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THE GEOGRAPHY OF SIMPLY-CONNECTED SYMPLECTIC MANIFOLDS

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Abstract. By using the Seiberg-Witten invariant we show that the region under the Noether line in the lattice domain $\mathbb{Z}\times\mathbb{Z}$ is covered by minimal, simply connected, symplectic 4-manifolds.

Keywords: Seiberg-Witten invariant, geography of symplectic 4-manifold

MSC 2000: 57N13, 58F05

0. Introduction

Let (X,ω) be a simply connected, symplectic 4-manifold with a symplectic form ω. Then X has an almost complex structure compatible with the symplectic structure. The Noether formula says that the number $c_1(X)^2+c_2(X)$ is divisible by 12. The rank $b_2^+(X)$ of the space $H^{2,+}(X;\mathbb{R})$ of self-dual harmonic 2-forms on X is odd because the space X is simply connected. For simplicity we denote $\chi(X)=\frac{1}{2}(1+b_2^+(X))$. A compact symplectic 4-manifold X is called minimal if it contains no symplectically embedded sphere with self-intersection number -1. Let $\mathscr{F}$ denote the set of all minimal, simply connected, symplectic 4-manifolds.

Define a map $f\colon \mathscr{F}\longrightarrow \mathbb{Z}\times\mathbb{Z}$ by

$$X\longmapsto(\chi(X),c_1^2(X)).$$

It is known that $\chi(X)>0$ and $c_1^2(X)\geqslant 0$ if $X\in\mathscr{F}$ with $b_2^+(X)>1$ (for details see [14]). It is also well known that a complex surface X is either rational,

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elliptic, or a surface of general type. The simply connected, minimal rationals X are diffeomorphic to $\mathbb{CP}^2$, $S^2 \times S^2$ or $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}^2}$(the Hirzebruch surfaces). Then $b_2^+(X) = 1$ and $c_1^2(X) = 9$ or 8. Hence $f(X) = (1, 9)$ or $(1, 8)$. If X is minimal elliptic, then $f(X) = (\chi(X), c_1^2(X)) = (n, 0)$ for a natural number $n \in \mathbb{N}$. For surfaces X of general type we know that $c_1^2(X) > 0$ and the two famous inequalities, the Noether inequality and the Bogomolov-Miyaoka-Yau inequality, give constraints for $c_1^2(X)$ in terms of $\chi(X)$:

$$(*) \qquad 2\chi(X) - 6 \leqslant c_1^2(X) \leqslant 9\chi(X).$$

It is known that most of the points in the region $(*)$ correspond to some minimal surfaces of general type. That is, for any $(a, b) \in \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid 2a - 6 \leqslant b \leqslant 9a\}$, there is a minimal surface X of general type such that $(a, b) = (\chi(X), c_1^2(X))$. In this paper we will show that the region under the Noether line $c_1^2 = 2\chi - 6$ can be covered by minimal, simply connected, symplectic 4-manifolds by using the properties of the Seiberg-Witten invariant and the fiber sum.

1. The irreducibility of 4-manifold

In this section we review the definitions and the basic properties of the Seiberg-Witten invariants.

First we recall briefly the Seiberg-Witten invariant for a compact, oriented, Riemannian 4-manifold X with $b_2^+(X) > 1$. A Spin^c-structure s is defined by a triple (W^+, W^-, ϱ), where $W^\pm$ are Hermitian 2-plane bundles and $\varrho\colon T^*X \to \mathrm{Hom}(W^+, W^-)$ satisfies the Clifford relation

$$\varrho^*(e)\varrho(e) = |e|^2 \,\mathrm{Id}_{W^+}.$$

Let $L = \det(W^+)$ be a determinant line bundle of W^+. In particular, when X is a symplectic manifold, the Spin^c-structure on X which corresponds to a given complex line bundle L is characterized by the fact that its bundle W^+ is given by

$$W^+ = E \oplus (K^{-1} \otimes E),$$

where K is the canonical bundle of X. A connection A of the line bundle on L with the Levi-Civita connection on T^*X defines a covariant derivative $\nabla_A\colon \Gamma(W^+) \to \Gamma(W^+ \otimes T^*X)$. The composition of the covariant derivative ∇_A and the Cliffold multiplication defines a Dirac operator

