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SPHERICAL AND CLOCKWISE SPHERICAL GRAPHS

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Abstract. The main subject of our study are spherical (weakly spherical) graphs, i.e. connected graphs fulfilling the condition that in each interval to each vertex there is exactly one (at least one, respectively) antipodal vertex. Our analysis concerns properties of these graphs especially in connection with convexity and also with hypercube graphs. We deal e.g. with the problem under what conditions all intervals of a spherical graph induce hypercubes and find a new characterization of hypercubes: G is a hypercube if and only if G is spherical and bipartite.

Keywords: spherical graph, hypercube, antipodal vertex, interval

MSC 2000: 05C75, 05C12

1. Introduction

A hypercube graph looks like a sphere in the following sense: each of its vertices has an antipodal vertex, i.e. a vertex at maximal distance. To carry this metaphor one step further: all of its subcubes are sphere-like as well. On the other hand, a hypercube is nothing else than the interval between two of its antipodal vertices, where an interval is the collection of all geodesics between its ends. So in a hypercube all intervals are spherical. This salient property defines the class of spherical graphs, the focus of the paper. Amongst the spherical graphs are such nice distance-regular graphs as the hypercubes, the Johnson graphs, the Schläfli graph and the Gosset graph. But in general, spherical graphs need not be distance-regular, since the Cartesian product of two spherical graphs is again spherical. Thus the spherical

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graphs and the weakly spherical graphs, which are defined quite naturally in terms of just two basic graph functions—the distance function and the interval function—seem to constitute an interesting and highly non-trivial class of graphs.

The second key notion we are interested in is that of an interval monotone graph, or, more exactly, that of convexity. We introduce certain modifications of it (clockwise convexity etc.) trying to better understand the role it plays when dealing with spherical graphs. The paper may also be looked upon as a part of the research area investigating interval properties of certain specific classes of graphs (cf. [8] and [9]), namely those that contain the hypercubes as a subclass; the main goal is then to obtain—by adding suitable conditions—new characterizations of hypercubes and to describe the role hypercubes play when describing the graphs under consideration.

2. Definitions and notation

Our graphs will be finite, undirected, connected, and without loops or multiple edges. We will use several well-known notions and symbols in their usual meaning: e.g. if G is a graph, then V denotes its vertex-set and E its edge-set, $d(u,v)$ is the distance between u and v in V and $N(u)$ is the set of neighbors of u, i.e.

$$N(u) = \{v \in V\colon\ uv \in E\}.$$

The maximum distance in G is the *diameter* of G, and two vertices of G are *diametrical* if their distance equals the diameter of G. The concepts, notation and basic facts concerning intervals are used in accordance with [8], e.g. for u, v in V the interval $I(u,v)$ is defined as:

$$I(u,v) = \{w \in V\colon\ d(u,v) = d(u,w) + d(w,v)\}.$$

Furthermore, for $0 \leqslant i \leqslant d(u,v)$, we denote

$$N_i(u,v) = \{w \in I(u,v)\colon\ d(u,w) = i\}.$$

Note that $N_i(u,v) = N_{d(u,v)-i}(v,u)$ for $0 \leqslant i \leqslant d(u,v)$. The sets $N_i(u,v)$ are the *levels* of the interval $I(u,v)$.

We introduce the following notions, see also [5]. Let u, v be vertices of G, and let w, $\overline{w}$ be vertices in $I(u,v)$. We say that $\overline{w}$ is an *antipodal vertex of w within $I(u,v)$* if $d(w,\overline{w}) = d(u,v)$. Clearly, if $\overline{w}$ is an antipodal vertex of w, then w is an antipodal vertex of $\overline{w}$. The following facts easily follow from basic properties of intervals: if $w, \overline{w} \in I(u,v)$ and $\overline{w}$ is an antipodal vertex of w, then

(i) $d(u,w) = d(v,\overline{w})$; especially, if $w \in N(u)$, then $\overline{w} \in N(v)$,

(ii) if $u \neq w \neq v$, then $I(u,w) \cap I(v,\overline{w}) = \emptyset$.

We call an interval $I(u,v)$ *weakly spherical*, if every vertex of $I(u,v)$ has at least one antipodal vertex within $I(u,v)$. We call an interval $I(u,v)$ *spherical*, if every vertex of $I(u,v)$ has exactly one antipodal vertex within $I(u,v)$. A connected graph G is called (*weakly*) *spherical* if each of its intervals is (weakly) spherical. Note that there is no relation between our notion of spherical graphs and that of antipodal graphs, see e.g. Berman and Kotzig [1], or that of diametrical graphs, see [8].

A subset W of the set of vertices V is *convex* if $I(x,y) \subseteq W$ for all x, y in W. A graph G is *interval monotone*, see [8], if every interval of G is convex. See [7] for the latest results on interval monotone graphs. We call an interval $I(u,v)$ *clockwise convex*, if the following holds: whenever $w \in N_1(u,v)$ and $\overline{w}$ is an antipodal vertex of w within $I(u,v)$, then $I(w,\overline{w}) \subseteq I(u,v)$. A graph G is called *clockwise convex*, if every interval of G is clockwise convex. We call a graph *clockwise spherical*, if it is both clockwise convex and spherical.

