEAR DIFFERENTIAL EQUATIONS

Oleg Palumbíny, Trnava

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Abstract. The paper deals with oscillation criteria of fourth order linear differential equations with quasi-derivatives.

Keywords: linear differential equation, quasi-derivative, monotone solution, Kneser solution, oscillatory solution

MSC 2000: 34C10, 34C11, 34D05

1. Introduction

Consider the linear differential equation of the fourth order with quasi-derivatives

(L) $$L(y) \equiv L_4 y + P(t) L_2 y + Q(t) y = 0,$$

where

$$\begin{aligned} L_0 y(t) &= y(t), \\ L_1 y(t) &= p_1(t) y'(t) = p_1(t) \frac{\mathrm{d}y(t)}{\mathrm{d}t}, \\ L_2 y(t) &= p_2(t)(p_1(t) y'(t))' = p_2(t)(L_1 y(t))', \end{aligned}$$

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$$L_3y(t) = p_3(t)(p_2(t)(p_1(t)y'(t))')' = p_3(t)(L_2y(t))',$$
$$L_4y(t) = (p_3(t)(p_2(t)(p_1(t)y'(t))')')' = (L_3y(t))',$$

$P(t)$, $Q(t)$, $p_i(t)$, $i = 1, 2, 3$, are real-valued continuous functions on an interval $I_a = [a, \infty)$, $-\infty < a < \infty$. It is assumed throughout that

(A) $$P(t) \leqslant 0,\ \ Q(t) \leqslant 0,\ \ p_i(t) > 0,\ \ i = 1, 2, 3,\ \ t \in I_a \text{ and } Q(t) \text{ is not identically zero in any subinterval of } I_a.$$

We note that in the whole paper we will use the notation $I_b = [b, \infty)$, b is any real number.

In [4] sufficient conditions for (L) to be oscillatory have been stated. In this paper we will deal with other ones. We will describe two oscillation criteria for (L), which create the content of Theorem 3 and 4.

Theorem 3 asserts that (L) is oscillatory as a consequence of the fact that the binomial third order linear differential equation (see Eq. (L*))

$$L_3^* x + \frac{\theta\mu(t)Q(t)}{p_1(t)} x = 0$$

is oscillatory.

Theorem 4 is a special case of Theorem 3 because it states a sufficient condition for the just mentioned third order differential equation to be oscillatory (and for (L), of course, too).

The paper is concluded by two examples illustrating the results mentioned above. In the end of the introduction we want to note that our theorems generalize some results which J. Regenda derived in [5] as well as in [6].

2. Definitions and preliminary results

Definition 1. A solution $y(t)$ of (L) on I_a is called positively (negatively) nonoscillatory iff there exists $t_0 \geqslant a$ such that $y(t) > 0$ ($y(t) < 0$), $t \geqslant t_0$.

Definition 2. A solution $y(t)$ of (L) on I_a is called nonoscillatory iff $y(t)$ is positively or negatively nonoscillatory.

Definition 3. The equation (L) is called nonoscillatory iff every nontrivial solution of (L) on I_a is nonoscillatory.

Definition 4. A nontrivial solution $y(t)$ of (L) on I_a is called oscillatory on I_a iff the set of all its zeros on I_a is not bounded from above.

Definition 5. The equation (L) is called oscillatory iff there exists at least one oscillatory solution of (L) on I_a.

Definition 6. A positively nonoscillatory solution $y(t)$ of (L) on I_a such that $y(t) > 0$ for $t \geqslant t_0 \geqslant a$ is called a monotone (Kneser) solution on $[t_0, \infty)$ iff $L_k y(t) > 0$ ($(-1)^k L_k y(t) > 0$), $k = 0, 1, 2, 3$, $t \geqslant t_0$.

Lemma 1 [1, Lemma 2.2]. *Let $f(t)$ be a real valued function defined in $[t_0, \infty)$ for some real number $t_0 \geqslant 0$. Suppose that $f(t) > 0$ and that $f'(t)$ and $f''(t)$ exist for $t \geqslant t_0$. Suppose also that if $f'(t) \geqslant 0$ eventually, then $\lim\limits_{t\to\infty} f(t) = A < \infty$. Then*

$$\liminf_{t\to\infty} |t^\alpha f''(t) - \alpha t^{\alpha-1} f'(t)| = 0$$

for any $\alpha \leqslant 2$.