$$D_A\colon \Gamma(W^+) \longrightarrow \Gamma(W^-).$$

For a connection A of L and a section $\Phi \in \Gamma(W^+)$ of W^+, the equations

$$\begin{cases} D_A\Phi = 0, \\ F_A^+ = \dfrac{1}{4}\tau(\Phi \otimes \Phi^*) \end{cases}$$

are called the Seiberg-Witten equations. Here F_A^+ is the self-dual part of the curvature of A and $\tau\colon \mathrm{End}(W^+) \to \Gamma^+(T^*X) \otimes \mathbb{C}$ is the adjoint of the Cliffold multiplication. The gauge group $\mathscr{G} = C^\infty(X, U(1))$ of the complex line bundle L acts on the space of solutions of the SW-equations. The quotient of the space of solutions by the gauge group is called the moduli space of the line bundle L. Then the moduli space is generically a compact smooth manifold with the dimension

$$\frac{1}{4}(c_1^2(L) - (2e(X) + 3\sigma(X))),$$

where $e(X)$ is the Euler characteristic and $\sigma(X)$ is the signature of X. The moduli space defines a diffeomorphic invariant on X which is the so called Seiberg-Witten invariant $SW_X(L)\colon \mathrm{Spin}^c(X) \to \mathbb{Z}$. Here $\mathrm{Spin}^c(X)$ is the set of isomorphism classes of Spin^c-structures on X. For details, see [12].

Definition 1.1. A cohomology class $c = c_1(L) \in H^2(X;\mathbb{Z})$ is called a *basic class* if $SW_X(L) \neq 0$. The manifold X is said to be of *simple type* if $c^2 = 2e(X) + 3\sigma(X)$ for every basic class $c \in H^2(X;\mathbb{Z})$.

Theorem 1.2 [16]. *Let (X,ω) be a symplectic 4-manifold with its orientation given by the volume form $\omega\wedge\omega$, and let $b_2^+(X) \geqslant 2$. If K is the canonical line bundle of X associated to ω, then its Seiberg-Witten invariant $SW_X(K) = \pm 1$ is non-zero.*

Theorem 1.3 [16]. *Every compact symplectic 4-manifold X with $b_2^+(X) \geqslant 2$ is of simple type.*

A smooth 4-manifold X is said to be irreducible if the space X cannot be decomposed into a smooth connected sum $X = X_1 \sharp X_2$ with non-spheres.

Proposition 1.4. *Let X be a simply connected 4-manifold with nontrivial Seiberg-Witten invariants. If for any basic classes K_i, K_j on X*

$$(K_i - K_j)^2 \neq -4,$$

then the space X is irreducible.

P r o o f. Since $SW_X \not\equiv 0$, there is a basic class of X. Assume that X is reducible. Then $X = X_1 \sharp X_2$ and one of the X_i's, say X_2, has negative definite intersection form. By Donaldson there is an element $e \in H^2(X_2)$ such that $e \cdot e = -1$. If K is a basic class of X_1, then $K \pm e$ are also basic classes on X, where $e \in H^2(X_2)$ with $e \cdot e = -1$. Therefore $\{(K+e) - (K-e)\}^2 = (2e)^2 = -4$ gives a contradiction. $\square$

Corollary 1.5. *Let X be a simply connected 4-manifold satisfying the assumption of Proposition 1.4. Then X is minimal.*

2. Fiber sums of elliptic surfaces

Let X be a closed, oriented, smooth 4-manifold with a basic class $c_1(L) \in H^2(X;\mathbb{Z})$ and let x_0 be a fixed point in X.

Definition 2.1. The space

$$\hat{\mathscr{M}}_X(L) = \{(A,\psi,\varphi) \mid F_A^+ = \tfrac{1}{4}\tau(\psi \otimes \psi^*),\ D_A\psi = 0,\ |\varphi| = 1,\ \varphi \in W^+|_{x_0}\}/\mathscr{G}$$

is called the *framed Seiberg-Witten moduli space.* Here $\mathscr{G}$ is the gauge group $C^\infty(X, U(1))$ of the complex line bundle L.