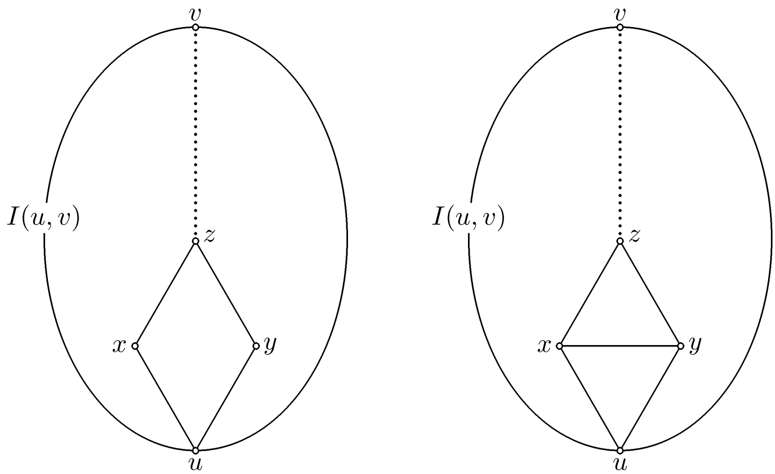
Let u,v be vertices of a clockwise convex graph G. If $\overline{w}$ is an antipodal vertex of w within $I(u,v)$ with w in $N_1(u,v)$, then, by definition, $I(w,\overline{w}) \subseteq I(u,v)$. It is easy to verify that in this case u lies in $N_1(w,\overline{w})$ and that v is an antipodal vertex of u within $I(w,\overline{w})$. Hence $I(u,v) \subseteq I(w,\overline{w})$ and $I(u,v) = I(w,\overline{w})$. Loosely speaking, in a clockwise convex graph we can *shift* intervals *clockwise*.

Similarly, if u,v are vertices of an interval monotone graph G and w and $\overline{w}$ are antipodal vertices within $I(u,v)$, then $I(u,v) = I(w,\overline{w})$.

We say that an interval $I(u,v)$ has the *quadrangle property*, if for any two different non-adjacent vertices x,y from $N_1(u,v)$ there is a common neighbor z of x and y such that $d(z,v) = d(u,v) - 2$, see Fig. 1a. Note that z lies in $I(x,v) \cap I(y,v) \subseteq I(u,v)$.

Calling the graph $K_{1,1,2}$ a *kite*, we say that an interval $I(u,v)$ has the *kite property*, if for any two different adjacent vertices x, y from $N_1(u,v)$ there is a common neighbor z of x and y in $I(u,v)$ such that $d(z,v) = d(u,v) - 2$, see Fig. 1b. A graph G is said to have the *quadrangle (kite) property*, if every interval of G has the quadrangle (kite) property.

Finally, recall that the *Cartesian product* $G \,\Box\, H$ of two graphs $G = (V,E)$ and $H = (V',E')$ has $V \times V'$ as its vertex-set, where (u,u') and (v,v') are adjacent if and only if either $u = v$ and $u'v' \in E'$ or $uv \in E$ and $u' = v'$. Note that the interval between (u,u') and (v,v') in $G \,\Box\, H$ is just the Cartesian product of the sets $I_G(u,v)$ and $I_H(u',v')$.



a. The quadrangle property b. The kite property

Fig. 1

3. Examples

It is easy to see that both the class of weakly spherical graphs and that of spherical graphs are closed under the operation of Cartesian product.

Every complete k-partite graph $K_{m_1,\ldots,m_k}$ for $k, m_1, \ldots, m_k \geqslant 2$ is weakly spherical. If, furthermore, $\max(m_1, \ldots, m_k) > 2$, then $K_{m_1,\ldots,m_k}$ is not spherical.

Clearly, complete graphs K_n and hyperoctahedra are simple examples of spherical graphs. Here a hyperoctahedron is a K_{2m} minus a perfect matching, $m \geqslant 3$ or, equivalently, $K_{2,2,\ldots,2}$. Hypercubes Q_n with $n \geqslant 1$, as another example of spherical graphs, are Cartesian products of n copies of K_2.

A less trivial example of spherical graphs are the so-called *extended odd graphs* E_k for $k \geqslant 2$, see [8], which are also called *Laborde-Mulder graphs* [4]. Write $N_{2k-1} = \{1, \ldots, 2k-1\}$. The vertex-set of E_k is $\{A \subseteq N_{2k-1} \mid 0 \leqslant |A| \leqslant k-1\}$, two vertices being adjacent if their symmetric difference consists either of 1 or $2k-2$ elements. Clearly, E_2 is K_4 and E_k may be obtained in either of the following two ways:

- take Q_{2k-1} and identify every two diametrical vertices of it; therefore they are also called *folded* $(2k-1)$-*cubes*, see [3];
- take Q_{2k-2} and add to it 2^{2k-3} "diagonals" (i.e. new edges, joining diametrical vertices of Q_{2k-2}).

As still another example of spherical graphs consider the second powers of hypercubes Q_n^2 for $n \geqslant 2$. They belong to the class of the so called cube-like graphs, see below. The graph Q_n^2 arises from Q_n by adding $2^{n-1}\binom{n}{2}$ new edges, joining any two vertices whose Hamming distance is 2. It is easy to see what are the intervals of Q_n^2 and that all of its intervals are spherical.