Lemma 2 [7, Lemma 3]. *Let* (A) *and* $\int^{\infty} (1/p_1(t))\,\mathrm{d}t = \infty$ *hold. Then for every nonoscillatory solution $y(t)$ of* (L) *there exists a number $t_0 \geqslant a$ such that either*

$$\big(y(t)L_1y(t) > 0,\ \ y(t)L_2y(t) > 0\big) \ \text{ or } \ \big(y(t)L_1y(t) < 0,\ \ y(t)L_2y(t) > 0\big)$$
$$\text{or } \ \big(y(t)L_1y(t) > 0,\ \ y(t)L_2y(t) < 0\big) \quad \text{for all } t \geqslant t_0.$$

Lemma 3 [4, Lemma 6]. *Let* (A) *hold. If every positively nonoscillatory solution of* (L) *on I_a is either monotone or Kneser, then* (L) *is oscillatory.*

Lemma 4. *Consider a linear differential equation*

$$\text{(M)} \qquad M(y) \equiv a_1(t)y''' + a_2(t)y'' + a_3(t)y' + a_4(t)y = 0,$$

where the functions $a_i(t)$, $i = 1, 2, 3, 4$, are continuous on $[b, c]$, $b < c$. Then every nontrivial solution of (M) *on $[b, c]$ admits at most two zeros on this interval if and only if there exist functions $z_1(t)$, $z_2(t)$, both from the class $C^3([b, c])$, such that*

$$z_1(t) > 0,\ \ z_2(t) > 0,\ \ W(z_1, z_2) > 0,\ \ M(z_1) \geqslant 0,\ \ M(z_2) \leqslant 0,\ \ t \in [b, c],$$

where $W(z_1, z_2)$ denotes Wronski's determinant of $z_1(t)$, $z_2(t)$.

P r o o f. The lemma is the special case of [2, Theorem 4.1] for $n = 3$. □

Lemma 5 [3, Theorem]**.** *Consider the linear differential equation*

$$\text{(N)} \qquad N(y) \equiv (p_3(t)(p_2(t)y')')' + r(t)y = 0,$$

where $p_i(t)$, $i = 2, 3$ *are positive and continuous on* I_a, $r(t)$ *is a function nonpositive and continuous on* I_a, $r(t)$ *is not identically zero in any subinterval of* I_a. *If*

$$\int_a^\infty \frac{1}{p_2(s)}\,\mathrm{d}s = \int_a^\infty \frac{1}{p_3(s)}\,\mathrm{d}s = \int_a^\infty -r(s)\,\mathrm{d}s = \infty,$$

then (N) *is oscillatory.*

Lemma 6. *Let the function* $L_3y(t)$ *be continuous on an interval* I_c *and let* $y(t) > 0$, $L_1y(t) > 0$, $L_2y(t) > 0$, $L_3y(t) < 0$, $p_1'(t) \geqslant 0$, $p_2'(t) \geqslant 0$ *on* I_c. *Then*

$$y(t) > \frac{t-c}{2p_1(t)} L_1y(t)$$

for all $t \in I_c$.

P r o o f. Let us consider the function $f(t) = L_1y(t) - (t-c)(L_1y(t))'$ for $t \in I_c$. It is obvious that $f(c) = L_1y(c) > 0$. Then

$$\begin{aligned} f'(t) &= (L_1y(t))' - (L_1y(t))' - (t-c)(L_1y(t))'' \\ &= -(t-c)\Big(\frac{L_2y(t)}{p_2(t)}\Big)' \\ &= -(t-c)\frac{p_2(t)L_3y(t)/p_3(t) - L_2y(t)p_2'(t)}{p_2^2(t)} > 0 \quad \text{for } t > c. \end{aligned}$$

This implies that $f(t)$ is nondecreasing on I_c and $f(t) \geqslant f(c) > 0$ on I_c, i.e.

