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INVARIANT METRICS ON G-SPACES

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Abstract. Let X be a G-space such that the orbit space X/G is metrizable. Suppose a family of slices is given at each point of X. We study a construction which associates, under some conditions on the family of slices, with any metric on X/G an invariant metric on X. We show also that a family of slices with the required properties exists for any action of a countable group on a locally compact and locally connected metric space.

Keywords: G-space, invariant metric, slice

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1. Introduction

Let G be a topological group acting continuously on a topological space X. Existence of a G-invariant metric on X compatible with the topology of X is a distinct property of the action and provides an important technical tool. The obvious necessary condition is metrizability of X, and we assume this in the sequel. The standard case when an invariant metric is given by a simple formula, is the case of a compact group G endowed with a finite measure ν invariant under (say, right) translations. Then from any metric d on X we obtain an invariant metric by averaging:

$$\tilde{d}(x,y) = \int_G d(gx, gy)\nu(\mathrm{d}g).$$

The most important example is the Haar measure on a compact Lie group.

For a topological group G acting on itself by (say, left) translations, the theorem of Birkhoff [2] and Kakutani [4] (cf. [6]) gives a sufficient condition for the existence of an invariant metric, namely the existence of a countable base of open sets at the

unit. The general case is less known. Koszul [5] and Palais [8] showed that for proper (in a sense) actions of Lie groups on a separable and metrizable space there exists an invariant metric.

A similar existence problem can be stated for riemannian metrics and actions on smooth manifolds. Sufficient conditions for the existence of an invariant riemannian metric were given by Koszul [5], Palais [8] and Alekseevski [1].

In the present paper we follow ideas coming from differential geometry. In order to explain this in a comprehensible way we discuss first in some detail the case of totally discontinuous actions (in particular, the group G is discrete). This is equivalent to assuming that the projection $p\colon X \to X/G$, where X/G is the space of orbits with the quotient topology, is a covering map. Assuming X, X/G and the projection to be smooth, one can lift any riemannian metric from X/G to X, since the map p is a local diffeomorphism. To measure the distance between points $x, y \in X$, we have to measure the length of geodesics joining x with y This can be done in X/G, since the projection p becomes a *local isometry*, in particular, geodesics are preserved *locally*. The resulting metric is invariant by construction, since it is defined in terms of the quotient space.

One can follow this observation to construct an invariant metric for a general covering space with a metrizable base. So for any metric d on X/G and a given family $\mathscr{U}$ of open subsets of X define

$$(1) \qquad \varrho(x,y) = \inf \sum_{i=1}^{k-1} d(px_i, px_{i+1}),$$

where $x_1, \ldots, x_k$ is an *allowable sequence* of points of X, which means that $x_1 = x$, $x_k = y$ and any two adjacent points lie in one set belonging to the family $\mathscr{U}$. The infimum is taken over all allowable sequences. Whatever family $\mathscr{U}$ we are given, we will call its elements *small sets*.

The minimal requirement for $\mathscr{U}$ is that a small set U should be elementary, i.e. it is mapped homeomorphically by p onto its image (which we also call an elementary set). We also want the family to be G-invariant, to ensure G-invariance of ϱ. The definition of ϱ is the analogue of lifting geodesics piece by piece when they are divided so that each segment lies in an elementary subset. By the definition of a covering map, the quotient space has a base of open elementary sets. However, if we allow all elementary sets in (1), then ϱ is only a pseudometric. One can see this easily even in the simplest example of the action of integers on the real line, which gives a covering of the circle by $\mathbb{R}$. For locally connected spaces it is not difficult to find a remedy. We consider all balls which are elementary and balls of radius 4 times larger are still elementary. The detailed discussion of that case together with a generalization (to coverings in the category of spaces with group actions) is given in Section 3.

We want to explain when the formula (1) works, hence in fact we investigate the problem of lifting a given metric on X/G to X instead of mere existence of an invariant metric. The assumption of metrizability of X/G is therefore natural. Moreover, it is easy to observe that any invariant metric comes from X/G if the latter is metrizable.

One can observe the strength of the assumption considering those actions on the real line $\mathbb{R}$ which have metrizable orbit spaces. There are only few possibilities for the latter: the line itself, the halfline, the closed interval, the circle and the point. According to this, we may classify groups which act effectively on the real line with a metrizable quotient. Any such action is equivalent to one of the following:

  i) the (trivial) action of the trivial group;

 ii) the involution $x \rightarrow -x$;

iii) the action of integers by $x \rightarrow x + n$;

iv) the action of the subgroup of isometries of $\mathbb{R}$ generated by two elements: $x \rightarrow x + 1$ and $x \rightarrow -x$;

 v) a transitive action.

This implies that for any effective action on $\mathbb{R}$ with a metrizable quotient there exists an invariant metric except for those examples in case v) when the group is in a sense larger than the orbit (e.g. the whole affine group). This shows that the group must be related to the topology of the orbit, if we want to find an invariant metric. One consequence of this is that the action must be proper in a sense (there are various meanings of this word in literature). We use the name perfect action (as considered by Koszul and Alekseevski) for the type of properness we use. It is not difficult to give examples of non-proper actions with a metrizable quotient. We observe that any perfect action of a countable group on a locally compact space has a metrizable orbits space. This and other topological preliminaries are given in Section 2. Then, as a consequence of our construction and some classical theorems of P. A. Smith and Mongomery-Zippin, we show that any discrete action (cf. Definition 3) on a manifold is perfect.

The main aim of the present paper is to extend the construction of the lifting beyond totally discontinuous actions.