We can supply still other examples of spherical graphs. For $k > 0$ and $0 \leqslant d \leqslant k$ the *Johnson graph* $J(k,d)$ [3] has as its vertex-set the set of all $0,1$-vectors of length k containing d ones and $k-d$ zeros; two vertices are adjacent if they differ in exactly 2 coordinates. For $m \geqslant 1$, a *folded Johnson graph* $J''(2m,m)$ is obtained from $J(2m,m)$ by identifying every pair of its diametrical vertices. It is not difficult to verify that both the Johnson graphs and the folded Johnson graphs are spherical. The *Gosset graph* ([3]) has as vertices the vectors of length 8, either consisting of two 1's and six 0's, or consisting of six $+\frac{1}{2}$ and two $-\frac{1}{2}$; e.g. $(1,1,0,0,0,0,0,0)$, $(+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2})$, and $(+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2})$ are vertices of the Gosset graph. Two vertices are adjacent if and only if their inner product is exactly $+1$ (so the first and the second, as well as the second and the third vector of the above three are adjacent). The *Schläfli graph* [3] is the subgraph of the Gosset graph consisting of the $0,1$-vectors with one 1 at the last two places and the $(+\frac{1}{2},-\frac{1}{2})$-vectors with minus signs at the first six places only. It is straightforward to verify that the Gosset graph and the Schläfli graph are spherical as well.

An example of a class of weakly spherical graphs is provided by the *cube-like graphs*. These are defined as follows, see [6]. Let V be the set of all $0,1$-vectors of length $n \geqslant 1$, and let σ be a nonempty subset of V. Then the cube-like graph $Q_n(\sigma)$ has V as its vertex-set, and two vectors u and v are adjacent whenever there exists a vector e in σ such that $v = u \oplus e$, where $\oplus$ denotes the coordinate-wise addition modulo 2. Note that $Q_n(\sigma)$ is just the n-dimensional hypercube if σ consists of all $0,1$-vectors of weight one. To verify that $Q_n(\sigma)$ is weakly spherical we first observe that this is trivially true if $n = 1$. Let $n \geqslant 2$, let u and v be vertices of $Q_n(\sigma)$ with $d(u,v) = k \geqslant 2$. Then there exist vectors $e_1, e_2, \ldots, e_k$ in σ such that

$$v = u \oplus e_1 \oplus e_2 \oplus \ldots \oplus e_k$$

and, for any $f_1, \ldots, f_l$ in σ with $1 \leqslant l < k$, we have

$$v \neq u \oplus f_1 \oplus \ldots \oplus f_l.$$

Choose any x in $I(u,v)$ with $u \neq x \neq v$. Then there are vectors

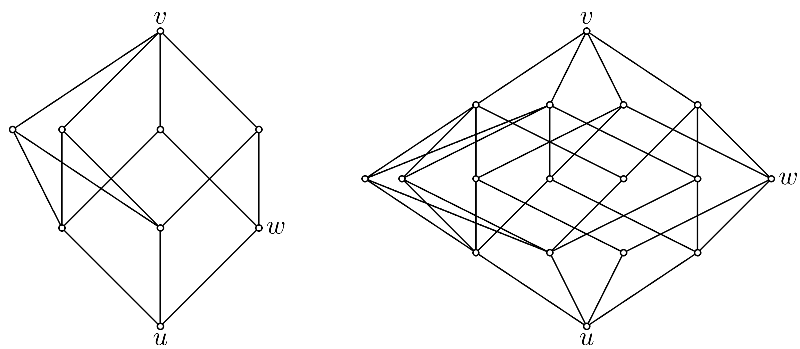
$$g_1, \ldots, g_p, g_{p+1}, \ldots, g_k$$

in σ with $1 \leqslant p < k$ such that

$$x = u \oplus g_1 \oplus \ldots \oplus g_p \quad \text{and} \quad v = x \oplus g_{p+1} \oplus \ldots \oplus g_k.$$

It follows that for $y = u \oplus g_{p+1} \oplus \ldots \oplus g_k$ we have $y \in I(u,v)$ and $d(x,y) = k$.

We close this section with an operation which produces new weakly spherical graphs from old ones. Let G be a graph and let u be a vertex of G. We say that we *split u into two vertices* if we add a new vertex u' and make it adjacent to all neighbors of u. Recall that a vertex u is *dominated* by a vertex v if v is adjacent to u and all its neighbors. Now let G be a (weakly) spherical graph, and let u be a vertex of G. Then it is easily verified that splitting u results in a weakly spherical graph if and only if u is a non-dominated vertex in G, see Fig. 2 for examples of splitting a vertex.



a. Splitting a vertex of Q_3 b. Splitting a vertex of Q_4

Fig. 2

4. Basic facts

In this section we prove some basic facts on weakly spherical and spherical graphs. Recall that a set W of vertices in a graph G is *independent* if the vertices of W are mutually non-adjacent.