Let M be a 3-manifold embedded respectively in X and Y with zero self-intersections. If there is only the trivial solution of Seiberg-Witten equations on $\mathbb{R} \times M$, then $\hat{\mathscr{M}}(X \underset{M}{\cup} Y)$ satisfies a gluing law in the limit as the length of the neck goes to infinity ([1]).

Let $X_\infty = X \underset{M}{\cup} ([0,\infty) \times M)$, $M_\infty = \mathbb{R} \times M$, and $Y_\infty = Y \underset{M}{\cup} ([0,\infty) \times M)$. For R large enough applying the neck-streching argument, we have

$$\hat{\mathscr{M}}(X \underset{M}{\cup} [0,R] \times M \underset{M}{\cup} Y) \cong \hat{\mathscr{M}}(X_\infty) \underset{\hat{\mathscr{M}}(M_\infty)}{\times} \hat{\mathscr{M}}(Y_\infty).$$

Let $E(1)$ be the elliptic surface over $\mathbb{CP}^1$ which is diffeomorphic to $\mathbb{CP}^2 \sharp 9\overline{\mathbb{CP}^2}$. By repeating the fiber sums we have

$$E(n) = E(n-1) \underset{f}{\sharp} E(1) \quad \text{for } n \geqslant 2,$$

where f is a generic fiber. That is,

$$E(n) = (E(n-1) \backslash N(f)) \underset{\partial(N(f))}{\cup} (E(1) \backslash N(f)),$$

where $N(f)$ is a tubular neighbourhood of a generic fiber f lying in a cusp neighbourhood.

There is only the trivial solution on $\mathbb{R}\times T^3$ because it has zero scalar curvature. Let $X = E(n-1)\backslash N(f)$, $Y = E(1)\backslash N(f)$ and $M = \mathbb{R}^3$. Then by the definition above $E(n)$, $X_\infty = (E(n-1)\backslash N(f)) \cup [0,\infty)\times T^3$, $Y_\infty = (E(1)\backslash N(f))\cup[0,\infty)\times T^3$, and $M_\infty = \mathbb{R}\times T^3$. Since there is only the trivial (static) solution on $\mathbb{R}\times T^3$, we have

$$\hat{\mathscr{M}}(E(n)) \cong \hat{\mathscr{M}}(E(n-1)) \underset{\hat{\mathscr{M}}(\mathbb{R}\times T^3)}{\times} \hat{\mathscr{M}}(E(1)).$$

In [12] it is shown that

$$\mathscr{M}_L(E(1)) = \mathscr{M}_L(\mathbb{C}P^2 \sharp 9\overline{\mathbb{C}P^2}) \cong \begin{cases} \{(A,0)\} & \text{if } c_1(L)\cdot[\omega_g] > 0,\\ \{(A,\psi) \mid \psi \not\equiv 0\} & \text{if } c_1(L)\cdot[\omega_g] < 0,\end{cases}$$

where ω_g is the symplectic form depending on a generic g on $E(1)$. If K is a basic class of $E(n-1)$, then

$$\begin{aligned}\hat{\mathscr{M}}_{K+f}(E(n)) &\cong \hat{\mathscr{M}}_K(E(n-1)) \underset{\hat{\mathscr{M}}(\mathbb{R}\times T^3)}{\times} \hat{\mathscr{M}}_f(E(1))\\ &\cong \hat{\mathscr{M}}_K(E(n-1) \times \mathscr{M}_f(E(1))\\ &\cong \hat{\mathscr{M}}_K(E(n-1)).\end{aligned}$$

Similarly, we get $\hat{\mathscr{M}}_{K-f}(E(n)) \cong \hat{\mathscr{M}}_K(E(n-1))$. So we have

Lemma 2.1. *For $n \geqslant 3$*

$$\mathscr{M}_{K+f}(E(n)) \cong \mathscr{M}_{K-f}(E(n)) \cong \mathscr{M}(E(n-1)).$$

Theorem 2.2. *The basic classes of $E(n)$ are of the form*

$$\{kf \mid k = -(n-2), -(n-4), \dots, n-4, n-2\}, \quad (n \geqslant 2).$$

Proof. We prove Theorem 2.2 by induction on n,

(1) $n = 2$, in [2] the only basic class of $E(2)$ is 0.