In the presence of nontrivial isotropy (while still assuming the orbits to be discrete) we have to change formula (1) slightly. This is because p can not be a local isometry in general, hence in an allowable sequence we have to impose a stronger condition. The condition corresponds to the radial distance preserving, which means that in any small set there is a base point (origin) such that the map p preserves the distance between the base point and any other point of the set. One can see the problems we encounter even in the simple example of the finite cyclic group acting on the plane $\mathbb{R}^2$

by rotations. One can see how the properties of ϱ depend on various choices of the class of small sets.

The next level is the case when the orbits (hence the group) are not discrete. Like in the case of locally trivial bundles, we have to introduce into the formula the metric along the orbit. Such a family of metrics in orbits we call an *orbital distance* when some subcontinuity conditions are satisfied (cf. Section 4). The small sets have to be more refined, in order to remain related to the topology of the orbit space. The appropriate requirement is that small sets should be slices (cf. Definition 2).

The usefulness of slices is clear since the Palais paper [8] and in fact this notion plays the role analogous to horizontal distributions in locally trivial smooth fibrations. From our point of view this is crucial, as the lift of a geodesic from the base to the total space of a fibration requires the use of a horizontal distribution (or a connection).

Having now a family of small slices $\{S_x \colon x \in X\}$ and an orbital distance d_O, we can again define the lift of a metric d on X/G.

Definition 1. We say that $x_1, \ldots, x_k$ is an *allowable* sequence of points in X if for any two consecutive points one of them belongs to a small slice at the other, or they are in the same orbit.

Let ϱ be the function

$$(2) \qquad \varrho(x, y) = \inf \sum_{i=1}^{k-1} (d(px_i, px_{i+1}) + d_O(x_i, x_{i+1})),$$

where the infimum is taken over all allowable sequences such that $x_1 = x$, $x_k = y$.

Our main result consists of two parts. The first is Theorem 19 which describes properties of the orbital metric and of the family of small sets sufficient to ensure that the formula (1.2) gives an (invariant) metric. The properties are independent of the given metric d, hence once we have such a family, any metric can be lifted.

Then we give an existence theorem. Theorem 22 shows that for any locally compact and locally connected metrizable space X and any discrete action of a countable group there exists a family satisfying the hypothesis of Theorem 19.

So the class of spaces for which the construction works is rather large, as it contains all manifolds and all locally finite polyhedra.

2. Topological preliminaries

Notation. Given a metric ϱ, by $K_\varrho(x, \varepsilon)$ we denote the ball of radius ε centered at x with respect to ϱ.

In the sequel, all actions will be assumed continuous. For an action of a group G on X we denote by X/G the orbit space and by $p\colon X \to X/G$ the quotient map. By G_x we denote the stabilizer (isotropy subgroup) of x; $G_x = \{g \in G\colon gx = x\}$.

$GA = \{ga\colon g \in G, a \in A\}$, hence $Gx = G\{x\}$ denotes the orbit of x.

The map $e_x\colon G \to Gx$; $g \mapsto gx$ will be called the *evaluation map*.

Definition 2. By a slice at $x \in X$ we mean a subset S_x such that

1. $x \in S_x$;
2. $gS_x \cap S_x \neq \emptyset$ implies $gx = x$;
3. $gS_x = S_x$ for $g \in G_x$, hence G_x acts on S_x;
4. the map $(G \times S_x)/G_x \to X$ given by the evaluation is a homeomorphic embedding onto an open neighbourhood of the orbit Gx.

Note that our definition is stronger than the usual one, since it includes condition 4, which implies that the evaluation map $G \to Gx$ is open for any x. Under this assumption the topology of G is strictly related to the topology of orbits. In particular, this excludes the examples given by changing the topology of a group acting on X, for instance from a connected group to a discrete group. Note the following properties of slices:

1) the image of any slice is open in X/G;
2) for any open set $U \subset X/G$, the intersection $p^{-1}U \cap S_x$ is open in S_x and, if $px \in U$, it is another slice at x.

For an action of a discrete group, the slice is an open subset. This leads to the following definition.

Definition 3. The action of a (discrete) group G on a topological space X is called discrete if for any orbit Gx there exists an open neighbourhood U of x such that for all $h \notin G_x$ we have $hU \cap U = \emptyset$ and every point $x \in X$ has a base of open G_x-invariant neighbourhoods.

Any orbit of a discrete action is discrete in X. If the action is free and discrete, then it is totally discontinuous.

The following observation is well known (cf. [8], p. 319).

Proposition 4. *Let Γ be a group acting by isometries on a metric space (X, ϱ) such that each orbit Γx is closed in X. Then the formula*

$$d(px, py) = \inf\{\varrho(a, b)\colon a \in \Gamma x, \ b \in \Gamma y\}$$

defines a metric on the orbit space.

In the sequel we will need the following fact. If X/G is T_1, then any orbit is a closed subset of X. Thus any accumulation point of Gx belongs to that orbit. Therefore either there are no accumulation points or, by homogeneity, every point of Gx is its accumulation point. When X is locally compact, in the latter case we would get an uncountable orbit. This yields the following lemma.

Lemma 5. *Let G be a countable group acting on locally compact Hausdorff space X such that the quotient space is a T_1-space. Then any orbit is discrete in X.*

When we want to apply our construction, the following notion arises naturally. It is stronger than the usual notion of proper actions.

Definition 6. We call an effective action of a group G on a Hausdorff space X perfect if for any compact sets $U, V \subset X$ the set $\{g \in G\colon gU \cap V \neq \emptyset\}$ is compact.

Any perfect action has compact stabilizers (hence finite, if the group is countable). If a countable group acts perfectly on a locally compact space X, then any orbit is discrete in X.