Lemma 1. *Let G be weakly spherical, let u, v be vertices of G. Then*

(i) *if $w \in N_1(v,u)$, then $|N_1(u,v)| \geqslant |N_1(u,w)| + 1$,*

(ii) *there are at least $d(u,v)$ independent vertices in $N_1(u,v)$,*

(iii) $|N_1(u,v)| \geqslant d(u,v)$,

(iv) *if $|N_1(u,v)| = d(u,v)$, then $|N_1(u,w)| = d(u,w)$ for any $w \in I(u,v)$,*

(v) *if $|N_1(u,v)| = d(u,v)$, then for any $w \in N_1(v,u)$ there is exactly one $\overline{w} \in I(u,v)$ such that $d(w,\overline{w}) = d(u,v)$.*

P r o o f. (i) Let w be in $N_1(v,u)$. Then we have $I(u,w) \subseteq I(u,v)$. Since $I(u,v)$ is weakly spherical, there is a vertex $\overline{w}$ in $I(u,v)$ such that $d(w,\overline{w}) = d(u,v)$. Obviously, $\overline{w}$ is in $N_1(u,v) - I(u,w)$.

(ii) We use induction on $d(u,v)$. The statement is trivial for $d(u,v) = 1$. Let $d(u,v) > 1$, and let w be in $N_1(v,u)$. Then $N_1(u,w) \subseteq N_1(u,v)$, and $d(u,w) = d(u,v) - 1$ whence by the induction hypothesis, there are at least $d(u,w)$ independent vertices in $N_1(u,w)$. Consider an antipodal vertex $\overline{w}$ of w within $I(u,v)$. Then $\overline{w}$ lies in $N_1(u,v) - I(u,w)$ and $\overline{w}$ is not adjacent to any vertex of $N_1(u,w)$, for otherwise $d(\overline{w},w) < d(u,v)$.

(iii) This follows trivially from (ii).

(iv) Assume that $|N_1(u,w)| > d(u,w)$ for some w in $I(u,v)$. Then, by induction and using (i), we have

$$|N_1(u,v))| \geqslant |N_1(u,w)| + d(w,v) > d(u,v),$$

which yields a contradiction.

(v) Let w be in $N_1(v,u)$. By (iv), we have

$$|N_1(u,w)| = d(u,w) = d(u,v) - 1.$$

Furthermore, from

$$\overline{w} \in I(u,v) \quad \text{and} \quad d(w,\overline{w}) = d(u,v)$$

we deduce

$$\overline{w} \in N_1(u,v) - I(u,w),$$

and hence (v) follows. □

The example of the weakly spherical graph from Fig. 2.a, arising by splitting a vertex of Q_3, shows in contrast to (v) that the antipodal vertex $\overline{w}$ of w need not be determined uniquely for w not in $N_1(v,u)$. We cannot guarantee sphericity of $I(u,v)$ (that is, uniqueness of the antipodal vertex within $I(u,v)$ for every vertex of $I(u,v)$) even by asking

$$(*) \qquad |N_1(u,v)| = |N_1(v,u)| = d(u,v).$$

As an example consider a weakly spherical graph arising by splitting a vertex of Q_4, see Fig. 2.b.

This example also answers in the negative the following question posed by Bezrukov [2]. "Is it true that, if G is weakly spherical and $(*)$ holds for u,v in V, then $I(u,v)$ induces a hypercube $Q_{d(u,v)}$?"

In the case of spherical graphs the question is answered in the following section.

Lemma 2. *Let G be spherical. Then*

(i) $|N_k(u,v)| = |N_{d(u,v)-k}(u,v)|$ *for any two vertices* u *and* v *of* G *and for* $0 \leqslant k \leqslant d(u,v)$;

(ii) G *is regular.*

P r o o f. (i) In Section 2 we have already observed that the antipodal vertex $\overline{w}$ of w in $N_k(u,v)$ within $I(u,v)$ lies in $N_{d(u,v)-k}(u,v)$. Equality follows from the unicity of antipodes.

(ii) Let u and v be adjacent vertices of G. Take any vertex x in $N(u)-N(v)$. Then $d(x,v)=2$, so by sphericity, there is a unique vertex y in $I(x,v)$ that is not adjacent to u. Note that y is in $N(v)-N(u)$. This establishes an injection of $N(u)-N(v)$ into $N(v)-N(u)$. Similarly, we can obtain an injection of $N(v)-N(u)$ into $N(u)-N(v)$. Hence $|N(u)|=|N(v)|$, so that G is regular, being a connected graph. □

Note that regularity of spherical graphs was proved independently by Nomura ([10]).

Theorem 3. *Let* G *be a spherical graph. Then the vertex connectivity of* G *equals the degree of* G.

P r o o f. Let κ be the vertex connectivity of G and let δ be the degree of G. Then, by definition, we have $\kappa \leqslant \delta$.

Let S be a vertex cutset of minimum size, separating G into two parts A and B. Since S is of minimum size, every vertex in S has neighbors in A as well as in B. Let x be any vertex in S with, say, a neighbors in A and b neighbors in B. Then x has $s=\delta-a-b$ neighbors in S. Without loss of generality, we may assume that $1 \leqslant a \leqslant b$. Consider any neighbor u of x in A and any neighbor v of x in B. Then we have $d(u,v)=2$. Hence, by sphericity of G, there is a unique common neighbor y_{uv} of u and v distinct from x and not adjacent to x. Clearly, y_{uv} lies in S. Now, $d(x,y_{uv})=2$, and u and v are non-adjacent vertices in $I(x,y_{uv})$. So x and y_{uv} cannot have any other common neighbor in A or B. This implies that, for two distinct pairs u,v and u',v' of neighbors of x with u, u' in A and v, v' in B, the vertices y_{uv} and $y_{u'v'}$ are distinct. Let T be the set of vertices of type y_{uv}. Then it follows that $|T|=ab$ and, moreover, $N(x)\cap T=\emptyset$. Since $[N(x)\cap S]\cup\{x\}\cup T$ is a subset of S, we have the inequality

$$(**) \qquad s+1+ab \leqslant \kappa \leqslant \delta = s+a+b.$$

Hence we have $ab \leqslant a+b-1$. Since $1 \leqslant a \leqslant b$, we infer that $a=1$, so that we have equality in $(**)$. □

As promised, we will address the question where the hypercubes are located within the commonwealth of spherical graphs. We will use Theorem 2.2.1 from [8]. A $(0,2)$-*graph* is a connected graph G such that $|N(u) \cap N(v)| = 0$ or 2 for any two vertices u, v of G.