$$\text{(1)} \qquad L_1y(t) > (t-c)(L_1y(t))' \quad \text{for } t \geqslant c.$$

The integration of (1) over $[c,t]$, $c < t$ yields

$$\begin{aligned} \int_c^t p_1(s)y'(s)\,\mathrm{d}s &> \int_c^t (s-c)(L_1y(s))'\,\mathrm{d}s \\ &> (t-c)L_1y(t) - \int_c^t L_1y(s)\,\mathrm{d}s \\ &> (t-c)L_1y(t) - \int_c^t p_1(s)y'(s)\,\mathrm{d}s. \end{aligned}$$

This implies that

$$\int_c^t p_1(s)y'(s)\,\mathrm{d}s > \frac{(t-c)}{2}L_1y(t),$$

$$p_1(t)y(t) - p_1(c)y(c) + \int_c^t -p_1'(s)y(s)\,\mathrm{d}s > \frac{(t-c)}{2}L_1y(t),$$

$$p_1(t)y(t) > \frac{(t-c)}{2}L_1y(t),$$

$$y(t) > \frac{(t-c)}{2p_1(t)}L_1y(t)$$

for all $t \geqslant c$. The assertion is proved. □

Theorem 1. *Let* (A), $\int_a^\infty (1/p_1(t))\,\mathrm{d}t = \int_a^\infty (1/p_2(t))\,\mathrm{d}t = \infty$ *hold. If* (L) *is nonoscillatory on* I_a, *then there exists* $t_0 \in I_a$ *and a solution* $y(t)$ *of* (L) *on* I_{t_0} *such that either* $\big(y(t) > 0,\ L_1y(t) > 0,\ L_2y(t) < 0\big)$ *or* $\big(y(t) > 0,\ L_1y(t) > 0,\ L_2y(t) > 0,\ L_3y(t) < 0\big)$ *on* I_{t_0}.

P r o o f. According to Lemma 3 there exists a positively nonoscillatory solution $y(t)$ of (L) on I_a such that $y(t)$ is neither monotone nor Kneser. Lemma 2 yields that for this $y(t)$ there exists $b \in I_a$ such that

$$\begin{aligned} &\big(y(t) > 0,\ L_1y(t) > 0,\ L_2y(t) > 0\big)\\ \text{or }&\big(y(t) > 0,\ L_1y(t) < 0,\ L_2y(t) > 0\big)\\ \text{or }&\big(y(t) > 0,\ L_1y(t) > 0,\ L_2y(t) < 0\big)\end{aligned} \tag{2}$$

for $t \geqslant b$. Now (in accordance with (2)) let us assume that $y(t) > 0$, $L_1y(t) > 0$, $L_2y(t) > 0$ on some I_{t_0}. Then $L_4y(t) = -P(t)L_2y(t) - Q(t)y(t) \geqslant 0$, $t \geqslant t_0$ and $L_4y(t) = 0$ holds at most at isolated points (according to (A)). This implies that $L_3y(t)$ is increasing on I_{t_0}. Two cases may now occur:

$$L_3y(t) > 0 \ \text{ or } \ L_3y(t) < 0 \tag{3}$$

on some I_{t_1}, $t_1 \geqslant t_0$. However, the assertion in the first expression of (3) cannot be true, because $y(t)$ would be monotone on I_{t_1}, which is impossible.

If (in accordance with (2)) for this positively nonoscillatory solution $y(t)$ the inequalities $y(t) > 0$, $L_1y(t) < 0$, $L_2y(t) > 0$ held on some I_{t_0}, then (for the same reason) (3) would hold on some I_{t_2}, $t_2 \geqslant t_0$. However, the second inequality of (3) cannot be true, because $y(t)$ would be Kneser on I_{t_2}, which is impossible, either. If the first inequality of (3) were true, then

$$L_2y(t) = L_2y(t_2) + \int_{t_2}^t \frac{L_3y(s)}{p_3(s)}\,\mathrm{d}s \geqslant L_2y(t_2) > 0, \quad t \geqslant t_2$$

and

$$L_1y(t) = L_1y(t_2) + \int_{t_2}^{t} \frac{L_2y(s)}{p_2(s)}\,\mathrm{d}s$$
$$\geqslant L_1y(t_2) + L_2y(t_2)\int_{t_2}^{t} \frac{\mathrm{d}s}{p_2(s)} \to \infty \quad \text{for} \quad t \to \infty,$$

which would contradict $L_1y(t) < 0$ on I_{t_2}. The theorem is established. □

Remark 1. A part (viz. the necessary condition for (L) to be nonoscillatory) of [5, Theorem 1.1] is a special case of Theorem 1, where $p_i(t) \equiv 1$, $i = 1, 2, 3$, $t \in I_a$.