(2) Assume that the set of basic classes of $E(n-1)$ is

$$\begin{aligned}\{kf \mid k = &-((n-1)-2), -((n-1)-4), \dots, (n-1)-4, (n-1)-2\}\\ &= \{kf \mid k = -(n-3), -(n-5), \dots, (n-5), (n-3)\}.\end{aligned}$$

Then by Lemma 2.1, the set of basic classes of $E(n)$ is

$$\{kf \mid k = -(n-2), -(n-4), \dots, n-4, n-2\}.$$

□

3. Rational blow-up

The elliptic surface $E(1)$ can be constructed by blowing up $\mathbb{CP}^2$ at 9 intersection points of a generic pencil of cubic curves. The fiber class of $E(1)$ is $f = 3h - e_1 - e_2 - \ldots - e_9$ where $3h$ is the class of the cubic in $H_2(\mathbb{CP}^2; \mathbb{Z})$. The nine exceptional curves e_i are disjoint sections of the elliptic fibration

$$E(1) \longrightarrow \mathbb{CP}^1.$$

The elliptic surface $E(n)$ can be obtained as the fiber sum of n copies of $E(1)$ and these sums can be made so that the sections glue together to give nine disjoint sections of $E(n)$, each of square $-n$.

Consider $E(4)$ with nine disjoint sections of square -4. Each of the nine sections gives an embedded configuration C_2. Therefore $E(4)$ contains disjoint nine configuration space C_2. Let Y_i denote the space obtained by the rational blow downs of the first i-th sections, $1 \leqslant i \leqslant 9$. For $i \leqslant 8$, Y_i is simply connected. In [11] Gompf showed that all these manifolds admit symplectic structures. Therefore Y_i $(1 \leqslant i \leqslant 8)$ is a simply connected symplectic 4-manifold.

To find the basic classes of Y_i we can use the rational blow-down formula of Fintushel and Stern [9].

Theorem 3.1 (Rational blow-down [9]). *Let the rational blow-up Y of Z denote $Y = X \cup C_p$ and let the rational blow-down Z of Y denote $Z = X \cup B_p$ where B_p is a rational ball. If $K_Y \in H^2(Y; \mathbb{Z})$ and $K_Z \in H^2(Z; \mathbb{Z})$ are characteristic elements so that $K_Y^2 \geqslant 2e(Y) + 3\sigma(Y)$ and $i_Y^* K_Y = i_Z^* K_Z$ where $i_Y \colon X \to Y$ and $i_Z \colon X \to Z$, then*

$$SW_Y(K_Y) = SW_Z(K_Z).$$

Proposition 3.2. *The basic classes of Y_i are of the form*

$$\pm(2f + e_1 + e_2 + \ldots + e_i) \quad i = 1, \ldots, 8$$

where e_j is the hyperplane class in the j-th copy of the $\mathbb{CP}^2$'s $(1 \leqslant j \leqslant i)$.

Proof. First, consider the basic classes of Y_1 and consider the configuration C_2 in $\mathbb{CP}^2$ where the sphere represents $2e_1 = u_1$ where e_1 is the hyperplane class in $\mathbb{CP}^2$.

Let $Y = E(4) = X \cup C_2$ and $Z = X \cup B_2 = Y_1$. Let $i \colon X \to Y$ be the inclusion. Over the rational coefficient, the cohomology splits into

$$H^2(Y) = H^2(X) \oplus H^2(C_2).$$

It follows that i^*K is just the projection of K into $H^2(X)$. In other words,

$$i^*K = K + a_1 u_1$$

where a_1 is the unique rational number such that $i_Y^* K \cdot u_1 = 0$. With the rational coefficient,

$$H^2(Y_1) = H^2(X) \oplus H^2(B_2) \cong H^2(X).$$

Since the basic classes of Y are 0, $\pm 2f$, we can consider

$$i_Y^*(0), \quad i_Y^*(+2f) \quad \text{and} \quad i_Y^*(-2f)$$

as the candidates for the basic classes of Y_1 by the rational blow-down formula of Theorem 3.1. Since $u_1^2 = -4$ and $K_{E(4)} \cdot u_1 = 2$, by simple calculation, we obtain

$$i_Y^*(0) = 0, \;\; i_Y^*(2f) = 2f + e_1 \;\text{ and }\; i_Y^*(-2f) = -2f - e_1 = -(2f + e_1).$$