Theorem 4 (Mulder, 1980). *Let H be a bipartite $(0,2)$-graph of diameter D, and let u, v be vertices of H with $d(u,v) = n$. If, for any w of H*

$$|N_1(w,u)| = d(w,u) \quad \text{and} \quad |N_1(w,v)| = d(w,v),$$

then H is isomorphic to Q_D.

The main part of the proof of this theorem consists in showing, by induction on m, that $\bigcup\limits_{k \leqslant m} N_k(u,v)$ induces a subgraph isomorphic to the subgraph induced by the $m+1$ "lower levels" of Q_D.

Proposition 5. *Let G be spherical. Then, for any u, v in V with*

$$|N_1(u,v)| = d(u,v),$$

the subgraph of G induced by $I(u,v)$ is isomorphic to $Q_{d(u,v)}$.

Proof. Let $|N_1(u,v)| = d(u,v) = n$, and let H be the subgraph of G induced by $I(u,v)$.

First, it follows from Lemma 1 (iv) and Lemma 2 that, for any w in $I(u,v)$, we have

$$|N_1(w,u)| = d(w,u), \quad \text{and} \quad |N_1(w,v)| = d(w,v).$$

Assume that H is not bipartite, and let n be as small as possible under this condition. Note that an odd cycle in H cannot consist only of edges between levels of $I(u,v)$. Hence there exists an edge xy in some $N_k(u,v)$. By minimality of n we have

$$I(u,x) \cap I(u,y) = \{u\} \quad \text{and} \quad I(x,v) \cap I(y,v) = \{v\}.$$

Note that we have

$$|N_1(u,x)| = |N_1(u,y)| = k, \quad \text{and} \quad |N_1(v,x)| = |N_1(v,y)| = n-k.$$

Now any vertex w in $N_1(v,x)$ must have its antipodal vertex within $I(u,v)$ in $N_1(u,v) - [N_1(u,x) \cup N_1(u,y)]$. So $N_1(u,v)$ contains at least $2k + n - k = n + k > n$ vertices, which is impossible. So H is bipartite.

Finally, let x and y be vertices in H having a common neighbor in H. Since H is bipartite, we have either that x and y are in the same level $N_k(u,v)$, or that they are in different non-consecutive levels $N_{k-2}(u,v)$ and $N_k(u,v)$. In the latter case $I(x,y) \subseteq I(u,v)$, so, by sphericity, x and y have exactly two common neighbors in H. So assume x and y are in $N_k(u,v)$ and have a common neighbor, say, z in $N_{k+1}(u,v)$. By the first observation in the proof, we know that $|N_{k-1}(u,x)| = k = |N_k(u,z) - \{x\}|$. By sphericity, each vertex in $N_{k-1}(u,x)$ must have a unique neighbor in $N_k(u,z) - \{x\}$. In particular, x and y have a second common neighbor in $N_{k-1}(u,x)$.

Thus we have shown that H satisfies the conditions of Theorem 2.2.1 from [8], whence H induces a $Q_{d(u,v)}$. □

Theorem 6. *A graph G is a hypercube if and only if G is bipartite and spherical.*

Proof. Let G be bipartite and spherical of degree δ. By induction on $d(u,v)$ we prove that $|N_1(u,v)| = d(u,v)$. For $d(u,v) \leqslant 2$, this is obvious. So assume $d(u,v) = n > 2$, and let w be in $N_1(v,u)$. Then $d(u,w) = n-1$ and, by induction, $|N_1(u,w)| = n-1$. Clearly, the antipodal vertex of w lies in $N_1(u,v) - N_1(u,w)$. Take any y in $N_1(u,v) - N_1(u,w)$. Then $d(y,w) > n-2$, so, G being bipartite, we infer that $d(y,w) = n$. Therefore $|N_1(u,v) - N_1(u,w)| = 1$, so that $|N_1(u,v)| = n$.

Now let x and y be vertices of G with $d(x,y)$ equal to the diameter of G. Since G is bipartite, the set of neighbors of x is precisely $N_1(x,y)$. Therefore the diameter of G equals δ. By Proposition 3 and the regularity of spherical graphs we conclude that G is a hypercube of dimension δ. □

6. Spherical graphs and clockwise convexity

All examples of spherical graphs that we have presented so far are interval monotone (i.e. have convex intervals). Whether it is true for spherical graphs in general remains to be seen. Note that in the class of weakly spherical graphs there are graphs that are not interval monotone. These are first of all the complete k-partite graphs $K_{m_1,\ldots,m_k}$ for $k, m_1, \ldots, m_k \geqslant 2$ and $\max(m_1,\ldots,m_k) > 2$. But there are more interesting examples, like the cross product of the icosahedron with K_2, and the cross product of the Shrikhande graph with K_2, see [3] and [8] for the Shrikhande graph. To get a better understanding of what it means that intervals are convex in a spherical graph, we present the following two theorems.