Now we will show that the quotient space of any perfect action of a countable discrete group on a locally compact metrizable space is metrizable. Note, however, that perfectness is not a necessary condition. The following example shows a nonperfect action with a decent quotient.

Consider the subset $Y = \{(x, 0)\colon x \in \mathbb{R}\} \cup \bigcup_n Y_n$ of the plane, where Y_n is the sum of two closed intervals with a common initial point $(n, 0)$, for every $n \in \mathbb{Z}$. The group acting on Y is generated by homeomorphisms g_n which exchange the two intervals attached at $(n, 0)$ and fix the rest of Y.

Theorem 7. *Let G be a countable group acting on a locally compact metrizable space X. If the action is perfect, then the group is discrete and the quotient space X/G is metrizable.*

P r o o f. First note that by the remark above, each orbit Gx is closed, so X/G is a T_1 space. Note also that under the assumptions of the theorem, for any compacts $K_1, K_2 \subset X$ the set $\{g \in G\colon gK_1 \cap K_2 \neq \emptyset\}$ is finite.

Now let A be a closed subset of X/G and Gx_0 any orbit which does not belong to A. Then there exists $\delta > 0$ such that $\overline{K(z,\delta)}$ is compact and $\overline{K(z,\delta)} \cap p^{-1}(A) = \emptyset$, where $z \in Gx_0$.

For any $y \in p^{-1}(A)$ one can find $K(y,\delta_y)$ which is relatively compact and such that $Gx_0 \cap \overline{K(y,\delta_y)} = \emptyset$. Then

$$B = \{g \in G \colon g(\overline{K(z,\delta)}) \cap \overline{K(y,\delta_y)} \neq \emptyset\}$$

is finite, so there exists $\varepsilon_y \leqslant \delta_y$ such that $K(y,\varepsilon_y)$ is disjoint with $\bigcup_{g \in G} g\overline{K(z,\delta)}$. Since $\left(\bigcup_{g \in G} g(K(z,\delta))\right) \cap \left(\bigcup_{g \in G} g(K(y,\varepsilon_y))\right) = \emptyset$, we see that the space X/G is regular.

Now the following result of A. H. Stone (cf. [3, 4.5.17]) completes the proof. $\qquad\square$

Theorem 8. *If $f\colon X \to Y$ is a continuous and open mapping from a locally separable metric space X onto a regular space Y and the set $f^{-1}(y)$ is separable for every $y \in Y$, then Y is a metrizable space.*

3. COVERING SPACES

In this section we consider totally discontinuous actions of discrete groups, so that the projections $p\colon \tilde{X} \to X = \tilde{X}/G$ are coverings. Our aim is twofold. First we want to explain our construction in a relatively simple case. Secondly we show the existence of a lifted metric for coverings in the category of spaces with group actions.

Definition 9. Consider a group epimorphism $\varphi\colon \tilde{G} \to G$. Suppose that a group G acts on X and $\tilde{G}$ acts on $\tilde{X}$. By a covering in the category of spaces with group actions we mean a covering map $p\colon \tilde{X} \to X$ such that

$$p(\tilde{g}(\tilde{x})) = \varphi(\tilde{g})(p\tilde{x})$$

for any $\tilde{x} \in \tilde{X}$ and $\tilde{g} \in \tilde{G}$.

In the following we assume that $\tilde{X}$ is connected. If not, one can apply our construction to every connected component of $\tilde{X}$.

Definition 10. Let $p\colon \tilde{X} \to X$ be a covering map. Any open set $U \subseteq X$ such that its counterimage $p^{-1}(U)$ is homeomorphic to a disjoint sum $\bigcup_{j \in J} U_j$ and p restricted to U_j is a homeomorphism onto U, is called elementary. Any of its homeomorphic copies U_j will be also called elementary.

Theorem 11. *For any covering map $p\colon \tilde{X} \to X$ of a localy connected space X and for any metric d on X there exists a metric ϱ on $\tilde{X}$ such that p is locally isometric. Moreover, if X is a G-space and $\tilde{X}$ is a $\tilde{G}$-space covering of (X, G), then any G-invariant metric lifts to a $\tilde{G}$-invariant metric on $\tilde{X}$.*

P r o o f.   Since X is locally connected, thus each point of X (and thus of $\tilde{X}$) has a base of connected elementary neighbourhoods.

Let $x \in X$. Take a connected elementary neighbourhood U_x of x. Then there exists a ball $K(x, \varepsilon_x) \subseteq U_x$ and a connected open set $V_x \subseteq K(x, \varepsilon_{x/4})$ containing x. Define now a family $\mathscr{V}$ of small sets as the family of all connected components of sets $p^{-1}V_x$ for all $x \in X$. Note that all such small sets are elementary, since all V_x are. As in Definition 2 define now

$$\varrho(a, b) = \inf\left\{\sum_{i=1}^{k-1} d(p\tilde{x}_i, p\tilde{x}_{i+1})\right\},$$

where the infimum is taken over all $\mathscr{V}$-admissible sequences $\xi = \{a = \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_k = b\}$ in $\tilde{X}$, i.e. such that any two consecutive points of the sequence belong to the same small set.

It is easy to check that ϱ is a pseudometric and $0 < d(pa, pb) \leqslant \varrho(a, b)$ when $pa \neq pb$.

Consider two sets $A, B \in \mathscr{V}$ such that $A \cap B \neq \emptyset$. By the definition of $\mathscr{V}$, we have $pA = V_x$, $pB = V_y$ for some $x, y \in X$. When we assume that $\text{diam}(V_y) \leqslant \text{diam}(V_x)$, then $\text{diam}(V_y) \leqslant \text{diam}(V_x) \leqslant \varepsilon_x/2$ and $V_x \cap V_y \neq \emptyset$, so we see that $V_x \cup V_y \subseteq K(x, \varepsilon_x) \subseteq U_x$. Thus $A \cup B$ is elementary and because of the connectedness of V_x, V_y and U_x it follows that $A \cup B$ is included in some component of $p^{-1}U_x$.