Since Y_1 is a symplectic manifold with $b_2^+ > 1$, Y is of simple type by Theorem 1.3. Therefore $i_Y^*(0)$ is not a basic class of Y_1 because of $c_1^2(Y_1) = 1$. By Theorem 1.2, $\pm(2f + e_1)$ are the only basic classes of Y_1.

To repeat the above process, let $Y = Y_1 = X \cup C_2$ and $Z = X \cup B_2 = Y_2$. Here the configuration $C_2 \subset \mathbb{CP}^2$ in which the sphere represents $2e_2 = u_1$ where e_2 is the hyperplane class in $\mathbb{CP}^2$.

Repeating the above method, the basic classes of Y_2 are

$$\pm(2f + e_1 + e_2).$$

Similarly, if we repeat the above process $i - 2$ times, then the basic classes of Y_i are

$$\pm(2f + e_1 + e_2 + \ldots + e_i) \quad i = 1, \ldots, 8$$

where e_j is the hyperplane class in the j-th copy of the $\mathbb{CP}^2$'s. □

Lemma 3.3 [9]. *For $n \geqslant 4$, the elliptic surface $E(n)$ contains a pair of disjoint configurations C_{n-2} in which the spheres u_j $(1 \leqslant j \leqslant n-1)$ are sections of $E(n)$ and for $1 \leqslant j \leqslant n-2$, $u_j \cdot f = 0$. Furthermore, the rational blow-down of this pair of configurations is the Horikawa surface $H(n)$.*

The first case $n = 4$ gives the example $H(4) = Y_2$. The Horikawa surfaces $H(n)$ lie on the Noether line $2\chi - 6 = c_1^2$.

Proposition 3.4. *The basic classes of* $H(n)$ *are of the form*

$$\pm((n-2)f + e_1 + e_2 + \ldots + e_{n-3} + {e_1}' + {e_2}' + \ldots + {e_{n-3}}')$$

where $e_1, \ldots, e_{n-3}$ *and* ${e_1}', \ldots, {e_{n-3}}'$ *are the exceptional classes in* $H(n)$.

Proof. By Lemma 3.3, the Horikawa surface $H(n)$ is the rational blow-down of the pair of configurations C_{n-3} in $E(n)$. The configurations C_{n-3} embed into $(n-3)\overline{\mathbb{CP}^2}$ representing the elements

$$u_1 = 2e_1 + e_2 + \ldots + e_{n-3}, \; u_2 = e_2 - e_1, \; \ldots, \; u_{n-3} = e_{n-3} - e_{n-4},$$

where e_i is the hyperplane class in the i-th copy of the $\overline{\mathbb{CP}^2}$'s $(1 \leqslant i \leqslant n-3)$. Also the other configurations C_{n-3} embed into $(n-3)\overline{\mathbb{CP}^2}$ representing the elements

$${u_1}' = 2{e_1}' + {e_2}' + \ldots + {e_{n-3}}', \; {u_2}' = {e_2}' - {e_1}', \; \ldots, \; {u_{n-3}}' = {e_{n-3}}' - {e_{n-4}}',$$

where ${e_i}'$ is the hyperplane class in the i-th copy of the $\overline{\mathbb{CP}^2}$'s $(1 \leqslant i \leqslant n-3)$. By Theorem 2.2, the basic classes of $E(n)$ are of the form

$$\{kf \mid k = -(n-2), -(n-4), \ldots, n-4, n-2\}.$$

Let $Y = E(n) = X \cup C_{n-3}$ and $Z = X \cup B_{n-3} \equiv Y(n)$. Let $i\colon X \to E(n)$ be the inclusion. Over the rational coefficient, the cohomology splits into