Theorem 7. *Let $G = (V,E)$ be a spherical graph. Then the following conditions are equivalent:*

(i) *G is clockwise convex,*
(ii) *G is interval monotone,*
(iii) *G has the quadrangle property.*

Proof. (i) ⇒ (ii) Take any interval $I(u,v)$ and any two vertices x, y in $I(u,v)$. Let $\overline{x}$ be the antipode of x within $I(u,v)$. We will show that $I(x,\overline{x}) = I(u,v)$. This suffices, since, by the basic properties of intervals, $I(x,y) \subseteq I(x,\overline{x})$. Let $d(u,x) = k$. If $k = 0$, then we are trivially done. So let $k \geqslant 1$, and let $u = x_0, x_1, \ldots, x_k = x$ be the consecutive vertices of a shortest path between u and x in $I(u,x) \subseteq I(u,v)$, and let $\overline{x_i}$ be the antipode of x_i within $I(u,v)$ for $1 \leqslant i \leqslant k$. Note that $\overline{x_1}$ is adjacent to $\overline{x_0} = v$. By clockwise convexity, we have $I(x_i,\overline{x_i}) = I(x_{i-1},\overline{x_{i-1}})$, so that x_{i+1} and $\overline{x_{i+1}}$ are antipodes within $I(x_i,\overline{x_i})$. Furthermore, $\overline{x_{i+1}}$ is adjacent to $\overline{x_i}$ for $i = 1, 2, \ldots k-1$. In particular, it follows that $I(x,\overline{x}) = I(u,v)$, and we are done.

(ii) ⇒ (iii) Let $I(u,v)$ be any interval and let x, y be non-adjacent neighbors of v in $I(u,v)$. Then, by sphericity, there must be a common neighbor z of x and y which is not adjacent to v. By interval monotonicity, z lies in $I(u,v)$, whence $d(u,z) = d(u,v) - 2$.

(iii) ⇒ (i) Assume the contrary. Let u and v be vertices of G, with $n = d(u,v)$ as small as possible, such that there are antipodal vertices within $I(u,v)$ with w in $N_1(u,v)$, with $\overline{w}$ in $N_{n-1}(u,v)$, and with $I(w,\overline{w}) - I(u,v) \neq \emptyset$. In the proof we will use the quadrangle property over and over again, in most cases to construct an interval of length 2 which is not spherical. We use the following format: we apply the quadrangle property to non-adjacent vertices p and q in $N_1(a,b)$ to find a common neighbor r of p and q in $I(p,b)$ with

$$d(r,b) = d(p,b) - 1 = d(a,b) - 2.$$

We use the interval $I(p,b)$ to decide where r is located with respect to $I(u,v)$ or $I(w,\overline{w})$.

Choose any x in $I(w,\overline{w}) - I(u,v)$ with $k = d(w,x)$ as small as possible. Note that u and v are in $I(w,\overline{w})$, so that

$$I(u,\overline{w}) \cup I(w,v) \subseteq I(u,v) \cap I(w,\overline{w}).$$

Hence x does not lie in $I(u,\overline{w}) \cup I(w,v)$. This implies that

$$\begin{aligned} k \leqslant (u,x) &\leqslant k+1, \\ n-k \leqslant d(x,v) &\leqslant n-k+1, \\ n+1 \leqslant d(u,x) + d(x,v) &\leqslant n+2. \end{aligned}$$

First assume that $k = 1$. If x were adjacent to u, then the above inequalities would imply that $d(x, v) = n$. But then v would have u and x as distinct antipodes in $I(w, \overline{w})$. So we have $d(u, x) = 2$. By the quadrangle property applied to u and x in $N_1(w, \overline{w})$, we find a common neighbor p in $I(u, \overline{w}) \subseteq I(u, v)$ of x and u with $d(p, \overline{w}) = n - 2$. Then we have $d(p, w) = 2$ and $d(v, p) = n - 1$. Now we apply the quadrangle property to p and w in $N_1(u, v)$ and find a common neighbor q of p and w with $d(v, q) = n - 2$. But now u has q and x as distinct antipodes in $I(p, w)$, which is impossible. So $k \geqslant 2$.

By Lemma 1 (ii) we find two non-adjacent vertices y and z in $N_1(x, w)$. By the quadrangle property, we find a common neighbor s of y and z with $d(w, s) = k - 2$. Note that, by minimality of k, all of y, z, s are in $I(u, v)$. Since $d(u, x) \geqslant k$ and $d(w, s) + 1 = d(w, y) = d(w, z) = k - 1$, we have

$$k - 1 \leqslant d(u, y) \leqslant k,$$
$$k - 1 \leqslant d(u, z) \leqslant k,$$
$$k - 2 \leqslant d(u, s) \leqslant k - 1.$$