Theorem 2. *Let* (A), $p_2'(t) \geqslant 0$ *on* I_a, $p_2(\infty) < \infty$, $p_3(\infty) < \infty$, *and* $t^2P(t) + (2tp_3(t))' \geqslant 0$ *for* $t > t_0 \geqslant \max\{a, 0\}$ *hold. Then the equation* (L) *does not admit a solution* $y(t)$ *on* I_a *such that* $y(t) > 0$, $L_1y(t) > 0$, $L_2y(t) < 0$ *for* $t > t_1 \geqslant t_0$.

P r o o f. Let us assume for a while the existence of such a solution $y(t)$ of (L) on some (t_1, ∞). Then

$$L_4y(t) + P(t)L_2y(t) + Q(t)y(t) \equiv 0 \quad \text{on } [t_1, \infty). \tag{4}$$

Now we shall prove that there exists no $t_2 \geqslant t_1$ such that $L_3y(t) < 0$ on I_{t_2}. So, assume that such t_2 exists. Then

$$L_1y(t) = L_1y(t_2) + \int_{t_2}^{t} \frac{L_2y(s)}{p_2(s)}\,\mathrm{d}s$$
$$= L_1y(t_2) + \int_{t_2}^{t} \frac{1}{p_2(s)}\left(L_2y(t_2) + \int_{t_2}^{s} \frac{L_3y(r)}{p_3(r)}\,\mathrm{d}r\right)\mathrm{d}s$$
$$\leqslant L_1y(t_2) + \int_{t_2}^{t} \frac{L_2y(t_2)}{p_2(s)}\,\mathrm{d}s \to -\infty \quad \text{for} \quad t \to \infty,$$

which contradicts $L_1y(t) > 0$ on (t_1, ∞) and implies the existence of $t_3 > t_1$ such that $L_3y(t_3) \geqslant 0$. Multiplying (4) by t^2 and integrating over $[t_3, t]$ yields

$$\begin{aligned} &t^2L_3y(t) - t_3^2L_3y(t_3) - 2tp_3(t)L_2y(t) + 2t_3p_3(t_3)L_2y(t_3) \\ &\quad + \int_{t_3}^{t} (s^2P(s) + (2sp_3(s))')L_2y(s)\,\mathrm{d}s + \int_{t_3}^{t} s^2Q(s)y(s)\,\mathrm{d}s = 0. \end{aligned} \tag{5}$$

Let us denote $A(t) = t^2L_3y(t) - 2tp_3(t)L_2y(t) = p_3(t)(t^2(L_2y(t))' - 2tL_2y(t))$. Then from (5) we obtain (6), where

$$\begin{aligned} A(t) - t_3^2L_3y(t_3) + 2t_3p_3(t_3)L_2y(t_3) + \int_{t_3}^{t} (s^2P(s) + (2sp_3(s))')L_2y(s)\,\mathrm{d}s& \\ + \int_{t_3}^{t} s^2Q(s)y(s)\,\mathrm{d}s = 0.& \end{aligned} \tag{6}$$

Since the second, fourth and fifth terms on the left-hand side of (6) are nonpositive and the third is negative we have

$$|A(t)| \geqslant A(t) \geqslant -2t_3 p_3(t_3) L_2 y(t_3) > 0, \quad t \geqslant t_3. \tag{7}$$

In Lemma 1 let us put $\alpha = 2$, $f'(t) = \frac{\mathrm{d}f(t)}{\mathrm{d}t} = L_2 y(t)$ on I_{t_3}. Then $f'(t) < 0$, $t \geqslant t_3$, and on I_{t_3} we have

$$\begin{aligned} f(t) &= f(t_3) + \int_{t_3}^{t} p_2(s)(L_1 y(s))' \,\mathrm{d}s \\ &\geqslant f(t_3) + p_2(\infty) \int_{t_3}^{t} (L_1 y(s))' \,\mathrm{d}s \\ &\geqslant f(t_3) + p_2(\infty) \int_{t_3}^{\infty} (L_1 y(s))' \,\mathrm{d}s \\ &\geqslant f(t_3) + p_2(\infty) L_1 y(\infty) - p_2(\infty) L_1 y(t_3) \\ &\geqslant f(t_3) - p_2(\infty) L_1 y(t_3) > -\infty. \end{aligned} \tag{8}$$