$$H^2(E(n)) = H^2(X) \oplus H^2(C_{n-3}).$$

It follows that i^*K is just the restriction of the canonical class $K \in H^2(E(n))$ into $H^2(X)$. In other words,

$$i^*K = K + a_1u_1 + a_2u_2 + \ldots + a_{n-3}u_{n-3},$$

where a_j are the unique rational numbers such that $i^*K \cdot u_j = 0$ for all $1 \leqslant j \leqslant n-3$. With the rational coefficients we have

$$H^2(Y(n)) = H^2(X) \oplus H^2(B_{n-3}) = H^2(X).$$

Since the basic classes of $E(n)$ are $kf \quad (k = -(n-2), -(n-4), \ldots, n-4, n-2)$, we can consider $i^*(kf) \quad (k = -(n-2), -(n-4), \ldots, n-4, n-2)$ as the candidates for the basic classes of $Y(n)$.

First, let $i^*(f) = f + a_1u_1 + a_2u_2 + \ldots + a_{n-3}u_{n-3}$. Since $i^*(f) \cdot u_j = 0$ for all $1 \leqslant j \leqslant n-3$, we have

$$\begin{aligned} i^*(f) \cdot u_1 = 0 &\quad \Rightarrow \quad 1 - na_1 + a_2 = 0, \\ i^*(f) \cdot u_2 = 0 &\quad \Rightarrow \quad a_1 - 2a_2 + a_3 = 0, \\ i^*(f) \cdot u_3 = 0 &\quad \Rightarrow \quad a_2 - 2a_3 + a_4 = 0, \\ &\quad \vdots \\ i^*(f) \cdot u_{n-3} = 0 &\quad \Rightarrow \quad a_{n-4} - 2a_{n-2} = 0. \end{aligned}$$

Then we get

$$a_1 = \frac{n-3}{(n-2)^2}, \quad a_2 = \frac{n-4}{(n-2)^2}, \quad \ldots, \quad a_{n-4} = \frac{2}{(n-2)^2}, \quad a_{n-3} = \frac{1}{(n-2)^2}.$$

Therefore,

$$\begin{aligned} i^*(f) &= f + a_1u_1 + \ldots + a_{n-3}u_{n-3} \\ &= f + \frac{n-3}{(n-2)^2}u_1 + \ldots + \frac{1}{(n-2)^2}u_{n-3} \\ &= f + \frac{1}{n-2}e_1 + \frac{1}{n-2}e_2 + \ldots + \frac{1}{n-2}e_{n-3}. \end{aligned}$$

Similarly $i^*(kf) = k(f + \frac{1}{n-2}e_1 + \frac{1}{n-2}e_2 + \ldots + \frac{1}{n-2}e_{n-3})$ for all $k = -(n-2)$, $-(n-4), \ldots, n-4, n-2$. Since $Y(n)$ is a symplectic manifold with $b_2^+ > 1$, $Y(n)$ is of simple type. And by Theorem 1.2, $\pm((n-2)f + e_1 + e_2 + \ldots + e_{n-3})$ are the only basic classes of $Y(n)$ and $(\pm((n-2)f + e_1 + e_2 + \ldots + e_{n-3}))^2 = n-3$.

To repeat the above process, let $Y = Y(n) = X \cup C_{n-3}$ and $Z = X \cup B_{n-3} = H(n)$. The basic classes $H(n)$ are

$$\pm((n-2)f + e_1 + \ldots + e_{n-3} + {e_1}' + \ldots + {e_{n-3}}')$$

and

$$\pm((n-2)f + e_1 + \ldots + e_{n-3} + {e_1}' + \ldots + {e_{n-3}}')^2 = 2n-6.$$

□

Remark. In the proof of Proposition 3.4, $Y(n)$ $(n \geqslant 4)$ is a simply connected, symplectic 4-manifold and $Y(n)$ are not homotopy equivalent to any complex surface.