We now show that $d(u, y) \neq d(u, z)$. First assume that $d(u, y) = d(u, z) = k$, so that $d(u, s) = k - 1$. Then, the vertices y, z, s being in $I(u, v)$, we have $d(v, y) = d(v, z) = n - k$ and $d(v, s) = n - k + 1$. We apply the quadrangle property to y and z in $N_1(s, v)$ and find a common neighbor t of y and z in $I(y, v) \subseteq I(u, v)$ with $d(v, t) = d(v, s) - 2 = n - k - 1$. Now s has x and t as distinct antipodes in $I(u, z)$. So we cannot have $d(u, y)$ and $d(u, z)$ both equal to k. If $d(u, y) = d(u, z) = k - 1$, then we have $d(u, x) = k$. We apply the quadrangle property to y and z in $N_1(x, y)$ and find a common neighbor s' of y and z in $I(u, y) \subseteq I(u, v)$ with $d(u, s') = k - 2$, so that $d(v, s') = n - k + 2$. Now we apply the quadrangle property to y and z in $N_1(s'v)$ and find a common neighbor t' in $I(y, v) \subseteq I(u, v)$ of y and z with $d(v, t') = d(v, s') - 2$. But now s' has t' and x as distinct antipodes in $I(y, z)$. Thus we have shown that $d(u, y)$ and $d(u, z)$ are unequal. Without loss of generality, we may assume that

$$d(u, y) = k,$$
$$d(u, z) = k - 1.$$

This implies that we have

$$d(u, s) = k - 1,$$
$$d(u, x) = k,$$
$$d(v, x) = d(v, z) = d(v, s) = n - k + 1,$$
$$d(v, y) = n - k.$$

Note that s and y are in $I(w,v)$, so $I(w,s) \subseteq I(w,y) \subseteq I(w,v)$. Recall that $I(w,s) \subseteq I(w,x) \subseteq I(w,\overline{w})$. We consider two cases.

Case 1. $k \geqslant 3$.

Let w_1 be a neighbor of w in $I(w,s)$. Then w_1 is not adjacent to u. We apply the quadrangle property to u and w_1 in $N_1(w,\overline{w})$ and find a common neighbor u_1 in $I(u,\overline{w})$ of u and w_1 with $d(u_1,\overline{w}) = n-2$. Then we have the following situation: $d(w_1,\overline{w}) = n-1$ and x is in $I(w_1,\overline{w})$, whereas x is not in $I(u_1,v) \subseteq I(u,v)$ and, finally, w_1 and $\overline{w}$ are antipodes in $I(u_1,v)$. This is in conflict with the minimality of n, by which the Case 1 is settled.

Case 2. $k = 2$.

Now we have the following situation: $w = s$ and w is adjacent to y and z, and u is adjacent to z, and

$$\begin{aligned} d(u,y) = d(u,x) &= 2,\\ d(v,y) &= n-2,\\ d(v,x) &= n-1,\\ d(y,\overline{w}) = d(z,\overline{w}) = d(u,\overline{w}) &= n-1,\\ d(x,\overline{w}) &= n-2. \end{aligned}$$

We apply the quadrangle property to u and y in $N_1(w,\overline{w})$ and find a common neighbor u' in $I(u,\overline{w}) \subseteq I(u,v)$ of u and y with $d(u',\overline{w}) = n-2 = d(u',v)-1$. Note that u' is distinct from x and z. If u' were adjacent to z, then w would have x and u' as distinct antipodes in $I(z,y)$, so u' is not adjacent to z.

Suppose u' is adjacent to x. By the quadrangle property applied to u' and z in $N_1(u,v)$ we find a common neighbor z' in $I(u',v) \subseteq I(u,v)$ of u' and z with $d(v,z') = n-2$. Then u has x and z' as distinct antipodes in $I(u',z)$. So u' is not adjacent to x.

We apply the quadrangle property to u' and x in $N_1(y,\overline{w})$ and find a common neighbor x' in $I(u',\overline{w}) \subseteq I(u,\overline{w})$ of u' and x with $d(x',\overline{w}) = n-3$. Then we also have $d(v,x') = n-2 = d(v,y) = d(v,x)-1$. Observe that, since $d(y,\overline{w}) = n-1$, the vertices x' and y are not adjacent. We apply the quadrangle property to x' and y in $N_1(x,v)$ and find a common neighbor y' in $I(y,v) \subseteq I(u,v)$ of x' and y with $d(y',v) = n-3$. Since y' is in $I(u,v)$, we have $I(u,y') \subseteq I(u,v)$ so that x is not in $I(u,y')$. Now we have

$$d(u,y) = d(u,x') = 2 = d(u,y') - 1.$$

We apply the quadrangle property to x' and y in $N_1(y',u)$ and find a common neighbor r in $I(y,u) \subseteq I(u,v)$ of x' and y with $d(u,r) = 1$. Then y' has x and r as distinct antipodes in $I(x', y)$. This is impossible, by which Case 2 is settled and the proof is complete. $\square$

The next theorem provides some extra information on clockwise spherical graphs.