It follows from (8) that $f(t_3)$ can be chosen such that $f(t) > 0$ on I_{t_3}. Then Lemma 1 yields $\liminf\limits_{t\to\infty} |A(t)| = 0$. However, (7) implies that $\liminf\limits_{t\to\infty} |A(t)| > 0$. This contradiction proves the theorem. □

Remark 2. [5, Theorem 1.3] is a special case of Theorem 2, where $p_i(t) \equiv 1$, $i = 1, 2, 3$, $t \in I_a$.

3. Oscillation criteria

Theorem 3. *Let a function $\mu(t)$ be positive and continuous in (T, ∞), $T > \max\{a, 0\}$, and such that $\liminf\limits_{t\to\infty}((t - t_0)/\mu(t)) \geqslant 2$ for any $t_0 \geqslant a$. Let* (A) *hold and let $p_i'(t) \geqslant 0$, $i = 1, 2$, $t^2 P(t) + (2tp_3(t))' \geqslant 0$ for $t \geqslant T > \max\{a, 0\}$, $p_j(\infty) < \infty$, $j = 2, 3$, $\int_a^\infty (1/p_1(t)) \,\mathrm{d}t = \infty$. For every $\tau > T$ let there exist $\tau_1 > \tau$ such that $p_1(t)p_2'(t)p_3'(t) + p_1(t)p_2''(t)p_3(t) + \theta\mu(t)Q(t)(t - \tau) \leqslant 0$ for all $t \geqslant \tau_1$, $\theta \in (0, 1)$. If the differential equation*

$$L^*(x) \equiv L_3^* x + \frac{\theta\mu(t)Q(t)}{p_1(t)} x \equiv (p_3(t)(p_2(t)x')')' + \frac{\theta\mu(t)Q(t)}{p_1(t)} x = 0 \tag{L*}$$

(where $x = x(t)$, $x' = \mathrm{d}x/\mathrm{d}t$) is oscillatory for some $\theta \in (0, 1)$, then (L) *is oscillatory.*

Proof. Let us assume (L) to be nonoscillatory. It is clear that $\int_a^\infty (\mathrm{d}t/p_2(t)) = \infty$. Then Theorem 1 and Theorem 2 yield the existence of a solution $y(t)$ of (L) on

$[t_0, \infty)$, where $t_0 \geqslant T \geqslant a$, such that $y(t) > 0$, $L_1 y(t) > 0$, $L_2 y(t) > 0$, $L_3 y(t) < 0$, $t \geqslant t_0$. Thus, according to Lemma 6, $y(t)$ satisfies

$$
\begin{aligned}
0 &\equiv L_4 y(t) + P(t) L_2 y(t) + Q(t) y(t) \\
&\leqslant L_4 y(t) + P(t) L_2 y(t) + \frac{(t - t_0) Q(t) L_1 y(t)}{2 p_1(t)} \\
&\leqslant L_4 y(t) + \frac{(t - t_0) Q(t) L_1 y(t)}{2 p_1(t)} \quad \text{on } I_{t_0}.
\end{aligned} \tag{9}
$$

Let us put $z_1(t) \equiv L_1 y(t)$. Then

$$
\begin{aligned}
L_3^* z_1(t) = L_4 y(t) &= \big(p_3(t)(p_2(t) z_1'(t))'\big)' \\
&= p_2(t) p_3(t) z_1'''(t) + (2 p_3(t) p_2'(t) + p_3'(t) p_2(t)) z_1''(t) \\
&\quad + (p_2'(t) p_3'(t) + p_3(t) p_2''(t)) z_1'(t) \quad \text{on } I_{t_0}.
\end{aligned} \tag{10}
$$

It follows from (9) and (10) that (11) holds, where

$$
0 \leqslant L_3^* z_1(t) + \frac{(t - t_0) Q(t) z_1(t)}{2 p_1(t)} \quad \text{on } I_{t_0}. \tag{11}
$$