4. Main theorem

Let X be a simply connected, symplectic 4-manifold. Let X contain a torus f with square 0 lying in a cusp neighbourhood. Taking the fiber sum of X with the regular elliptic surface $E(n)$ along f, the fiber sum $X \underset{f}{\sharp} E(n)$ is a simply connected, symplectic 4-manifold. We know the following relation:

$$(\chi(X \underset{f}{\sharp} E(n)), c_1^2(X \underset{f}{\sharp} E(n))) = (\chi(X) + n, c_1^2(X)).$$

Denote $D \equiv \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid 0 < b < 2a - 6\}$.

Theorem 4.1. *If $(a, b) \in D$ is a point in the region under the Noether line, then there is a minimal, simply connected, symplectic 4-manifold X such that $(\chi(X), c_1^2(X)) = (a, b)$.*

Proof. First, to prove Theorem 4.1 we only have to show that for every $b > 0$, there is a simply connected symplectic manifold X which contains a torus f with square 0 lying in a cusp neighbourhood.

Suppose that b is even. Then by Lemma 3.3, the Horikawa surface $H(n)$ satisfies the above statement. That is, the Horikawa surface $H(n)$ is the simply connected, symplectic manifold which contains a torus f with square $f \cdot f = 0$ lying in a cusp neighbourhood. And $H(n)$ lies on the Noether-line $2\chi - 6 = c_1^2 = b$.

Suppose that b is odd. If $b \leqslant 7$, then the manifolds Y_b ($b = 1, 3, 5, 7$) are the simply connected, symplectic manifolds which contain a torus f with square $f \cdot f = 0$ lying in a cusp neighbourhood. If $b \geqslant 9$, then the manifold $Y_7 \underset{f}{\sharp} H(n)$ ($n \geqslant 4$) lies on the line $2\chi - 7 = c_1^2$ and is simply connected, symplectic 4-manifold which contains a torus f with square $f \cdot f = 0$ lying in a cusp neighbourhood.

Therefore, for every $b > 0$, there is a simply connected manifold X ($= H(n), Y_b$ ($b = 1, 3, 5, 7$), $Y_7 \underset{f}{\sharp} H(n)$ ($n \geqslant 4$)) which contains a torus f with square $f \cdot f = 0$ lying in a cusp neighbourhood.

To complete the proof of Theorem 4.1, we have to show that $X \underset{f}{\sharp} E(n)$ is irreducible when X is either $H(n)$, Y_b, or $Y_7 \underset{f}{\sharp} H(n)$. By Proposition 3.2 and Proposition 3.4, the basic classes of X are only $\pm K_X$ when X is $H(n)$ or Y_i. Therefore the set of basic classes of $X \underset{f}{\sharp} E(n)$ is

$$\{\pm K_X + kf \mid k = -(n-2), -(n-4), \ldots, n-4, n-2\}.$$

The differences of two basic classes are $k_1 f$ or $\pm(2K_X + k_2 f)$ for some integers k_1, k_2. The squares of these are

$$(k_1 f)^2 = 0,$$
$$(\pm(2K_X + k_2 f))^2 = 4K_X^2 > 0.$$

Therefore, by Proposition 1.4, $X \sharp_f E(n)$ is irreducible when X is $H(n)$ or Y_i.

Similarly, the basic classes of $Y_7 \sharp_f H(n)$ are $\pm(K_{Y_7} \pm K_{H(n)})$. Therefore the set of basic classes of $(Y_7 \sharp_f H(n)) \sharp_f E(m)$ is

$$\{\pm(K_{Y_7} \pm K_{H(n)}) + kf \mid k = -(m-2), -(m-4), \dots, m-4, m-2\}.$$

Then the differences of two basic classes are $k_1 f$ or $\pm(2(K_{Y_7} \pm K_{H(n)}) + k_2 f)$ for some integers k_1, k_2. The squares of these are

$$(k_1 f)^2 = 0,$$
$$(\pm(2(K_{Y_7} \pm K_{H(n)}) + k_2 f))^2 = 4(K_{Y_7} \pm K_{H(n)})^2 > 0.$$

Therefore, by Proposition 1.4, $(Y_7 \sharp_f H(n)) \sharp_f E(m)$ is irreducible. □

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