Now take any points $\tilde{x}, \tilde{\tilde{x}}$ such that $p\tilde{x} = p\tilde{\tilde{x}} = x$. We will show that every admissible sequence joining $\tilde{x}$ and $\tilde{\tilde{x}}$ contains $z \notin p^{-1}(V_x)$. Assume the contrary. Then in such an admissible sequence there exist $\tilde{x}_i, \tilde{x}_{i+1}$ which belong to different components of $p^{-1}(V_x)$, and belong to a component of $p^{-1}(V_z)$ for some $z \in X$. We have now two cases:

i) $V_x \cup V_z \subseteq S_x$. Then some component of $p^{-1}(S_x)$ intersects two different components of $p^{-1}(V_x)$, which is impossible.

ii) $V_x \cup V_z \subseteq U_z$. Then each component of $p^{-1}(V_x)$ is contained in precisely one component of $p^{-1}(U_z)$ ($p\colon \tilde{X} \to X$ is a covering map), but at least one component of $p^{-1}(V_z)$ intersects two different components of $p^{-1}(V_x)$, which is a contradiction.

Because V_x is open, so $0 < \delta \leqslant \varrho(\tilde{x}, \tilde{\tilde{x}})$. Note also that p restricted to any component of any admissible set is an isometry. This yields compatibility of ϱ with the topology of $\tilde{X}$.

To prove the second part of the theorem take as small sets the family of the connected components of the sets $\{p^{-1}(gV_x)\colon g \in G, \ x \in X\}$. First note that if U is elementary then gU is also elementary, because the action of G is covered by the action of $\tilde{G}$. Thus $\mathscr{V}$ is $\tilde{G}$-invariant. Since G acts by isometries, so ϱ defined as before is $\tilde{G}$-invariant. By the same argument as in part one of the theorem we obtain that ϱ is a metric on $\tilde{X}$ compatible with the topology on $\tilde{X}$. $\qquad\square$

4. Orbital distance

When a G-space is endowed with an invariant metric, then we have in particular a continuous family of metrics in orbits. On the other hand, an important ingredient of our construction is a family of invariant metrics in orbits. We follow the construction which works for locally product bundles. In this case, with help of a horizontal connection, a metric on the total space can be obtained from a metric on the base and a fiberwise family of metrics.

We encounter here two main difficulties. Even when the metric on an orbit is induced from a metric on the group G, then there is no canonical way to do this, because the identification of the orbit with the quotient G/G_x requires an apriori choice of the base point x in the orbit. Once an orbit has a base point x, one may want to use the slice S_x to endow the orbits in the neighbourhood GS_x of Gx with base points. But neighbour orbits in general cut the slice at many points, so we are forced to assume that any such base point gives the same metric (i.e, that the metric on G is right invariant with respect to G_x). The other problem is that the family of metrics coming from G is not continuous in general. It has, however, some subcontinuity properties which are sufficient for our purposes.

Let d_G be a left invariant metric on G. Consider first a single orbit with a base point x in it and assume that the evaluation map $e_x\colon G/G_x \to Gx$ given by $e_x(g) = gx$ is a homeomorphism. Using this identification, we want to endow the orbit with an invariant metric. For a subgroup $K \subset G$ define a left G-invariant pseudometric on G/K by the formula

$$(3) \qquad\qquad d_K(g_1K, g_2K) = \inf\{d_G(g_1u, g_2v)\colon \ u, v \in K\}.$$

In particular,

$$d_K(g_1K, g_2K) \leqslant d_G(g_1, g_2)$$

for any $g_1, g_2 \in G$.

Lemma 12. *Let H be a closed subgroup of the group G. If d_G is left G-invariant and right H-invariant, then for any closed subgroup $K \subset H$ we have*

$$(4) \qquad d_K(g_1 K, g_2 K) = \inf\{d_G(g_1, g_2 u) \colon u \in K\},$$

thus formula (3) defines a G-invariant metric on G/K, compatible with the quotient topology. The same is true if K is a normal subgroup of G. When we replace K by $K' = hKh^{-1}$, $h \in H$, then G/K and G/hKh^{-1} with the metrics d_K, $d_{hKh^{-1}}$ are isometric under the natural map

$$\varphi(gK) = gh^{-1}K'.$$

P r o o f. Both formula (4) and the isometry follow directly from the right invariance. For instance, we have $d_K(g_1 K, g_2 K) = d_K(g_1 h^{-1} hK, g_2 h^{-1} hK) = d_{hKh^{-1}}(\varphi(g_1 K), \varphi(g_2 K))$. Since we assumed K closed, thus d_K is a metric. $\square$

Under the identification of the orbit Gy with G/G_y by the evaluation map, the metric d_{G_y} is a well-defined metric in the orbit. The translation of the orbit by $h \in G_x$ which changes the point y to hy corresponds to the above homeomorphism $\varphi \colon G/G_y \to G/G_{hy}$. Given a slice S_x at x and a metric in G which is left G-invariant and right G_x-invariant we have a well defined metric in any orbit passing through S_x, since we can use any of the points of intersection $Gy \cap S_x$ as the base point of Gy (the stabilizer G_x corresponds to H and S_y to K). We denote the whole family of metrics by d_x to stress which slice was used to define the metrics.