The assumption $\liminf\limits_{t \to \infty} ((t - t_0)/\mu(t)) \geqslant 2$ implies the existence of $\tau > t_0$ such that $\frac{t - t_0}{\mu(t)} > 2\theta$ for all $t \geqslant \tau$, where $\theta \in (0, 1)$ is an arbitrary fixed real number (the number τ depends on θ). This result and (11) imply (12), where

$$
0 \leqslant L_3^* z_1(t) + \frac{\theta \mu(t) Q(t) z_1(t)}{p_1(t)} \quad \text{on } [\tau, \infty). \tag{12}
$$

Let us put

$$
\begin{aligned}
a_1(t) &\equiv 1, \\
a_2(t) &\equiv \frac{2 p_2'(t) p_3(t) + p_2(t) p_3'(t)}{p_2(t) p_3(t)}, \\
a_3(t) &\equiv \frac{p_2'(t) p_3'(t) + p_2''(t) p_3(t)}{p_2(t) p_3(t)}, \\
a_4(t) &\equiv \frac{\theta \mu(t) Q(t)}{p_1(t) p_2(t) p_3(t)}
\end{aligned}
$$

in Lemma 4. Then owing to (10) and (12) we obtain

$$
M(z_1) \equiv \frac{L_3^* z_1(t) + \frac{\theta \mu(t) Q(t) z_1(t)}{p_1(t)}}{p_2(t) p_3(t)} \geqslant 0 \quad \text{on } [\tau, \infty).
$$

Let us put $z_2(t) \equiv t-\tau$. Then by the assumptions of the theorem we have

$$\begin{aligned} M(z_2) &\equiv \frac{p_2'(t)p_3'(t)+p_2''(t)p_3(t)}{p_2(t)p_3(t)}\cdot 1+\frac{\theta\mu(t)Q(t)}{p_1(t)p_2(t)p_3(t)}\cdot(t-\tau)\\ &= \frac{p_1(t)p_2'(t)p_3'(t)+p_1(t)p_2''(t)p_3(t)+\theta\mu(t)Q(t)(t-\tau)}{p_1(t)p_2(t)p_3(t)}\\ &\leqslant 0 \quad \text{for } t\geqslant\tau_1>\tau. \end{aligned}$$

It is obvious that $z_1(t)\equiv L_1y(t)>0$, $z_2(t)\equiv t-\tau>0$ for $t\geqslant\tau_1$. Similarly

$$\begin{aligned} W(z_1,z_2) &= \begin{vmatrix} z_1(t) & z_2(t)\\ z_1'(t) & z_2'(t)\end{vmatrix} = \begin{vmatrix} z_1(t) & t-\tau\\ z_1'(t) & 1\end{vmatrix}\\ &= z_1(t)-(t-\tau)z_1'(t)\\ &= z_1(\tau)+(t-\tau)z_1'(\xi)-(t-\tau)z_1'(t)\\ &= z_1(\tau)+(t-\tau)(z_1'(\xi)-z_1'(t))\\ &= z_1(\tau)+\frac{(t-\tau)}{p_2(t)}(p_2(t)z_1'(\xi)-p_2(t)z_1'(t))\\ &\geqslant z_1(\tau)+\frac{(t-\tau)}{p_2(t)}(p_2(\xi)z_1'(\xi)-p_2(t)z_1'(t))>0 \quad \text{for } t\geqslant\tau_1,\ \ \xi\in(\tau,t), \end{aligned}$$

where we have used Lagrange's mean value formula and the fact that the function $p_2(t)z_1'(t)$ is positive and decreasing (because $(p_2(t)z_1'(t))'=L_3y(t)/p_3(t)<0$). Then Lemma 4 yields that (L^*) does not admit a nontrivial solution on I_a having more than two zeros on I_{τ_1}, i.e. (L^*) is nonoscillatory. The theorem is proved. □

Remark 3. A part (with the condition (1)) of [6, Theorem 5] is a special case of Theorem 3, where $p_i(t)\equiv 1$, $i=1,2,3$, $t\in I_a$.

By combining the previous theorem and Lemma 5 we obtain another oscillation criterion.