Theorem 8. *Let G be a clockwise spherical graph. Then*

(i) *G has the kite property,*

(ii) *if u, v are vertices of G, then the maximum independent set in $N_1(u, v)$ has size $d(u, v)$.*

Proof. (i) Let x, y be adjacent neighbors of v in the interval $I(u, v)$. Assume that x and y have no common neighbor at distance $d(u, v) - 2$ from u. Let $d(u, v)$ be as small as possible under this condition. Then we have

$$I(u, x) \cap I(u, y) = \{u\},$$

and $d(u, v) \geqslant 3$. Let z be any neighbor of y in $I(u, y)$, so that z is not adjacent to x. By sphericity there is a common neighbor p of z and v, which is not adjacent to y. Note that now v and z are non-adjacent vertices in the interval $I(p, y)$. Hence p cannot be adjacent to x. By the quadrangle property, we find a common neighbor q of x and p at distance $d(u, v) - 2$ from u. Now, either by the quadrangle property or by the minimality of $d(u, v)$, there must be a vertex adjacent to q as well as z at distance $d(u, v) - 3$ from u. Since $I(u, x) \cap I(u, y) = \{u\}$, this vertex must be u itself. Consider the antipode $\overline{p}$ of p within $I(u, v)$. By clockwise convexity, $I(\overline{p}, p) = I(u, v)$. Since $d(p, x) = d(p, y) = 2$, it follows that x and y are adjacent to $\overline{p}$. This contradicts our choice of u, v, x, y, whence the kite property is established.

(ii) Let G be clockwise spherical, let u, v be vertices of G. Because of (ii) of Lemma 1, it suffices to prove that every independent subset A of $N_1(u, v)$ has at most $d(u, v)$ elements. We prove it by induction on $d(u, v)$. The statement is trivial for $d(u, v) \leqslant 1$. Hence assume $d(u, v) = n \geqslant 2$, and let A be an independent subset of $N_1(u, v)$. Choose w in A arbitrarily and let $\overline{w}$ be the antipodal vertex of w within $I(u, v)$. Now it is easy to check that

$$x \in N_1(u, v) - I(u, \overline{w}) \ \& \ x \neq w \rightarrow xw \in E.$$

In fact, for such a vertex x we easily verify $d(\overline{w}, x) = n - 1$ (because $d(\overline{w}, x) \leqslant n - 2$ would imply $x \in I(u, \overline{w})$ whereas $d(\overline{w}, x) = n$ would mean that x is a second antipodal vertex of $\overline{w}$ within $I(u, v)$, a contradiction). From clockwise sphericity of G we have $I(u, v) = I(w, \overline{w})$, hence $x \in I(w, \overline{w})$, therefore $d(x, w) = 1$. So, all vertices of A different from w belong to $N_1(u, \overline{w})$ and, by the induction hypothesis, we have $|A| - 1 \leqslant d(u, \overline{w}) = n - 1$. $\square$

Corollary 9. *Let G be a kite-free and clockwise spherical graph. Then each interval in G induces a hypercube.*

P r o o f. In view of Proposition 4, we only need to prove that $|N_1(u,v)| = d(u,v)$ for any two vertices u and v of G. This is accomplished by induction on $d(u,v)$. For $d(u,v) \leqslant 2$ it is obvious. So let $d(u,v) = n \geqslant 3$. Take any vertex w in $N_1(u,v)$ with its antipode $\overline{w}$ within $I(u,v)$. By induction, we have $|N_1(u,\overline{w})| = n-1$. If there were another vertex z besides w in $N_1(u,v) - N_1(u,\overline{w})$ then, by clockwise sphericity, z would be in $I(w,\overline{w})$. Hence $d(z,\overline{w}) = n-1$, which implies that z is adjacent to w. By the kite property applied to w, u, z in $I(w,\overline{w})$ we produce a kite. This contradiction settles the proof. □

We would like to conclude this paper by two open problems. First there is a question whether all spherical graphs are interval monotone or not. In view of the rather complex proof of (iii) ⇒ (i) in Theorem 6 we expect that this is not the case. But so far we have no examples of spherical graphs that are not interval monotone. Concerning to Corollary 8, we ask whether it is possible to replace it by a stronger statement. We suggest the following conjecture: *if G is spherical and triangle-free, then all intervals of G induce hypercubes.*

References

[1] *A. Berman and A. Kotzig*: Cross-cloning and antipodal graphs. Discrete Math. *69* (1988), 107–114.
[2] *S. L. Bezrukov*: Private communication.
[3] *A. E. Brouwer, A. M. Cohen and A. Neumaier*: Distance-regular graphs. A Series of Modern Surveys in Mathematics. Springer Verlag, Berlin-Heidelberg-New York, 1989.
[4] *G. Burosch*: Hasse Graphen spezieller Ordnungen. In: K. Wagner, R. Bodendiek: Graphentheorie I. Anwendungen auf Topologie, Gruppentheorie und Verbandstheorie, B. I. Wissenschaftsverlag, Mannheim-Wien-Zurich, 1989, pp. 157–235.
[5] *G. Burosch, I. Havel and J.-M. Laborde*: Distance monotone graphs and a new characterization of hypercubes. Discrete Math. *110* (1992), 9–16.
[6] *F. Harary*: Four difficult unsolved problems in graph theory. Recent Advances in Graph Theory (M. Fiedler, ed.). Academia, Praha, 1974, pp. 249–256.
[7] *M. Mollard*: Interval-regularity does not lead to interval monotonicity. Discrete Math. *118* (1993), 233–237.
[8] *H. M. Mulder*: The interval function of a graph. Mathematical Centre Tracts 132. Mathematisch Centrum Amsterdam, 1980.
[9] *H. M. Mulder*: Interval-regular graphs. Discrete Math. *44* (1982), 253–269.
[10] *K. Nomura*: Private communication.

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