Corollary 13. *Let d_G be an invariant metric on G. Given a point x, assume either that d_G is right G_x-invariant or that all stabilizers of points in S_x are normal. Then we get from d_G a family d_x of G-invariant metrics in all orbits intersecting S_x. For any point $y \in S_x$ and $g_1, g_2 \in G$ we have the inequalities*

$$(5) \qquad d_x(g_1 x, g_2 x) \leqslant d_x(g_1 y, g_2 y) \leqslant d_G(g_1, g_2).$$

If $y \in g_0 S_x$ for some $g_0 \in G$, then

$$(6) \qquad d_x(y, gy) \leqslant d_G(g_0, g g_0).$$

P r o o f. To prove (5) we start from the inequality $d_x(x, gx) \leqslant d_x(y, gy)$ which holds, by the definition of d_x, for any $y \in S_x$ and any g. Then we use the left invariance of d_x.

Inequality (6) follows from (5). $\square$

Thus, assuming that the action has slices, our construction works locally for the following classes of actions:

- any action of a discrete group, or more generally of a group having a biinvariant metric,
- perfect actions, and in general for actions with compact isotropy subgroups,
- actions whose all stabilizers are normal in G.

Now we globalize the procedure by gluing together the metrics defined locally with the use of a decomposition of unity. Let a G-space X be given having a slice S_x at each point such that $S_{gx} = gS_x$ and suppose that X/G is metrizable and endowed with a metric d.

For any open set $U \subset X/G$, the set $p^{-1}U \cap S_x$ is also a slice at x, and we say that it is a subslice (with respect to the given family of slices).

Under our assumptions, there exists a family S_α, each S_α a subslice of S_{x_α} for a point $x_\alpha \in X$, such that $\overline{S}_\alpha \subset S_{x_\alpha}$ and the family pS_α is a locally finite open cover of X/G. The cover admits a decomposition of unity $\{\chi_\alpha\}$. Denote by d_α the local family of metrics in orbits in GS_α and define the global family of metrics in orbits by the formula

$$(7) \qquad d_O(x, gx) = \sum_\alpha \chi_\alpha(px) d_\alpha(x, gx).$$

To describe it formally, denote $V = \{(x, y)\colon y = gx, \ g \in G\}$. Then $d_O\colon V \to \mathbb{R}$ is G-invariant and restricted to each orbit it is a metric. In the sequel we will consider the map as extended by zero map to the whole $X \times X$ and we will call it an *orbital metric*.

The orbital metric d_O is in each orbit compatible with the topology of the orbit (since each metric d_α is). However, it is not a continuous function even on V, since orbits with smaller isotropy do not converge to orbits of larger isotropy. For instance, d_O is not continuous at fixed points.

In order to get an orbital metric with nice properties, we assume that the given family of slices $\{S_x\}_{x \in X}$ satisfies the following condition:

$(*)$ \qquad for any $x \in X$ and $y \in S_x$, the intersection $S_x \cap S_y$ is open in S_y.

This implies in particular that any $y \in S_x$ has an open neighbourhood U such that $S_y \cap U \subset S_x$.

The following conditions are stronger than $(*)$.

(1) For any $y \in S_x$ we have $S_y \subset S_x$.
(2) For any $y \in S_x$ and $g \notin G_x$ we have $S_y \cap S_{gx} = \emptyset$.

First we show that assuming (∗) we have some subcontinuity properties for d_O. Let $S_x(\delta) = S_x \cap p^{-1}K_d(px, \delta)$.

Property A. *For any $x \in X$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $d_O(y, gy) < \varepsilon$ if $y \in S_x(\delta)$ and $d_G(g, 1) < \delta$.*

P r o o f.   A given point x is in the closure of a finite number of sets GS_α, say $GS_1, \ldots, GS_r$. It means that $x \in GS_{x_1} \cap \ldots \cap GS_{x_r}$, where S_{x_i} is the slice which by our assumption contains the closure of S_i. Let g_i be chosen such that $g_i x \in S_{x_i}$. By (∗) there exists $\delta > 0$ such that for all $y \in S_x(\delta)$ we have $g_i y \in S_{x_i}$. We can also assume that $S_x(\delta)$ does not intersect S_α except for $S_1, \ldots, S_r$. Thus

$$d_O(y, gy) = \sum_{i=1}^{r} \chi_i(px_i) d_{x_i}(y, gy) = \sum_{i=1}^{r} \chi_i(px_i) d(g_i y, g_i g g_i^{-1} g_i y)$$
$$\leqslant \sum_{i=1}^{r} \chi_i(px_i) d_G(1, g_i g g_i^{-1})$$

by (5). Since the conjugation by g_i in G is continuous, there exists δ with the required properties. □

Now we will show that the orbital distance is subcontinuous.

Property B. *For any point x there exists $\delta > 0$ such that for any $y \in S_x(\delta)$ and any $g_1, g_2 \in G$ we have*

(8) $$d_O(g_1 x, g_2 x) \leqslant d_O(g_1 y, g_2 y).$$

P r o o f.   This is straightforward from the definition of d_O (and from the fact that isotropy is subcontinuous) once we know (as in A) that the element $g_i \in G$ which brings x into S_i does the same with any element of $S_x(\delta)$. But this is true for δ small enough by (∗). □

Similarly, the assumption that the evaluation map $G/G_x \to Gx$ is a homeomorphism gives the following result.

Property C. *For any point $x \in X$ and any $\delta > 0$ there exists ε such that $d_O(x, gx) < \varepsilon$ implies that there exists $u \in G_x$ satisfying the inequality $d_G(1, gu) < \delta$.*

Given a G-space with slices and an invariant metric d_G in G, by an *associated orbital metric d_G* we mean an orbital metric which in each orbit restricts to a

G-invariant metric isometric to d_K for a stabilizer K in the orbit and has properties A, B and C.