Theorem 4. *Let a function $\mu(t)$ be positive and continuous in (T,∞), $T>\max\{a,0\}$, such that $\liminf_{t\to\infty}((t-t_0)/\mu(t))\geqslant 2$ for any $t_0\geqslant a$. Let* (A) *hold and let $p_i'(t)\geqslant 0$, $i=1,2$, $t^2P(t)+(2tp_3(t))'\geqslant 0$ for $t\geqslant T>\max\{a,0\}$, $p_j(\infty)<\infty$, $j=2,3$, $\int_T^\infty(-\mu(t)Q(t)/p_1(t))\,\mathrm{d}t=\int_a^\infty(1/p_i(t))\,\mathrm{d}t=\infty$, $i=1,3$. For every $\tau>T$ let there exist $\tau_1>\tau$ such that $p_1(t)p_2'(t)p_3'(t)+p_1(t)p_2''(t)p_3(t)+\theta\mu(t)Q(t)(t-\tau)\leqslant 0$ for all $t\geqslant\tau_1$, $\theta\in(0,1)$. Then* (L) *is oscillatory.*

P r o o f. According to Lemma 5 the equation (L^*) is oscillatory for all $\theta\in(0,1)$. Then Theorem 3 yields that (L) is oscillatory. The criterion is established. □

Example 1. The equation

$$\big(\arctan t((2-\mathrm{e}^{-t})(ty')')'\big)' - t^2(2-\mathrm{e}^{-t})(ty')' - 2t^3y = 0$$

is oscillatory according to Theorem 4, where $a = T = 1$, $\mu(t) = t/2$ because

$$\begin{aligned}
&\lim_{t\to\infty}\big(p_1(t)p_2'(t)p_3'(t) + p_1(t)p_2''(t)p_3(t) + \theta\mu(t)Q(t)(t-\tau)\big)\\
&\quad = \lim_{t\to\infty}\Big(t\mathrm{e}^{-t}\frac{1}{1+t^2} - t\mathrm{e}^{-t}\arctan t - \theta t^4(t-\tau)\Big) = -\infty,
\end{aligned}$$

i.e. for every $\tau > 1$ there exists $\tau_1 > \tau$ such that $p_1(t)p_2'(t)p_3'(t) + p_1(t)p_2''(t)p_3(t) + \theta\mu(t)Q(t)(t-\tau) \leqslant 0$ for all $t \geqslant \tau_1$, $\theta \in (0,1)$.

Example 2. Let us consider the equation

$$y^{(4)} - \frac{y''}{t^3} - \frac{y}{t^4} = 0, \quad t \in [1,\infty). \tag{13}$$

If the equation

$$x''' - \frac{91}{216t^3}x = 0 \tag{14}$$

is oscillatory, then Theorem 3 (where $\mu(t) = t/2$ and $\theta = 2\cdot\frac{91}{216} \in (0,1)$) yields that (13) is oscillatory. If we test (13) to be oscillatory, we cannot use Theorem 4, because

$$\int_1^\infty \frac{-\mu(t)Q(t)}{p_1(t)}\,\mathrm{d}t = \int_1^\infty \frac{1}{2t^3}\,\mathrm{d}t < \infty.$$

However, (14) is an Euler equation; the substitution $t = \mathrm{e}^u$ transforms (14) into

$$\frac{\mathrm{d}^3x}{\mathrm{d}u^3} - 3\frac{\mathrm{d}^2x}{\mathrm{d}u^2} + 2\frac{\mathrm{d}x}{\mathrm{d}u} - \frac{91}{216}x = 0, \tag{15}$$

which is oscillatory, because the characteristic equation for (15)

$$r^3 - 3r^2 + 2r - \frac{7\cdot 13}{6^3} \equiv \Big(\Big(r-\frac{5}{12}\Big)^2 + \frac{1}{48}\Big)\Big(r - \frac{13}{6}\Big) = 0$$

admits complex zeros. Therefore (15), (14) as well as (13) are oscillatory.

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Author's address: Department of Mathematics, Faculty of Materials Science and Technology, Slovak University of Technology in Bratislava, Paulínska 16, 917 24 Trnava, Slovakia, e-mail: palum@mtf.stuba.sk.