From the above considerations we know that an orbital metric exists for any G-space with slices such that there exists a metric on G which is left G-invariant and right G_x-invariant for any stabilizer G_x, or the stabilizers are normal, and the quotient X/G is metrizable.

Remark. The metric in X/G is not necessary in our construction, it is enough to assume the quotient is paracompact. More difficult it is to show that the condition $(*)$ can be removed when the quotient is locally compact. We will not go into the details, since we treat here the general case.

5. MAIN RESULTS

We will now prove the main theorems of this paper.

Consider an action of a group G on X and suppose that G admits a left invariant metric d_G. Assume we are given a family $\mathscr{S} = \{S_x\}_{x \in X}$ of slices in X such that $S_{gx} = gS_x$ for any x, g, an associated orbital metric d_O in X and a metric d in X/G.

Definition 14. A sequence $x_1, \ldots, x_n$ is called an $\mathscr{S}$-*allowable* (or simply *allowable*) sequence if for each $1 \leqslant i \leqslant n-1$ we have $x_i \in S_{x_{i+1}}$, or $x_{i+1} \in S_{x_i}$, or $x_{i+1} = gx_i$.

For an allowable sequence $\xi = \{x_1, \ldots, x_n\}$ in X denote

$$\Sigma_\xi = \sum_{i=1}^{n-1} d(px_i, px_{i+1}), \qquad \Phi_\xi = \sum_{i=1}^{n-1} d_O(x_i, x_{i+1}).$$

Definition 15.

$$\varrho(x, y) = \inf\{\Sigma_\xi + \Phi_\xi \colon \xi = \{x_1, \ldots, x_k\} \text{ is allowable, } x = x_1, y = x_k\}.$$

We say that a G-space *has small slices* if there exists a slice at every point $x \in X$ and for any open neighbourhood of x and any slice S_x there exists a slice $S'_x \subset S_x \cap U$.

Theorem 16. *Let G be a topological group with a left invariant metric d_G, acting on X. Suppose that X has small slices and we are given a family of slices $\mathscr{S} = \{S_x\}_{x \in X}$ satisfying*
 (i) *$gS_x = S_{gx}$ for any $x \in X$ and $g \in G$;*
 (ii) *for any $y \in S_x$, $S_y \cap S_{gx} \neq \emptyset$ implies $gx = x$.*

Suppose also a metric d in the orbit space X/G and an orbital distance d_O are given. Then formula (15) defines a lift of the metric d to the G-invariant metric ϱ in X, compatible with the topology of X.

P r o o f.   It is a direct consequence of the definition that ϱ is a pseudometric in X and $\varrho(x,z) \geqslant d(px,pz) > 0$ if $px \neq pz$. It remains to prove that ϱ distinguishes points in the same orbit and that it is compatible with the topology of X. For any $\varepsilon > 0$ we denote $S_x(\varepsilon) = S_x \cap p^{-1}K_d(px, \varepsilon)$ and by $B(\varepsilon)$ the ball in G with the center at the unit element and of radius ε. Recall that $S_x(\varepsilon)$ is again a slice at x.

Lemma 17. *For any point $x \in X$ and $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$B(\delta)S_x(\delta) \subset K_\varrho(x, \varepsilon).$$

Lemma 18. *For any point $x \in X$ and $\delta > 0$ there exists a positive number ε such that $K_d(px, \varepsilon) \subset pS_x$ and*

$$K_\varrho(x, \varepsilon) \subset B(\delta)S_x(\varepsilon).$$

P r o o f of Lemma 17.   Consider an allowable sequence $\xi = \{x, y, gy\}$, where $y \in S_x(\delta)$ and $g \in B(\delta)$. By Property A of orbital metrics for δ small enough, $\varrho(x, gy) \leqslant d_O(y, gy) + d(px, py) < \varepsilon$. $\square$

P r o o f of Lemma 18.   Since the projection p is open, there exists ε such that $K(px, \varepsilon) \subset pS_x$. Let $z \in K_\varrho(x, \varepsilon)$, $\xi = \{x = x_1, x_2, \ldots, x_k = z\}$ be an allowable sequence and $\Sigma_\xi + \Phi_\xi < \varepsilon$. If an element x_i of the sequence ξ does not belong to $GS_x(\varepsilon)$, then $\Sigma_\xi \geqslant d(px, px_i) \geqslant \varepsilon$. Thus any point x_i in the sequence ξ can be written as g_ix_i' for some $g_i \in G$, $x_i' \in S_x(\varepsilon)$. We can choose $x_1', \ldots, x_k'$ and $g_1, \ldots, g_k$ such that $x_i' = x_{i+1}'$ when $x_{i+1} = gx_i$ and $g_i = g_{i+1}$ if $x_{i+1} \in S_{x_i}$. In the last case we use the assumption (ii), which is equivalent to the inclusion $S_y \cap GS_x \subset S_x$ for any $y \in S_x$. It gives that for $x_{i+1} = g_{i+1}x_{i+1}'$ we have $g_i^{-1}g_{i+1} \in G_x$, so we have $x_{i+1} = g_ig_i^{-1}g_{i+1}x_{i+1}'$.

By Property B of the orbital metric, if ε is small enough then

$$d_O(x_i, x_{i+1}) = d_O(g_ix_i', g_{i+1}x_i') \geqslant d_O(g_ix, g_{i+1}x)$$

in the case $x_{i+1} \in Gx_i$, and we have zeros on both sides otherwise. It follows that

$$d_O(x, g_kx) \leqslant \sum_{i=1}^{k-1} d_O(g_ix, g_{i+1}x) \leqslant \Phi_\xi < \varepsilon,$$

and by Property C for ε small enough there exists $u \in G_x$ such that $d_G(1, g_ku) < \delta$. So we have $z = g_kx_k' = (g_ku)(u^{-1}x_k')$. This completes the proof of Lemma 18. $\square$

The last lemma implies that ϱ distinguishes points in orbits: if $\varrho(x, gx) = 0$, then there is a sequence $g_n \in G$ convergent to the unit of G such that $gx = g_n x$. Thus $gx = x$. Since the sets $B(\delta) \times S_x(\varepsilon)$ generate the topology of X, Lemmas 17 and 18 show that ϱ is compatible with the topology of X. $\qquad \square$

Remark. Assumption (ii) can be omitted if the quotient space is locally compact (cf. Section 4). If X is locally compact, then the assumption of existence of small slices is superfluous.

When one wants to find a class of actions for which the assumptions of Theorem 16 are satisfied, the first try would be discrete actions. In that case the orbital metric is trivially given by the (biinvariant) discrete metric on G, and we have an easy description of slices. A slice at x is an open neighbourhood U of x which is G_x-invariant and disjoint with any of its translations by any $g \notin G_x$. We prove that for any locally compact and locally connected G-space one can find the family of slices for actions with a Hausdorff orbit space. Note that the condition $(*)$ is easy to obtain once we have slices, because for any slice S_x, any $y \in S_x$ and a slice S_y, the set $S_x \cap S_y$ is a slice at y.

Theorem 19. *Let X be a metrizable, locally compact and locally connected space and let G be a countable group acting on X such that the quotient space X/G is Hausdorff. Then each orbit Gx is discrete in X and every point $x \in X$ has a base of connected, G_x-invariant open neighbourhoods.*

Moreover, there exists a family $U_x\colon\ x \in X$ of G_x-invariant, path connected open sets satisfying

i) $U_{gx} = gU_x$,

ii) *if $U_x \cap U_{gx} \neq \emptyset$ then $g \in G_x$.*

P r o o f.   Take any $x \in X$ and an open set U containing x. There exists $V \subseteq U$ such that $x \in V$, $\overline{V}$ is compact and $\overline{V} \cap Gx = \{x\}$. We claim that there exists an open set W such that $x \in W \subseteq \bigcap_{g \in G_x} gV$.

Assume it is not true. Then for any natural n there exist a connected neighbourhood $V_n \subseteq V$ of x such that $x \in V_n \subseteq K(x, (1/n))$ and $g_n \in G_x$ such that $(X - \overline{g_n V}) \cap V_n \neq \emptyset$.

We want to show that $\partial(g_n V) \cap V_n \neq \emptyset$. If this holds, then V_n can be decomposed as the sum of the sets $V_n \cap g_n V$ and $V_n \cap (X - \overline{g_n V})$ which are disjoint, open and non-empty, so V_n would be disconnected.

Pick any $d_n \in \partial(g_n V) \cap V_n$. Notice that $\varrho(d_n, x) < 1/n$, where ϱ denotes an auxiliary metric on X. It follows that $w_n = g_n^{-1} d_n \in \partial V$. The last set is compact, so up to passing to a subsequence we may assume $w_n \to w' \in \partial V$. Since $\overline{V} \cap Gx = \{x\}$ and X/G is Hausdorff, we get a contradiction.

An open set W contains a connected neighbourhood W' of x. Now $W'' = \bigcup_{g \in G_x} gW'$ is an open, path connected and G_x-invariant subset of V. Since V can be chosen arbitrarily small, the proof of the first part is complete.

Now the last part of the theorem.

Notice first that if a neighbourhood U_x of x is G_x-invariant then $U_{gx} = gU_x$ is G_{gx}-invariant.

Assume, on the contrary, that for every G_x-invariant open set S_x containing x there exists $g \notin G_x$ such that $gU_x \cap U_x \neq \emptyset$.

Choose then a G_x-invariant neighbourhood U of x such that $\overline{U} \cap Gx = \{x\}$. For every ball $K(x, (1/n)) \subseteq U$ there exist open $V_n \subseteq K(x, (1/n))$ which are connected and $g_n \notin G_x$ such that $g_n U \cap V_n \neq \emptyset$. The same argument as before yields a contradiction. $\qquad \square$

Theorem 20. *Let X be a locally compact and locally path connected metric space and let G be a countable group acting on X such that the quotient space X/G is a metric space with a metric d. Then there exists an open covering $\{S_x\}_{x \in X}$ of X such that*

A) $U_{gx} = gU_x$,

B) *if $U_x \cap U_{gx} \neq \emptyset$ then $g \in G_x$,*

C) *for any $y \in U_x$, if $U_y \cap U_{gx} \neq \emptyset$ then $g \in G_x$ for any $x \in X$ and $g \in G$.*

P r o o f.   We already know that a covering satisfying A) and B) exists. Further, $p \colon X \to X/G$ is open, so pS_x is open. Hence there exists $K(px, \varepsilon_x) \subseteq pS_x$. Now $S_x \cap p^{-1}(K(px, (\varepsilon_x/8)))$ is open and x belongs in it, so there exists a G_x-invariant, path connected and open set V_x contained in the set mentioned above. We will show that the covering $\{V_x\}_{x \in X}$ satisfies C), because it obviously satisfies A) and B).

Note first that for $y \in V_x$ the conditions $V_x \cup V_{gx} \subseteq V_y$ for some $x, y \in X$ and $g \notin G_x$ can not be satisfied simultaneously. To see this take $y \in S_x$; then $S_y \subseteq G_x$ because of B); but also $x \in V_x \subseteq V_y$, so $G_x = S_y$. Now $V_{gx} \subseteq V_y$ for some $g \notin G_x$ gives a contradiction by the argument described above for covering spaces.

Assume, on the contrary, that $V_y \cap V_{gx} \neq \emptyset$ and $y \in V_x$ for some $x, y \in X$ and $g \notin G_x$. We have two cases:

i) $\operatorname{diam}(pV_y) \leqslant \operatorname{diam}(pV_x)$. Then $pV_y \subseteq pS_x$ and for each $g \in G$ there exists $h \in G$ such that $V_{gy} \subseteq U_{hx}$.

Then it is a clear contradiction because of A), B) and the connectedness of all sets considered.

ii) $\operatorname{diam}(pV_x) \leqslant \operatorname{diam}(pV_y)$. Then $pV_x \subseteq pU_y$ and for every $g \in G$ there exists $h \in G$ such that $V_{gx} \subseteq U_{hy}$. Then $V_y \cup V_x \cup V_{gx} \subseteq U_y$, but it is not consistent with the remark made above. $\qquad \square$

Thus we have the following result.

Theorem 21. *Let (X, ϱ_0) be a locally compact and locally connected metric space, G a countable group acting discretely on X such that the quotient space X/G is metrizable and let us fix a metric d on it. Then the formula*

$$\varrho(x, y) = \inf \sum_{i=1}^{k-1} d(px_i, px_{i+1})$$

where the infimum is over all allowable sequences joining x with y endows X with a G-invariant metric ϱ topologically equivalent to ϱ_0.

The result has several interesting corollaries.

First, it is possible to weaken the assumptions in Theorem 21, since it is enough to assume that X is a metrizable, locally compact, locally connected space and X/G is Hausdorff to ensure the existence of an invariant metric. This follows from Stone's theorem, since the image of a locally compact space under an open map is again locally compact.

One has also the following consequence.

Corollary 22. *If a countable group G acts effectively on a compact connected manifold M such that M/G is Hausdorff, then G is finite.*

This and the next corollary are related to the classical results of Newman [7], Smith [9], and Montgomery-Zippin [6, Section 5.5.5].

Corollary 23. *Let M be a connected topological manifold, G a countable discrete group acting on M such that M/G is Hausdorff. Then*
   a) *(rigidity) any element $g \in G$ whose set of fixed points has nonempty interior is the identity transformation,*
   b) *the action is perfect and all isotropy groups are finite.*

P r o o f.   a) Let g be any element of M whose set of fixed points has nonempty interior. If $g \neq \mathrm{id}$ then one can find $x \in \partial(\mathrm{Fix}\, g)$ such that each ball centered at x intersects the interior of $\mathrm{Fix}\, g$. It follows from the existence of a relatively compact slice U_x and the theorems listed before the corollary that $U_x \subset \mathrm{Fix}\, g$, which is a contradiction.

b) Let $x \in M$ and assume that G_x is infinite. Then by part a) G_x acts effectively on some relatively compact slice U_x at x. When we denote $M_g = \{y \in U_x : gy = y\}$, then by Baire's theorem there exists $z \in U_x - \bigcup_{g \in G_x} M_g$. $G_x z$ is infinite in U_x. Thus we obtain a contradiction.

Now we want to show that for any compacts $A, B \subset X$ the set $\{g \in G\colon\ gA \cap B \neq \emptyset\}$ is finite. If not, then the existence of slice yields the existence of an element of M with an infinite stabilizer. $\qquad\square$

Corollary 24. *Let X be a compact, locally connected, metrizable G-space. If G is a countable group and the orbit space is Hausdorff, then there exists an invariant metric on X.*

Corollary 25. *Let X be a compact, locally connected metric G-space, G a countable group. If X/G is Hausdorff, then any sequence $\{g_n\}$ in G contains a subsequence which is convergent in $\mathrm{Homeo}(X)$ (in the compact-open topology) to a homeomorphism.*

P r o o f. We take an invariant metric on X. Using similar arguments as in the classical Ascoli-Arzela's theorem we can find a subsequence convergent to some element of $\mathrm{Homeo}(X)$ in the compact-open topology. $\qquad\square$

Corollary 26. *Let X be a locally compact, locally connected, metrizable and connected G-space. If G is a countable group acting on X such that X/G is Hausdorff, then either all stabilizers are finite, or all are infinite.*

P r o o f. From the existence of a relatively compact slice at each point we see that the set of points with infinite stabilizers is open. Existence of a slice gives that the set of points with finite stabilizers is open. So for a connected space we see that only one type of stabilizers can appear. $\qquad\square$

References

[1] *D. W. Alekseevski*: On perfect actions of groups. Uspiekhi Math. Nauk *34* (1979), 219–220. (In Russian.)
[2] *G. Birkhoff*: A note on topological groups. Compositio Math. *3* (1936), 427–430.
[3] *R. Engelking*: General Topology. PWN, Warszawa, 1977.
[4] *S. Kakutani*: Über die Metrization der topologischen Gruppen. Proc. Imp. Acad. Japan *12* (1936), 82–84.
[5] *J. L. Koszul*: Lectures on Groups of Transformation. Tate Institute, Bombay, 1965.
[6] *D. Montgomery and L. Zippin*: Topological Transformation Groups. Interscience, London, New York, 1995.
[7] *M. H. Newman*: A theorem on periodic transformations of spaces. Quart. J. Math. *2* (1931), 1–8.
[8] *R. S. Palais*: On the existence of slices for actions of non-compact Lie groups. Ann. Math. *73* (1961), 295–323.
[9] *P. A. Smith*: Transformations of finite period III. Ann. Math. *42* (1941), 446–458.

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